THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE MATHEMATICAL TRAINING OF CHEMISTS[1]

Your committee has endeavored to emphasize ways in which the mathematical training of chemists can be improved. It considers improvement necessary because chemistry is rapidly becoming more mathematical and because adequate preparation becomes more vital when competition for positions is keener. Furthermore it hopes that early in his career the prospective student of chemistry will be told by his advisers of the importance of mathematics, and that the mathematical prerequisites will be definitely stated in the catalogs describing the course.

REQUIREMENTS FOR PHYSICAL CHEMISTRY

The demand for calculus as a universal prerequisite for physical chemistry has been gaining headway rapidly. In 1932 the student of Chemistry, who does not understand the use of calculus, is denied admission to any first class course in physical chemistry. In some cases it may still be necessary to admit students of the biological or agricultural sciences if they agree to study simultaneously the chemically-essential parts of calculus.

The standard courses in mathematics through analytical geometry and differential and integral calculus are sufficient for the first course in physical chemistry. The student of physical chemistry above all should understand fully the meaning and significance of calculus. He should have at immediate command a limited number of formulas (including the differentiation of x, x^n, e^x, a^x, $\log_e x$, uv, u/v, and the integration of x, x^{-1}, x^n, e^x; and the formulas for partial differentiation and for integration between limits) but he need not be ashamed to refer to his book and to a table of integrals for more complicated formulas. Too often the loss of confidence, resulting from a realization that many formulas have been forgotten, is a serious and unnecessary handicap. With this meager equipment the student of physical chemistry can get along, but his progress will depend largely on the adequacy of his mathematical training and insight. His success in the course will be greater, the greater the number of problems in mathematics which he has worked out. He must be thoroughly familiar with algebra and he must be able to make transpositions, solve quadratic equations and handle formulas without hesitation. The successful student of physical chemistry will be quick to translate a phenomenon into mathematical language and he will see clearly and completely the physical significance of his formulas. When properly trained, the student will use his slide rule efficiently for some problems and for others he will use extensive tables of logarithms with unfailing accuracy. He will not hesitate to use the exponential notation for very large

[1] This is a report of the Committee on Mathematics, Division of Chemical Education, American Chemical Society, presented at the Denver meeting, August 24, 1932. In addition to what is here given, the report contained a section of *Suggestions for the Journal of Chemical Education*, and a much longer list of typical problems. The committee consisted of Professor Farrington Daniels, chairman, and Professors George H. Bruce, Charles H. Stone, Walter T. Schrenck, and John H. Yoe.

and small numbers nor will he be disturbed when his graphs have unequal scales along the axes or when they fail to start at zero. He must acquire a sense of the order of accuracy required in different types of problems.

SUGGESTIONS FOR TEACHERS OF MATHEMATICS

Certain parts of the standard courses in mathematics are of more importance to the chemist than other parts. Teachers of mathematics realize this fact and would like to know the particular requirements for students of chemistry. In fact this Committee was appointed as a direct result of a letter from Professor E. R. Hedrick, then president of the American Mathematical Society, stating that suggestions from teachers of chemistry would be welcomed.

A few mathematical operations seem to give particular trouble in physical chemistry and it is hoped that more attention may be paid to them in the preparatory courses. Engineering students with their greater experience in mathematical problems have little trouble. Students can use their logarithm tables but they are sometimes troubled when the logarithm must be handled as a number in a formula, particularly when a negative characteristic is involved. Exponential equations sometimes offer difficulties, also. The idea of partial differentiation is particularly important in physical chemistry and thermodynamics and it is to be hoped that this subject will not be reserved for advanced courses nor placed at the end of a course but that it will receive adequate treatment in the standard courses in calculus.

The chemist usually graphs his data but he would like to fit them with a mathematical formula. This Committee feels that more attention should be paid to curve fitting in analytical geometry. Experience in exact graphical calculations and the finding of areas and tangents would be very welcome also. Chemists would find these things of more value than many theorems in trigonometry and analytical geometry. The compound interest law and the significance of e, the base of natural logarithms, are of special importance.

It is realized that these special needs may not be suitable for engineers and others, who constitute the majority of the classes in mathematics. When possible, an excellent service could be rendered by putting chemistry students into separate sections in the mathematics courses and giving them slightly different material, preferably under an instructor in mathematics who has had some training in the natural sciences.

Whether or not it is practical to put chemists into a separate section, it should be possible to include, along with the problems of interest to engineers and physicists, a few of special interest to chemists. Attached to this report is a list of problems which may be taken as suggestive of the type which chemists would like to see sprinkled through text books on mathematics, as special preparation for physical chemistry.

The committee feels that the responsibility for any action along these lines rests with the chemistry faculties in the colleges and universities. If they take the initiative in presenting their special needs to their department of mathematics, the latter will be glad to coöperate.

REQUIREMENTS FOR ADVANCED PHYSICAL CHEMISTRY

For advanced courses in physical chemistry, and research in the newer fields of physical chemistry the mathematical training through calculus is insufficient. Sooner or later the advanced physical chemist will be blocked by inadequate preparation in mathematics but the distance to which he can go will depend largely on his mathematical ability. The only advice to those who wish to prepare themselves for the new advances in physical chemistry and quantum theory is to go as far as possible with the mastery of advanced mathematics.

It may often be advantageous for a member of the chemistry or mathematics faculty to give a course on selected topics in advanced mathematics for those graduate students in physical chemistry who do not expect to do creative work in these fields.

REQUIREMENTS FOR ELEMENTARY CHEMISTRY

The mathematical requirements for elementary courses in chemistry are not extensive. A knowledge of algebra is sufficient but here again the mathematically-adept student or the person who has worked problems in pure mathematics by the thousands will advance more easily. The concept of ratio and proportion is of course very important (The form $a/b=c/d$ is used in preference to the form $a:b::c:d$).

SELECTED TYPICAL PROBLEMS

1. The dissociation constant K of hydro iodic acid is given by the expression

$$K = \frac{(a-x)(b-x)}{4x^2},$$

where a and b are constants representing the concentration of hydrogen and iodine. What is the value of x in terms of a, b, and K?

2. $10^x = 0.00932$. $x=?$

3. $e^{-0.04} = ?$

4. $0.700 = 0.02 \log (0.01/x)$. $x=?$

5. The rate of decay of radon is given by the expression $k=[2.303 \log (I_0/I)]/t$ where I_0 is the radio activity intensity at the beginning of the experiment, I is the intensity after time t, and k is a constant. If the intensity has fallen to one-fifth of its value in 12,800 minutes, what is the value of k? What will be the intensity after 100 minutes? At what time will the intensity have fallen to one-half its value. Solve the preceding problem with the equivalent formula $I=I_0e^{-kt}$.

6. The vapor pressure P of chloroform at different absolute temperatures T is as follows:

T	293°	303	313	323	333
P	160.5 mm	247.5	369.3	535.0	755.5

Plot $\log P$ against $1/T$ and find the equation for the best straight line passing through these points.

7. In a chemical reaction involving two types of molecules the amount of material reacting x in time t is related to the concentrations a and b of the original materials as shown by the relation $dx/dt = k(a-x)(b-x)$ where k is a constant. Integrate and solve for k.

HISTORICAL NOTE ON NEGATIVE NUMBERS[1]

By G. A. MILLER, University of Illinois

Negative numbers now play such an important rôle in mathematics that the question when these numbers began to be used systematically by European mathematicians is of fundamental importance in the history of our science. It may therefore be of interest to note here some widely different views as regards this point which appear in several of our most reliable works of reference. In particular, it is stated in the *Encyklopädie der Mathematischen Wissenschaften*, note 18 of the first article (1898), that the actual calculation with negative numbers begins with R. Descartes (1596–1650) and that he attributed to the same letter sometimes a positive and sometimes a negative value. In the corresponding note (149) of the French edition of this encyclopedia which began to appear in 1904 it is stated, on the contrary, that the systematic calculations with negative numbers are posterior to R. Descartes.

These contradictory statements seem to deserve emphasis here because they were naturally adopted and will probably be adopted in the future by various writers who separately may have consulted only one of these noted authorities. The view that the systematic use of negative numbers is posterior to R. Descartes is now supported by many authorities including J. Tropfke, *Geschichte der Elementar-Mathematik*, volume 2 (1921), page 77, where it is also stated that R. Descartes did not attribute sometimes a positive and sometimes a negative number to the same letter. He did, however, sometimes place a dot before a letter and then this letter with its dot might be replaced by him either by a positive or by a negative number. This is clearly not in strict accord with the statement of the preceding paragraph.

Not only is there a radical difference between the statements relating to the early European use of negative numbers which appear in the two modern mathematical encyclopedias noted above but both of these statements are at variance with the one which appears in a still more modern mathematical encyclopedia; viz., the Italian work entitled *Enciclopedia delle Matematiche Elementari* which began to be published in 1930. In the account of the development of negative numbers found on page 89 of the first part of this work it is stated that while signs of negative numbers appear in the Greek work of Diophantus and in the Indian works by Brahmagupta and Bhaskara, and then in the European works of N. Chuquet and M. Stifel, the conception may be said to be sufficient in H.

[1] Abstract from a lecture before the Mathematical Association of America (Illinois Section), May 6, 1932.

Cardan and is complete in A. Girard. It therefore results that according to this encyclopedia the conception of negative numbers was completed even before the appearance of the famous Géométrie by R. Descartes in 1637.

One of the most striking evidences of the fact that the conception of negative numbers was not fully completed before about the close of the eighteenth century is the view that these numbers are greater than infinity which was expressed even in the second half of the eighteenth century by L. Euler, who was one of the greatest mathematicians of all times and corresponded with leading mathematicians of his day. This view had been announced earlier by the English mathematician, J. Wallis, and was maintained for a long time notwithstanding its absurdity from our point of view. It is also interesting to note in this connection that the real reason which inspired the work of the noted French mathematician, L. N. M. Carnot (1753–1823), along the line of projective geometry was his aversion for negative numbers, which he rejected because he thought that their use led to erroneous conclusions. This was done even in the early part of the nineteenth century. Cf. *Encyclopédie des Sciences Mathématiques*, tome 3, volume 2, page 3.

Some of the contradictory historical statements relating to negative numbers are doubtless due to a failure on the part of various writers to distinguish between the correct practical use of these numbers and a satisfactory explanation of the theory upon which the correct usage should be based. The latter appeared long after the former. Our present object is, however, not the establishment of this late appearance but the exhibition of facts which tend towards an explanation of various contradictory statements relating thereto since they are apt to interfere with the formation of clear views on this important historical question. Fortunately the history of mathematics is becoming a science instead of a chronicle or reportage.

A MINIMUM PROBLEM

By J. V. USPENSKY, Stanford University

1. In the present note we give a solution of the following problem: To find the *best* approximate representation of $(x^2+y^2)^{\frac{1}{2}}$ by a linear function $\lambda x+\mu y$ in a rectangular domain $0\leqq x\leqq a$; $0\leqq y\leqq b$. The solution of this problem depends, of course, on the meaning we attach to the expression "the best representation." Here this expression is intended to mean that the deviation of the function $(x^2+y^2)^{\frac{1}{2}}-\lambda x-\mu y$ from 0 in the rectangle $0\leqq x\leqq a$; $0\leqq y\leqq b$ should be as small as possible. To avoid misunderstanding we agree to call deviation of a function in a given domain the greatest numerical value the function attains in that domain.

The problem being stated in precise terms, here is the final answer: Without restricting the generality we may assume that $b/a=n\leqq 1$. Then, (i) if $n\geqq(9-\sqrt{17})/8$, parameters λ, μ should be chosen as follows

$$\lambda = \frac{2a - b + (a^2 + b^2)^{1/2}}{3a}$$

$$\mu = \frac{2b - a + (a^2 + b^2)^{1/2}}{3b};$$

and (ii) if $n \leqq (9-\sqrt{17})/8$, we must take

$$\lambda = \frac{4a^2 - 2ab + 2b(ab)^{1/2}}{b^2 + 4a^2}$$

$$\mu = \frac{2a^2 + b^2 - ab - 2a(ab)^{1/2}}{b^2 + 4a^2}\cdot$$

The deviation of $(x^2+y^2)^{1/2}-\lambda x-\mu y$ from 0 in the first case is

$$\delta = [a + b - (a^2 + b^2)^{1/2}]/3$$

whereas in the second case

$$\delta = [ab^2 + 2a^2b - 2(ab)^{3/2}]/(b^2 + 4a^2).$$

2. A few preliminary remarks will greatly facilitate the proof of these statements. The function $\phi(x,y)=(x^2+y^2)^{1/2}-\lambda x-\mu y$ cannot attain its extreme values in an *interior* point of the rectangle. For if the extreme value of $\phi(x, y)$ were attained in an interior point, we would have at this point

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0$$

whence, by virtue of homogeneity, $\phi(x, y)=0$ which is impossible. Thus the extreme values of $\phi(x, y)$ are attained at points on the boundary. Such points are to be sought only among the following five:

$$x = a,\ y = 0;\ x = 0,\ y = b;\ x = a,\ y = b;$$

$$x = \frac{\lambda b}{(1 - \lambda^2)^{1/2}},\ y = b;\ x = a,\ y = \frac{\mu a}{(1 - \mu^2)^{1/2}}\cdot$$

The fourth and fifth points can be taken into consideration only if we have respectively

$$0 < \lambda < \frac{a}{(a^2 + b^2)^{1/2}} \text{ or } 0 < \mu < \frac{b}{(a^2 + b^2)^{1/2}}\cdot$$

The greatest numerical values of $\phi(x, y)$ can be sought only among the following five numbers:

$$\phi(a, 0) = a(1 - \lambda)$$

$$\phi(0, b) = b(1 - \mu)$$

$$\phi(a, b) = (a^2 + b^2)^{1/2} - \lambda a - \mu b$$

$$\phi\left(\frac{\lambda b}{(1-\lambda^2)^{1/2}}, b\right) = b\left[(1-\lambda^2)^{1/2} - \mu\right]$$

$$\phi\left(a, \frac{\mu a}{(1-\mu^2)^{1/2}}\right) = a\left[(1-\mu^2)^{1/2} - \lambda\right].$$

The last two numbers can be taken into account only when λ and μ are subject to the limitations mentioned above. Besides, if these numbers are not less in absolute value than the remaining three, they must be *negative* since they correspond to minima of the function $\phi(x, y)$; this can be readily inferred from the fact that

$$\frac{\partial^2\phi}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}} > 0, \quad \frac{\partial^2\phi}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{3/2}} > 0.$$

3. Now we can easily dispose of Case 1 in which $n \geqq (9-\sqrt{17})/8$. In this case, taking for λ and μ

$$\lambda = \frac{2 - n + (n^2+1)^{1/2}}{3}$$

$$\mu = \frac{2n - 1 + (n^2+1)^{1/2}}{3},$$

we find that the greatest numerical values of $\phi(x, y)$ are attained only at the points $(0, b)$; $(a, 0)$; (a, b) and

$$\phi(a, 0) = \phi(0, b) = -\phi(a, b) = \frac{a + b - (a^2+b^2)^{1/2}}{3}.$$

To prove this it suffices to notice that of the two inequalities

$$\frac{2 - n + (n^2+1)^{1/2}}{3} \geqq \frac{1}{(n^2+1)^{1/2}}$$

$$\frac{2n - 1 + (n^2+1)^{1/2}}{3n} \geqq \frac{n}{(n^2+1)^{1/2}}$$

the first one is satisfied for every positive $n \leqq 1$ whereas the second requires $n \geqq (9-\sqrt{17})/8$.

It remains to prove that for λ and μ chosen as prescribed, the function $\phi(x, y)$ has the least possible deviation from 0. To this end let $\phi_1(x, y)$ be another function of the same type whose deviation is $\leqq \delta$. The difference $\psi(x, y) = \phi(x, y) - \phi_1(x, y)$ is a linear function $\psi(x, y) = \alpha x + \beta y$ and by hypothesis

$$\psi(a, 0) = \alpha a \geqq 0, \psi(0, b) = \beta b \geqq 0,$$

$$\psi(a, b) = \alpha a + \beta b \leqq 0,$$

which is impossible unless $\alpha = \beta = 0$. Thus Case 1 is disposed of.

4. It is more difficult to dispose of Case 2 when $n \leqq (9-\sqrt{17})/8$. To this end we shall first prove the truth of the following statements:

THEOREM 1. *If* $0<\lambda<1$, *then* $b[\mu-(1-\lambda^2)^{1/2}]\geqq\lambda a+\mu b-(a^2+b^2)^{1/2}$, *the equality sign being possible only when* $\lambda=a/(a^2+b^2)^{1/2}$.

PROOF. From the evident inequality

$$[a-\lambda(a^2+b^2)^{1/2}]^2\geqq 0$$

it follows that

$$b^2(1-\lambda^2)\leqq[-\lambda a+(a^2+b^2)^{1/2}]^2,$$

whence

$$-b(1-\lambda^2)^{1/2}\geqq\lambda a-(a^2+b^2)^{1/2}$$

and

$$b[\mu-(1-\lambda^2)^{1/2}]\geqq\lambda a+\mu b-(a^2+b^2)^{1/2}.$$

THEOREM 2. *The system of equations*

$$a(1-\lambda)=b(1-\mu)=b[\mu-(1-\lambda^2)^{1/2}]$$

has a solution in real numbers λ, μ *such that*

$$\mu>b/(a^2+b^2)^{1/2};\ 0<\lambda\leqq a/(a^2+b^2)^{1/2}$$

provided $n\leqq(9-\sqrt{17})/8$.

PROOF. Putting $1-\mu=\rho$ we have

$$\lambda=1-n\rho,\ 1-2\rho=(1-\lambda^2)^{1/2},$$

and eliminating λ,

$$(4+n^2)\rho^2-2(2+n)\rho+1=0.$$

To satisfy the requirements we take

$$\rho=\frac{n+2-2n^{1/2}}{4+n^2}=\frac{1}{n+2+2n^{1/2}}\cdot$$

In fact the inequality

$$\mu>n/(n^2+1)^{1/2}$$

is equivalent to

$$n+2+2n^{1/2}>n^2+1+n(n^2+1)^{1/2}$$

and it is easy to verify that the last inequality holds for $0<n\leqq 1$. Next

$$\lambda=1-n\rho>0.$$

Finally the condition

$$\lambda\leqq\frac{1}{(n^2+1)^{1/2}}$$

is satisfied if

$$\theta(n)=n+2+2n^{1/2}-2(1+n^{1/2})(n^2+1)^{1/2}\geqq 0.$$

For small positive n

$$\theta(n) > 0$$

and this inequality will hold while n remains less than the least root of the equation $\theta(n)=0$. Setting $n=t^2$ the resulting equation in t is

$$t^2 + 2t + 2 = 2(1+t)(1+t^4)^{1/2}$$

whence

$$4t^4 + 8t^3 + 3t^2 - 4t - 4 = 0.$$

This quartic splits into two quadratics

$$2t^2 + 3t + 2 = 0, \; 2t^2 + t - 2 = 0.$$

The first has imaginary roots. The positive root of the second is

$$t = (-1 + \sqrt{17})/4,$$

and correspondingly

$$n = t^2 = (9 - \sqrt{17})/8.$$

Hence $\theta(n) \geqq 0$ if $n \leqq (9-\sqrt{17})/8$.

THEOREM 3. *Numbers λ and μ being chosen as in Theorem 2 we shall have*

$$(a^2 + b^2)^{1/2} - \lambda a - \mu b < 0.$$

PROOF. This inequality is equivalent to

$$(n^2 + 4)(n^2 + 1)^{1/2} < n^3 - n^2 + 4n^{3/2} + 4$$

which, as it is easy to show, holds for $0 < n \leqq 1$.

THEOREM 4. *The function $\phi(x, y)$ attains its greatest numerical value $b\rho$ three times if λ and μ are chosen as in Theorem 2 and $n \leqq (9-\sqrt{17})/8$.*

PROOF. Since

$$\mu > n/(n^2 + 1)^{1/2}$$

the greatest numerical value of $\phi(x, y)$ is either

$$a(1-\lambda) = b(1-\mu) = -b[(1-\lambda^2)^{1/2} - \mu]$$

or

$$\lambda a + \mu b - (a^2 + b^2)^{1/2}.$$

By Theorems 3 and 1 we have

$$b[(1-\lambda^2)^{1/2} - \mu] > \lambda a + \mu b - (a^2 + b^2)^{1/2} > 0;$$

hence the statement is proved.

5. It is easy now to dispose of Case 2. By Theorem 4 the function $\phi(x, y)$ determined as required in Case 2, attains its greatest numerical value three times; namely, for

$$x = a,\ y = 0;\ x = 0,\ y = b;\ x = \lambda b/(1 - \lambda^2)^{1/2},\ y = b;$$

and so that

$$\phi(a, 0) = \phi(0, b) = -\phi[\lambda b/(1 - \lambda^2)^{1/2}, b] = \delta.$$

Let $\phi_1(x, y)$ be another function of the same type as $\phi(x, y)$ whose deviation from 0 is $\leqq \delta$. Then, setting

$$\psi(x, y) = \phi(x, y) - \phi_1(x, y) = \alpha x + \beta y$$

we shall have

$$\psi(a, 0) = \alpha a \geqq 0,\ \psi(0, b) = \beta b \geqq 0$$

$$\psi[\lambda b/(1 - \lambda^2)^{1/2}, b] = \alpha\lambda b/(1 - \lambda^2)^{1/2} + \beta b \leqq 0.$$

But $\lambda > 0$ and hence necessarily $\alpha = \beta = 0$. The proof is now complete.

ON SUMS OF SQUARES[1]

By GORDON PALL, McGill University

1. We shall obtain several results concerning:

1) Numbers which are sums of three positive squares;

2) All numbers which are not sums of s unequal squares, $s = 4$ or 5;

3) A formula for $r_s(n^2c)/r_s(c)$, where $r_s(n)$ denotes the number of representations of n as a sum of s squares;

4) The number of representations of an integer as a sum of s squares with bases in arithmetic progression, $s = 2, 3, 4$;

5) The number $R_s(n)$ of representations of n as a sum of s odd squares, $2 \leqq s \leqq 8$;

6) Representation as a sum of four squares prime to m.

Since the writer duplicated Dubouis' work before noticing the report of it (Dickson's History II, p. 316), it may be well to mention his theorem that, if $s \geqq 6$, the only integers not equal to a sum of s positive squares are

$$1, 2, 3, \cdots, s - 1, \text{ and } s + B,\ (B = 1, 2, 4, 5, 7, 10, 13);$$

that the only $n > 0$ not sums of five positive squares are 1, 2, 3, 4, $5 + B$, and 33; and finally, Descartes' conjecture that the only $n > 0$ not a sum of four positive squares are

$$1, 3, 5, 9, 11, 17, 29, 41, 2\cdot 4^h, 6\cdot 4^h, 14\cdot 4^h\ (h \geqq 0). \tag{1}$$

---

[1] Most of this paper was written by the author as a National Research Fellow at the California Institute of Technology.

It is trivial, since they are not sums of two squares, that every positive $8n+3$ and $8n+6$ is a sum of three positive squares.[1] It is conjectured that every $2(8n+1)$ except 2 and every $8n+1$ except 1 and 25 is a sum of three positive squares. The last conjecture was verified for every $8n+1$ to 1521.

A. Hurwitz [*Dickson's History* II, p. 271] proved that the only squares not a sum of three positive squares are 4^h and $25 \cdot 4^h$. We readily generalize this in the following theorem.

THEOREM 1. *Except for the numbers $25 \cdot 4^h$, every positive integer which is a sum of three squares at all, and possesses an odd square factor >1, is a sum of three positive squares.*

2. The proof depends on the case $s=3$ of the following formula, which is valid in case

$$(2) \qquad s = 2, 3, 4, 5, 6, 7, 8,\ s = 10 \text{ if } c \neq S_2,\ s = 11 \text{ if } c \neq S_3,$$
$$s = 12 \text{ if } c \text{ is even},$$

where S_r denotes a sum of r squares. In each case (2), we have[2]

$$(3) \qquad \frac{r_s(cn^2)}{r_s(c)} = \frac{\psi_2(cn^2)}{\psi_2(c)} \cdot \frac{\Pi\psi_p(cn^2)}{\Pi\psi_p(c)},$$

the products extending to all odd primes p.

To define $\psi_q(n)\, [=\psi_{q,s}(n)]$ for any prime q and positive integers s and n, write

$$(4) \qquad n = 2^\alpha m = r^2 t,\ m \text{ odd},\ t \text{ simple}^3,\ \alpha \geqq 0.$$

---

[1] The reader will have no difficulty in proving Descartes' conjecture concerning (1) by using the following classical result, which was first stated in full by Fermat, and was first proved by Legendre in 1798. A positive integer is a sum of three squares if and only if it is not of the form $4^h(8n+7)$ where h and n are integers $\geqq 0$. (Cf. Dickson's History, II, p. ix and p. 302.)

[2] For $s=5, 7$, and for $s=11$ if $c \neq S_3$, formula (3) is proved in the Journal of the London Mathematical Society, vol. 5, (1930), 102–105; for $s=3$ it has appeared in many forms, for example as the Gauss Problem (Cf. *Dickson's History* III, p. 95).

To extend it to the even values s I have merely taken well-known formulas for $r_s(n)$ (Dickson's History, II, pp. 233, 285, 305 ff) and united the resulting expressions for the left member of (3). Since we require $r_2(n)$ in §3 I state a formula for it: write $n = 2^\alpha p_1^{a_1} \cdots p_r^{a_r}$ where the p_i are distinct odd primes, $\alpha \geqq 0$, $a_i \geqq 0$, and $r \geqq 0$; then

$$r_2(n) = 4\prod_i \phi(p_i^{a_i}),$$

where

$$\begin{aligned} \phi(p^a) &= a + 1 \text{ if } p \equiv 1 \pmod 4; \\ &= 1 \text{ if } p \equiv 3 \text{ and } a \text{ even}; \\ &= 0 \text{ if } p \equiv 3 \text{ and } a \text{ odd}. \end{aligned}$$

The reader may deduce (3) for the case $s=2$.

[3] A positive integer without square factors >1 is called *simple*.

Let $a(=a_p)$ denote the exponent to which an odd prime p divides n, and let b, β, σ denote the greatest integers $\leqq \frac{1}{2}a$, $\frac{1}{2}\alpha$, $\frac{1}{2}s$ respectively. For example, b is the exponent to which p divides r. For any positive integers s and n, let

$$\zeta_s(n) = \sum_{n=d\delta} d^s, \quad \rho_s(n) = \sum_{n=d\delta} (-1)^{(\delta-1)/2} d^s, \tag{5}$$

where, in the second summation, δ is odd. Then we define

$$\begin{aligned} \psi_2(n) &= 1 \quad (\alpha = 0,\ s \equiv 0, \bmod 4), \\ &= 2^{\alpha(\sigma-1)} + (-1)^{\sigma/2}\{\zeta_{\sigma-1}(2^{\alpha-1}) - 2\} \quad (\alpha > 0,\ s \equiv 0), \\ &= 2^{\alpha(\sigma-1)} + (-1)^{(m-\sigma)/2} 2^{-(\sigma-1)} \quad (s \equiv 2), \\ &= \left\{1 + (2 \mid s)\frac{2^{(s-3)/2}}{2^{s-2} - 1}\right\} (2^{(s-2)\beta} - 1) + 1 + k \quad (s \text{ odd}), \end{aligned} \tag{6}$$

where

$$\begin{aligned} k &= (2 \mid s) 2^{-\sigma} + (-1)^{(m-s)/4} 2^{-(s-2)} \text{ if } t \equiv s \pmod 4, \\ &= -(2 \mid s) 2^{-\sigma} \text{ otherwise;} \end{aligned}$$

$$\begin{aligned} \psi_p(n) &= \zeta_{\sigma-1}(p^a) \quad (s \equiv 0), \\ &= \rho_{\sigma-1}(p^a) \quad (s \equiv 2), \\ &= \zeta_{s-2}(p^b) - (\{-1\}^{\sigma} t \mid p) p^{(s-3)/2} \zeta_{s-2}(p^{b-1}) \quad (s \text{ odd}). \end{aligned} \tag{7}$$

Thus, if s is even, $\psi_p(n) = 1$ unless $p \mid n$;[1] if s is odd, $\psi_p(n) = 1$ unless $p^2 \mid n$.

3. Proof of Theorem 1.

Let p be an odd prime and c simple. As a particular case of (3) we have then

$$r_3(p^2c) = r_3(c)\{p + 1 - (-c \mid p)\}. \tag{8}$$

Now $r_2(p^2c) = \lambda r_2(c)$ where λ is 1, 2, or 3 according as $p \equiv 3$, $p \equiv 1$ and $p \mid c$, or $p \equiv 1$ and $p \nmid c$, the congruences being mod 4. Hence, if $r^*(n)$ denotes for the moment the number of representations of n as a sum of three squares one of which is zero,

$$\begin{aligned} r^*(p^2c) &= 6\{3 + 2(-1 \mid p)\} \text{ if } c = 1, \\ &= 3\lambda r_2(c) \text{ if } c > 1. \end{aligned} \tag{9}$$

Hence

$$r_3(p^2) - r^*(p^2) = 6p - 12 - 18(-1 \mid p) > 0$$

except for $p = 5$, when it is zero. If $c > 1$, then $r_3(c) \geqq 3r_2(c)$, and

$$r_3(p^2c) - r^*(p^2c) = r_3(c)\{p + 1 - (-c \mid p)\} - 3\lambda r_2(c) > 0,$$

if $r_3(c) > 0$, for every $p \geqq 3$.

---

[1] The symbols $p \mid n$ and $p \nmid n$ indicate respectively that p divides n and p does not divide n. The symbol $(a \mid b)$ is the Legendre-Jacobi symbol: for example, $(a \mid b) = 0$ if a and b have a g.c.d. > 1; $(-1 \mid p) = 1$ if p is a prime of the form $4n+1$, and $(-1 \mid p) = -1$ if p is a prime $4n+3$.

Theorem 1 evidently follows.

4. The simplest and most elegant special case of (3) is as follows.

Let n be simple and odd. Then for $s=3, 5, 7$, or $s=11$ if $n\equiv 7 \pmod 8$, and for every integer $h\geqq 0$,

$$\frac{r_s(n^{2h+1})}{r_s(n)} = \zeta_{s-2}(n^h). \tag{10}$$

For example, $r_{11}(343)=2^{10}\cdot 3^2\cdot 5\cdot 11\cdot 43\cdot 117307$.

5. THEOREM 2. *The only integers $n>0$ not sums of four unequal squares $\geqq 0$ are $4^h a$, where $h=0, 1, 2, \cdots$, and*

$$\begin{aligned} a = {} & 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 27, 31, 33, 37, 43, 47, 55, 67, 73, \\ & 97, 103, 2, 6, 10, 18, 22, 34, 58, 82. \end{aligned} \tag{11}$$

Let A denote a positive odd integer. The equations

$$\begin{aligned} X &= x+y+z+w, \quad & Y &= x+y-z-w, \\ Z &= x-y+z-w, \quad & W &= x-y-z+w, \end{aligned} \tag{12}$$

define a (1, 1) correspondence between the sets of integers x, y, z, w satisfying

$$A = x^2+y^2+z^2+w^2 \tag{13}$$

and the sets of integers X, Y, Z, W satisfying

$$4A = X^2+Y^2+Z^2+W^2,\ X+Y+Z+W\equiv 0 \pmod 4;\ X, Y, Z, W \text{ odd}. \tag{14}$$

Similarly, a (1, 1) correspondence between the integral solutions of (13) and those of

$$2A = s^2+t^2+u^2+v^2,\ s\equiv t\not\equiv u\equiv v \pmod 2 \tag{15}$$

is defined by the equations

$$s = x+y,\ t = x-y,\ u = z+w,\ v = z-w. \tag{16}$$

As immediate corollaries of these facts we have

LEMMA 1. *An odd integer A is a sum of four unequal squares if and only if $4A$ is a sum of four unequal odd squares.*

LEMMA 2. *An odd integer A is a sum of four positive squares if and only if $2A$ is a sum of four unequal squares.*

Now, if either $x^2=y^2$ or $z^2=w^2$ in (13), then

$$stuv = 0. \tag{17}$$

And, if either $x^2=z^2$ or w^2, or $y^2=z^2$ or w^2, then

$$e_1s + e_2t + e_3u + e_4v = 0, \tag{18}$$

where the $e_i=+1$ or -1. Hence we have

LEMMA 3. *If $2A$ possesses a representation*

$$2A = s^2 + t^2 + u^2 + v^2,\ s,\ t,\ u,\ v \neq 0,\ s^2 > 3A/2, \tag{19}$$

then A is a sum of four unequal squares.

For, the inequality implies the last step of

$$(|t| + |u| + |v|)^2 \leqq 3(t^2 + u^2 + v^2) = 3(2A - s^2) < s^2, \tag{20}$$

which makes (18) impossible.

In proving theorem 2 we can, by the first lemma, restrict ourselves to $a \not\equiv 0 \pmod 4$. Lemma 2 and (1) give the complete results for $a\equiv 2 \pmod 4$.

For A odd (especially $A\equiv 3 \pmod 4$) we shall use lemma 3. Now $2A-s^2\equiv 6 \pmod 8$ for one of two consecutive even values s. Hence for one of four consecutive choices s, $2A-s^2$ is an S_3 but not an S_2. But four consecutive squares lie between $1\frac{1}{2}A$ and $2A$ if $(2A)^{1/2}-(3A/2)^{1/2}>4$, $A>421$.

If $A\equiv 1 \pmod 4$ a different method gives a lower limit than 421. We can choose x^2 from 1, 9, 25, and 49 so that $(A-x^2)/4\equiv 5 \pmod 8$ if $A\equiv 5 \pmod 8$, and from 1, 9, $\cdots$, 225 so that $(A-x^2)/4\equiv 10$ or $14 \pmod{16}$ if $A\equiv 1 \pmod 8$. In either case $(A-x^2)/4$ is an S_3 but not of the form $\xi^2+2\eta^2$. Hence every $8v+1\geqq 225$ and every $8v+5\geqq 53$ is a sum of four unequal squares. Examination of the few values A below these limits easily gives (11).

In determining all numbers not sums of five unequal squares, of which process we need not give the details, it was convenient to use both the lists (1) and (11), and lemmas 1 and 2 again.

THEOREM 3. *The only integers $n>0$ not sums of five unequal squares are*:

(21) 1, $\cdots$, 29, 31, $\cdots$, 38, 40, $\cdots$, 45, 47, 48, 49, 52, 53, 56, 58, $\cdots$, 61, 64, 67, 68, 69, 72, 73, 76, 77, 80, 83, 89, 92, 96, 97, 101, 104, 108, 112, 124, 128, 136, 137, 188, 224.

6. We denote $N[n=x_1^2+\cdots+x_s^2]$ by $r_s(n)$, and

$$N[n = x_1^2 + \cdots + x_s^2;\ \text{all } x_i \text{ odd}] \text{ by } R_s(n).$$

Hence $R_s(n)=0$ unless $n\equiv s \pmod 8$. We easily prove that: if $n\equiv s \pmod 8$ and $4\leqq s\leqq 7$, then

$$r_s(n) = c_sR_s(n), \tag{22}$$

where $c_4=3/2$, $c_5=7/2$, $c_6=17/2$, $c_7=37/2$. Also, if $n=2^\alpha m$, $\alpha\geqq 3$, m odd, then

$$R_8(n) = 2^{3\alpha-1}\zeta_3(m), \tag{23}$$

the right member being an evaluation, from classical formulae, of $[r_8(n) - r_8(n/4)]/36$.

PROOF: By Jacobi's theorem for four squares,

$$N[n = x_1^2 + x_2^2 + x_3^2 + x_4^2;\text{ all } x_i \text{ even}] = \tfrac{1}{2}N[n = x_1^2 + x_2^2 + x_3^2 + x_4^2;\text{ all } x_i \text{ odd}]$$

if $n\equiv 4 \pmod 8$. Hence, if $4\leqq s\leqq 7$, and $n\equiv s \pmod 8$,

$$r_s(n) = R_s(n) + \binom{s}{4} N(n = x_1^2 + \cdots + x_s^2; C) = R_s(n) + \tfrac{1}{2}\binom{s}{4} R_s(n),$$

wherein C denotes the condition $x_1, \cdots, x_4$ even, $x_1^2 + \cdots + x_4^2 \equiv 4 \pmod 8$. This gives (22). Also, by a similar argument,

$$r_8(n) = R_8(n) + \tfrac{1}{2}\binom{8}{4} R_8(n) + N[n = x_1^2 + \cdots + x_8^2;\text{ all } x_i \text{ even}].$$

It may be observed, as a corollary, that $r_5(8n+5)$ is divisible by 7, $r_6(8n+6)$ by 17, $r_7(8n+7)$ by 37.

It is interesting to note that as a consequence of (22) and classical formulae for $r_s(n)$, we have

$$R_s(n) = \frac{32n^{(s-1)/2}}{\pi^{(s-1)/2}} \sum \left(\frac{(-1)^{(s-1)/2}n}{m}\right) \frac{1}{m^{(s-1)/2}} \tag{24}$$

if $n\equiv s \pmod 8$, when $s=3$, 5 or 7. The summation is over all positive integers m prime to $2n$.

A curious relation is

$$\begin{aligned} &N[24n + 4 = x_1^2 + \cdots + x_4^2;\text{ all } x_i \text{ prime to } 3] \\ &= N[24n + 4 = x_1^2 + \cdots + x_4^2;\text{ all } x_i \text{ prime to } 2] \\ &= 16\zeta_1(6n + 1). \end{aligned} \tag{25}$$

8. Partial results on sums of three or four squares.

Every $40q+3$ from 3 to 3800, except 43, 403, 763, and 3763, is a sum of three squares of the form $(10n+1)^2$. Every $40q+27$ from 1 to 1227, except 427 and 667, is a sum of three squares of the form $(10n+3)^2$.

It is known that every $3q+4\geqq 4$ is a sum of four squares prime to 3. For no value r is it true that every sufficiently large number $\equiv r \pmod 7$ is a sum of four squares prime to 7 (compare the residues of (1) modulo 7). I have no proof that every $11n>11$ is a sum of four squares prime to 11. Since 2.4^h and 6.4^h represent the residues $r=1$, 2, 3, 4 (mod 5), infinitely many numbers of each progression $5n+r$ are not expressible as a sum of four squares prime to 5. Hence it is interesting to show that every $5n$ except 5 is a sum of four squares prime to 5.

First we consider the possible residues modulo 5 of x^2, y^2, z^2 in the equation

$$5n - w^2 = x^2 + y^2 + z^2, \tag{26}$$

where $5 \nmid w$. Unless two of x, y, z, are zero, which we may suppose not to be the case unless $5n - w^2 = 4^h$ or 25.4^h (by Hurwitz's part of Theorem 1), it is plain that none or two of x, y, z are divisible by 5, and that in the latter case we may assume that $5 \mid y$, $5 \mid z$, $y^2 + z^2 > 0$. By a simple transformation such as $(4Y \pm 3Z)^2 + (3Y \mp 4Z)^2$, possibly repeated, it is easy to replace $(5Y)^2 + (5Z)^2$ by an equal sum of two squares prime to 5.

Hence it remains only to show that, except when $n = 1$, we can choose w prime to 5 so that $5n - w^2$ possesses a representation (26) with no two of x, y, z equal to zero. This will be effected if $5n - w^2$ is of neither of the forms $4^h(8v+7)$ nor 4^h, and is positive. Hence $w = 1$ is effective if $n \equiv 2, 3, 4, 6$, or $7 \pmod 8$; $w = 2$ if $n \equiv 1, 2, 3, 5$, or $6 \pmod 8$ but not $n = 1$; $w = 2^{\alpha+1}$ if $n = 2^{2\alpha+1}N$, N odd, $\alpha \geqq 0$; $w = 2^{\alpha-1}$ if $n = 2^{2\alpha}N$, $\alpha \geqq 1$.

9. Let m and n satisfy $0 \leqq n \leqq m$, and either

$$m, n \text{ are odd and relative prime;} \tag{27}$$

or

$$m/2,\ n/2 \text{ are relative-prime integers of opposite parities.} \tag{28}$$

Let $\rho_s(u)$ where u is any integer $\geqq 0$, denote the number of sets $(x_1, \cdots, x_s)$ of integers x_i satisfying

$$8mu + sn^2 = (2mx_1 + n)^2 + \cdots + (2mx_s + n)^2. \tag{29}$$

There is no loss of generality in assuming (27) or (28), since any sum of s squares whose bases are integers in an arithmetic progression may be reduced essentially to one of those two forms.

PROBLEM. *To express $\rho_s(u)$ by means of numbers of representations in forms with no conditions on the variables.*

We shall succeed in solving this problem only in certain cases where m is *primary*, that is, m is a power of a single prime; or where m is the double of an odd primary number. In the latter case let us call *m secondary.*

First let $s = 2$. Make the transformation $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$. Straightway $\rho_2(u)$ is equal to the number of solutions (y_1, y_2) in integers of

$$4mu + n^2 = (my_1 + n)^2 + m^2y_2^2,\ y_1 \equiv y_2 \pmod 2. \tag{30}$$

Consider then $4mu + n^2 = x^2 + m^2y^2$, in integral (x, y), in the two cases

(31) m primary, $m > 1$, with condition (27); m secondary, $m > 2$, with condition (28).

From $x^2 \equiv n^2 \pmod{m}$ follows $\pm x \equiv n \pmod{m}$ for just one sign of $\pm x$, since otherwise $2n$ would be divisible by an odd prime factor of m. In all cases $(\pm x - n)/m \equiv y \pmod{2}$. This proves that when (31) holds,

$$\rho_2(u) = \tfrac{1}{2}N[4mu + n^2 = x^2 + m^2y^2]. \tag{32}$$

Simpler formulae are easily found for $m = 1$, 2, and 4.

For $s = 3$ we use the transformation

$$\begin{gathered} 3x_1 = y_1 - y_2,\ 6x_2 = 2y_1 + y_2 - 3y_3,\ 6x_3 = 2y_1 + y_2 + 3y_3, \\ y_1 \equiv y_2 \pmod{3},\ y_2 \equiv y_3 \pmod{2}, \end{gathered} \tag{33}$$

whence integers x_i correspond to integers y_i and conversely. Hence $\rho_3(u)$ is the number of solutions (y_1, y_2, y_3) of

$$\begin{gathered} 24mu + 9n^2 = (2my_1 + 3n)^2 + 2m^2y_2^2 + 6m^2y_3^2, \\ y_1 \equiv y_2 \pmod{3},\ y_2 \equiv y_3 \pmod{2}. \end{gathered} \tag{34}$$

Now in each of the three cases

$$\begin{gathered} 3 \nmid m,\ m \text{ primary, with condition (27)}; \\ 3 \nmid m,\ m \text{ secondary, with condition (28)}; \\ m = 2^{\alpha+2},\ \alpha \geqq 0, \text{ with condition (28)}; \end{gathered} \tag{35}$$

the integers x, y, z satisfying

$$24mu + 9n^2 = x^2 + 2m^2y^2 + 6m^2z^2 \tag{36}$$

will satisfy, for every integer $u \geqq 0$, the condition $x \equiv 3n \pmod{2m}$ by, and only by, a choice of sign of x; the condition $y \equiv z \pmod{2}$ automatically; and the condition $x \equiv 3n + 2my \pmod{6m}$ automatically if $3 \mid y$ but only by choice of sign of y if $3 \nmid y$. Hence in cases (35),

$$\begin{aligned} \rho_3(u) = {} & \tfrac{1}{4}N[8mu + 3n^2 = 3x^2 + 6m^2y^2 + 2m^2z^2] \\ & + \tfrac{1}{4}N[24mu + 9n^2 = x^2 + 2m^2y^2 + 6m^2z^2]. \end{aligned} \tag{37}$$

Simpler like formulae hold if $m = 1$, 2, 3, or 6.

For $s = 4$ we apply the transformation

$$\begin{gathered} x_1 + x_2 + x_3 + x_4 = y_1,\ x_1 + x_2 - x_3 - x_4 = y_2, \\ x_1 - x_2 + x_3 - x_4 = y_3,\ x_1 - x_2 - x_3 + x_4 = y_4, \\ y_1 \equiv y_2 \equiv y_3 \equiv y_4 \pmod{2},\ y_1 + y_2 + y_3 + y_4 \equiv 0 \pmod{4}. \end{gathered} \tag{38}$$

Hence $\rho_4(u)$ is the number of integers sets $(y_1, \cdots, y_4)$ satisfying

$$\begin{gathered} 8mu + 4n^2 = (my_1 + 2n)^2 + m^2(y_2^2 + y_3^2 + y_4^2), \\ \textstyle\sum y_i \equiv 0 \pmod{4},\ y_2 \equiv y_3 \equiv y_4 \pmod{2}. \end{gathered} \tag{39}$$

For brevity write $\phi_m(k) = N[k = x^2 + m^2y^2 + m^2z^2 + m^2t^2]$. Then we easily see from (39) that, if m is primary and (27) holds,

$$\rho_4(u) = \tfrac{1}{4}\phi_m(8mu + 4n^2) + \tfrac{1}{4}\phi_m(2mu + n^2); \tag{40}$$

that if $m = 2M$, M primary, and (28) holds,

$$\rho_4(u) = \tfrac{1}{4}\phi_M(2mu + n^2) + \tfrac{1}{4}(-1)^u\phi_M(Mu + \tfrac{1}{4}n^2); \tag{41}$$

that, if u is odd, $m = 4M$, $n = 2N$, M primary, N odd, and M prime to N, then

$$\rho_4(u) = \tfrac{1}{4}N[2Mu + N^2 = 4x^2 + M^2(y^2 + z^2 + t^2)]. \tag{42}$$

Simpler formulae are easily obtained for $m = 1, 2, 4$.

A NEW BOUND FOR THE ZEROS OF POLYNOMIALS

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In the treatment of the equation with complex coefficients

$$z^n = \sum_1^n b_r z^{n-r}, \qquad b_r \text{ complex}, \tag{1}$$

it is convenient to let p_r represent the absolute value of b_r, and in the present paper it is necessary to adopt the convention that q_i represent the positive quantities $p_r^{1/r}$ arranged in order of decreasing magnitude.

Following this notation, a simple bound due to Carmichael[1] is written

$$|z| \leqq \sum_1^n q_i, \tag{a}$$

one obtained by Fujiwara[2] becomes

$$|z| < 2q_1, \tag{b}$$

while one due to the present author[3] becomes

$$|z| \leqq q_1 + q_2. \tag{c}$$

Comparison of the three bounds given above led the present author to form the expression

$$|z| \leqq \sum_1^n g_i q_i, \tag{d}$$

and to attempt to determine the optimum set of values for g_i for which (d) would always be true. The following theorem resulted:

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[1] Bulletin of the American Mathematical Society, Vol. 24 (1917–18), pp. 286–296.

[2] Tôhoku Mathematical Journal, Vol. 10 (1916), pp. 167–171.

[3] The American Mathematical Monthly, Vol. 38 (1931), pp. 30–35.

Theorem: *Every root of equation* (1) *must satisfy inequality* (*d*) *when the positive quantities* g_i *are obtained through the equations*

$$y_m = \sum_1^m g_i, \qquad m = 1, \cdots, n, \tag{2}$$

the positive quantities y_m *being determined by the equations*

$$y_m^m = \sum_1^m y_m^{m-r}, \qquad m = 1, \cdots, n. \tag{3}$$

For the proof of this theorem we employ the following previously developed lemmas:[1]

Lemma 1: *Any value of x satisfying the inequality*

$$x^n \leqq \sum_1^n p_r x^{n-r}, \qquad p_r \geqq 0,$$

will form a lower bound for the sole positive root of the positive term equation

$$x^n = \sum_1^n p_r x^{n-r}, \qquad p_r \geqq 0. \tag{4}$$

Lemma 2: *Any positive value of x satisfying the inequality*

$$x^n \geqq \sum_1^n p_r x^{n-r}, \qquad p_r \geqq 0,$$

will form an upper bound for the sole positive root of (4).

Lemma 3: *No root of equation* (1) *can be greater in absolute value than the positive root of* (4) *when the coefficients of* (1) *and* (4) *are related through the equalities*

$$p_r = | b_r | \qquad r = 1, \cdots, n.$$

From Lemma 3 we see that in order to prove our theorem, we need only show that the positive root of (4) must satisfy the inequality

$$x \leqq \sum_1^n g_i q_i, \tag{5}$$

where the q_i are simply the positive expressions $p_r^{1/r}$ arranged in order of decreasing magnitude.

Employing a previously developed inequality[2] for the positive root of (4),

$$q_1 \leqq x < 2q_1,$$

[1] See the author's paper, loc. cit.

[2] See the author's paper, loc. cit.

we see that the positive roots of equations (3) must satisfy the inequalities

$$1 \leqq y_m < 2, \qquad m = 1, \cdots, n. \tag{6}$$

Choosing one of equations (3) and multiplying it through by y_m^{n-m} we have

$$y_m^n = \sum_1^m y_m^{n-r}, \qquad 0 < m \leqq n. \tag{7}$$

Since y_m is not less than 1, decreasing the degree of any term in the right member of (7) will not increase its value. This gives us

$$y_m^n \geqq \sum_1^m y_m^{n-k_r}, \qquad r \leqq k_r \leqq n;$$

and, applying Lemma 2, we see that the positive root of the $(m+1)$-nomial equation

$$y^n = \sum_1^m y^{n-k_r}, \qquad r \leqq k_r \leqq n,$$

must satisfy the inequality

$$y \leqq y_m \qquad 0 < m \leqq n.$$

Replacing y by x/a, where a is an arbitrary positive quantity and multiplying through by a^n, we see that the positive root of the $(m+1)$-nomial equation

$$x^n = \sum_1^m a^{k_r} x^{n-k_r}, \qquad r \leqq k_r \leqq n,$$

must satisfy the inequality

$$x = ay \leqq ay_m = a \sum_1^m g_i \equiv \sum_1^n g_i q_i.$$

This is identical with inequality (5), and our theorem is proven for the restricted class of equations in which q_i are each equal to either zero or an arbitrary positive quantity a.

We will now show that if our theorem is true when each of the q_i is equal to either zero or any one of $s-1$ arbitrary positive quantities, $1 < s \leqq n$, then it is true when each of them is equal to either zero or any one of s arbitrary positive quantities. If we can do this our theorem is proven, since we have already shown it for one arbitrary positive quantity or zero.

In the equation

$$x^n = \sum_1^n q_i^{k_i} x^{n-k_i}, \tag{8}$$

where the k_i are simply the integers 1 to n inclusive arranged in any order whatsoever, we will take the q_i to satisfy the following relations:

$$
\begin{aligned}
&q_i = C + ew, && i = 1, \cdots, m_1;\\
&q_i = C - (C - A_{s-2})w, && i = m_1 + 1, \cdots, m_2;\\
&q_i = A_{s-2}, && i = m_2 + 1, \cdots, m_3;\\
&q_i = A_{s-3}, && i = m_3 + 1, \cdots, m_4;\\
&\cdots\cdots\cdots && \cdots\cdots\cdots\\
&q_i = A_1, && i = m_{s-1} + 1, \cdots, m_s;\\
&q_i = A_0 \equiv 0, && i = m_s + 1, \cdots, n,
\end{aligned}
\tag{9}
$$

where e is a positive quantity to be evaluated later, and where

$$
\begin{aligned}
&0 < A_1 < \cdots < A_{s-2} < C,\\
&1 \leqq m_1 < \cdots < m_s \leqq n,\\
&0 \leqq w \leqq 1,\\
&1 < s \leqq n.
\end{aligned}
\tag{10}
$$

If our theorem applies to this case, the positive root of (8) must satisfy inequality (5). Remembering the expression for y_m given by (2), this gives us

$$
\begin{aligned}
x \leqq (C + ew)y_{m_1} + (C - Cw + A_{s-2}w)(y_{m_2} - y_{m_1})&\\
+ A_{s-2}(y_{m_3} - y_{m_2}) + \cdots + A_1(y_{m_s} - y_{m_{s-1}})&.
\end{aligned}
\tag{11}
$$

We may now choose the arbitrary positive quantity e to satisfy the equation

$$
ey_{m_1} = (C - A_{s-2})(y_{m_2} - y_{m_1}),
\tag{12}
$$

so as to make the coefficient of w in (11) vanish. This gives us

$$
x \leqq Cy_{m_2} + A_{s-2}(y_{m_3} - y_{m_2}) + \cdots + A_1(y_{m_s} - y_{m_{s-1}}) = h,
\tag{13}
$$

which, according to Lemmas 1 and 2, is equivalent to saying that h satisfies the inequality

$$
h^n \geqq \sum_1^n q_i^{k_i} h^{n-k_i} \equiv F(w), \qquad 0 \leqq w \leqq 1.
\tag{14}
$$

By assumption, our theorem holds for $w=0$ and $w=1$, since for each of these values of w, scheme (9) reduces to $s-1$ arbitrary positive quantities and zero. It follows that the inequalities

$$
h^n \geqq F(0), \quad h^n \geqq F(1)
$$

must hold, and investigation shows that the second derivative of $F(w)$ is positive for values of w between 0 and 1; whence inequality (14) must be true and our theorem must hold for w within this range. Suppose, for instance, we set

$w=c$, where c is an arbitrary positive quantity less than 1, we may then write

$$(15)\qquad \begin{aligned} C + ec &\equiv A_s, && e > 0; \\ C - (C - A_{s-2})c &\equiv A_{s-1}, && 0 < c < 1, \end{aligned}$$

where the quantities A_s and A_{s-1} thus defined are seen to satisfy the conditions

$$(16)\qquad A_{s-2} < A_{s-1} < A_s.$$

We see, then, that our theorem is satisfied when the q_i are given by the relations

$$(17)\qquad \begin{aligned} q_i &= A_s, && i = 1, \cdots, m_1; \\ &\cdots\cdots\cdots && \\ q_i &= A_1, && i = m_{s-1} + 1, \cdots, m_s, \\ q_i &= A_0 \equiv 0, && i = m_s + 1, \cdots, n, \end{aligned}$$

where we have

$$(18)\qquad \begin{aligned} &0 < A_1 < A_2 < \cdots < A_s, \\ &1 \leqq m_1 < \cdots < m_s \leqq n, \\ &1 < s \leqq n. \end{aligned}$$

We now need only show that A_s and A_{s-1} are arbitrary within the range prescribed by (16). We may best show this by eliminating e between equations (12) and (15) and solving for C and c. For this purpose we write

$$e = (C - A_{s-2})R, \qquad R \equiv (y_{m_2} - y_{m_1})/y_{m_1}.$$

Setting this expression for e into (15) and solving for C and c, we obtain

$$\begin{aligned} C &= A_{s-1} + (A_s - A_{s-1})(1 + R), \\ c &= (C - A_{s-1})/(C - A_{s-2}). \end{aligned}$$

From (2) we see that R is positive, and from (6), that it is less than one.

Thus we see that, when A_s and A_{s-1} are arbitrary positive quantities satisfying (16), we will have

$$\begin{gathered} C > A_{s-1} > A_{s-2}, \\ 0 < c < 1; \end{gathered}$$

but C and c are arbitrary within these limits, so we see that inequalities (16) are the only restrictions imposed on A_s and A_{s-1}, and our theorem is proven.

For $m>2$, equations (2) and (3) are not as useful for the numerical calculation of g_i and y_m as are the equivalent equations

$$(19)\qquad y_m^{m+1} - 2y_m^m + 1 = 0, \qquad m = 1, \cdots, n;$$

$$(20)\qquad g_m = y_m - y_{m-1}, \qquad m = 1, \cdots, n,$$

where y_m is taken to be greater than one when m is greater than one.

The following table of values may prove useful in applying the theorem:

m	1	2	3	4	5	6	7
y_m	1.0000	1.6180	1.8393	1.9276	1.9651	1.9835	1.9919
g_m	1.0000	0.6180	0.2213	0.0883	0.0375	0.0185	0.0074

Since $y_m \doteq 2$ as $m \doteq \infty$, we may write

$$\sum_8^\infty g_i = 2 - y_7 = 0.0081,$$

and use as the upper bound simply

$$z \leqq 0.0081\, q_8 + \sum_1^7 g_i q_i. \tag{e}$$

To illustrate the use of the theorem and compare it with the bounds given by (a), (b), (c), and Lemma 3, we may apply each of them to the equation

$$z^9 + z^8 - 512z^6 - 16z^5 + 16807z^4 + z^2 - 1 = 0. \tag{20}$$

The resulting bounds may be tabulated as follows:

(a)	(b)	(c)	(d)	Lemma 3
20	16	15	13	9.265

The direct use of Lemma 3 is, of course, not practical, since it involves the solution or approximate solution of an n-th degree equation.

THE PRODUCT OF A CIRCULANT MATRIX AND A SPECIAL DIAGONAL MATRIX

By UDO WEGNER, Darmstadt

In this MONTHLY, for May, 1932, vol. 39, p. 280, J. Williamson published a note, in which he proved by means of complicated determinantal relations that the matrix ZA, where A is the cyclic matrix

$$\begin{pmatrix} a_0 & a_1 \cdots a_{n-1} \\ a_{n-1} & a_0 \cdots a_{n-2} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_1 & a_2 \cdots a_0 \end{pmatrix}$$

and Z the matrix

$$\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \omega \cdots 0 \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 \cdots \omega^{n-1} \end{pmatrix},$$

satisfies the equation

$$(ZA)^n - Det(A)\cdot E = 0,$$

where E is the unit matrix and ω a primitive nth root of unity. In what follows I shall show how this result can be obtained more simply.

Let B be written for the product ZA so that

$$B = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \omega a_{n-1} & \omega a_0 & \cdots & \omega a_{n-2} \\ \omega^2 a_{n-2} & \omega^2 a_{n-1} & \cdots & \omega^2 a_{n-3} \\ \cdot & \cdot & \cdot & \cdot \\ \omega^{n-1} a_1 & \omega^{n-1} a_2 & \cdots & \omega^{n-1} a_0 \end{pmatrix},$$

and let λ_0 be a latent root of B, i.e., Det $(B-\lambda_0 E) = |B-\lambda_0 E| = 0$. If the matrix $\omega(B-\lambda_0 E) = \omega B - \omega\lambda_0 E$ is transformed by the matrix

$$Q_1 = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right),$$

then

$$Q_1(\omega B - \omega\lambda_0 E)Q_1^{-1} = B - \omega\lambda_0 E,$$

since by the above transformation in ωB the first column is replaced by the last, the second by the first, the third by the second, and the first row by the last, the second by the first and so on. From the equation

$$0 = |B - \lambda_0 E| = \omega^n |B - \lambda_0 E| = |Q_1|\,|\omega B - \omega\lambda_0 E|\,|Q_1^{-1}| = |B - \omega\lambda_0 E|,$$

it follows that $\omega\lambda_0$ is a latent root of B.

If

$$Q_2 = Q_1^2 = \left(\begin{array}{cccc|cc} 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots 1 & 0 & 0 \end{array}\right),$$

then

$$Q_2(\omega^2 B - \omega^2\lambda_0 E)Q_2^{-1} = B - \omega^2\lambda_0 E,$$

so that

$$0 = |B - \lambda_0 E| = \omega^{2n} |B - \lambda_0 E| = |Q_2|\,|\omega^2 B - \omega^2\lambda_0 E|\,|Q_2^{-1}| = |B - \omega^2\lambda_0 E|.$$

Accordingly $\omega^2\lambda_0$ is a latent root of B. Similarly it can be shown that $\omega^3\lambda_0$, $\omega^4\lambda_0, \cdots, \omega^{n-1}\lambda_0$ are latent roots of B. But these roots are all distinct and so $\lambda_0, \omega\lambda_0, \cdots, \omega^{n-1}\lambda_0$ are all the latent roots of B. Therefore the characteristic equation of B, $\phi(\lambda)=0$, has the form

$$\lambda^n - \lambda_0^n = 0.$$

Since the product of the latent roots of B is, apart from sign, the determinant of B or ZA,

$$\lambda_0^n = (-1)^{n-1}|Z||A| = (-1)^{n-1}\omega^{n(n-1)/2}|A| = |A|.$$

Therefore the characteristic equation of B is

$$\lambda^n - |A| = 0$$

and accordingly

$$(ZA)^n - |A|E = 0.$$

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION. PART III

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

The elder Āryabhaṭa has given rules for the extraction of the square and cube roots but not for finding the square and cube of a number. Brahmagupta has given rules for finding the cube as well as the cube root of a number but not for finding the square or the square root. An examination of these rules shows that they were meant to apply to numbers expressed in the modern place-value notation.

Let us first consider Brahmagupta's rule for finding the cube of a number. The rule is as follows:

Sthāpyaḥ+antya-ghanaḥ+antya-kṛtiḥ+
tri-guṇā+uttara-saṅguṇā ca *tat-prathamāt*
uttara-kṛtiḥ+antya-guṇā tri-guṇā ca+
uttara-ghanaḥ+ca ghanaḥ.[1]

The following is the prose rendering of this rule:

Antya-ghanaḥ (cube of antya) sthāpyaḥ (should be set down); antya-kṛtiḥ (square of antya) tri-guṇā (multiplied by three) uttara-saṅguṇā (multiplied by uttara) ca (and) tat-prathamāt (outside the first place of that) [sthāpyā (should be set down)], uttara-kṛtiḥ (square of uttara) antya-guṇā (multiplied by antya) tri-guṇā [tat-prathamāt sthāpyā]; uttara-ghanaḥ (cube of uttara) ca [tat-prathamāt sthāpyaḥ]; [evam (thus)] ghanaḥ (cube) [syāt (is)].

[1] *Brāhma-sphuṭa-siddhānta*, edited by Sudhākar Dvivedī, Ch. XII, verse 6. The plus sign + is used to indicate that the letters separated by it have been joined in the text by the rules of *sandhi* (conjunction of letters); the hyphen (-) indicates that the words separated by it have been joined in the text by the rules of *samāsa* (compounding of words).

Kṛitiḥ = square; tri = three; guṇā = saṅguṇā = multiplied; tat = that; prathama = first. In Sanskrit the conjunction *ca* is placed after the two words joined by it and not between them.

The reader should note that this rule gives the cube of a number of two digits. But the implication of the rule is that by repeated application of it the cube of any number can be found. The words *antya* and *uttara* are very significant. *Antya* means 'what exists at the end' and *uttara* means 'what comes after.' In the Brāhmī notation which preceded the modern place-value notation in India the number fifty-four was expressed as (50) 4. (50) cannot be called *antya*, as the principle of place-value was not used in the notation; 4 occurs at the end and must be called *antya*. As nothing occurs after 4, there is no *uttara*. Hence the rule is not applicable to the Brāhmī notation. We need not consider the Kharosṭhi notation as it disappeared from India more than a century before Brahmagupta. In the modern notation the number fifty-four is written as 54. Here the first place—the units' place is called the first place, *ekaṃ tu prathamaṃ sthānam*—is occupied[1] by 4 and the second or last place by 5. Hence 5 occurs at the end and is, therefore, *antya*; 4 comes after the digit 5, and so 4 is *uttara*. For *uttara* Mahāvīra uses the word *śeṣa* which means the same thing.

Far more significant than the words *antya* and *uttara* is the compound word *tat-prathamāt* italicised in the above rule. To find its significance let us compare the wordings of the above rule with those of the rules given by Mahāvīra (c. 850 A.D.) and Śrīdhara (b. 991 A.D.).

Mahāvīra's rule for cubing a number is:

Antyasya ghanaḥ kṛtiḥ + api
sā tri-hatā + *utsārya* śeṣa-guṇitā vā[2]
śeṣa-kṛtiḥ + tri + anta-hatā
sthāpyā + *utsārya* + evam + atra vidhiḥ.[3]
antyasya = of antya,
api = also;
sā = she (kṛtiḥ)
hatā = guṇitā
= multiplied.

Śrīdhara's rule for cubing a number is:

Sthāpyaḥ + antya-ghanaḥ + antya-kṛtiḥ
sthāna + *ādhikyaṃ* tri-pūrva-guṇitā ca
ādya-kṛtiḥ + antya-guṇitā tri-guṇā ca
ghanaḥ + tathā + ādyasya.[4]
tathā = similarly;
ādyasya = of ādya.

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[1] Hence Śrīdhara calls it *ādya* (what occurs at the beginning) or *pūrva* (what occupies the preceding place).

[2] Here *vā* means 'and'. This word has been previously used in this sense by Āryabhaṭa (*Daśagītikā*, verse 2) and Varāhamihira (Vṛhat Jātaka, XII, 19).

[3] *Gaṇita-sāra-saṃgraha* (edited by Rangācārya), Text, p. 15, verse 47.

[4] *Triśatikā* (edited by S. Dvivedī), p. 6, Rule 14.

In Mahāvīra's rule the last three words 'evam atra vidhiḥ' means 'such is the rule here.' This is expressed very tersely by the last word 'ghanaḥ' (cube, i.e., hence the cube of the number) in Brahmagupta's rule. Śrīdhara's rule is wanting in any such expression. In the rule Brahmagupta has used the word *uttara* thrice. Mahāvīra uses the word *śeṣa* twice and the rule is incomplete[1] as it indicates the operations corresponding to the three terms $a^3+3a^2b+3ab^2$. In a preceding rule he gives the correct formula $3a^2b+3ab^2+a^3+b^3$ for $(a+b)^3$. For *uttara* Śrīdhara uses the two synonyms[2] *pūrva* and *ādya*, the former being employed once and the latter twice. But for the word *sthānādhikyam*[3] Śrīdhara's rule could have been regarded as quite algebraical and applicable to any notation based on the additive principle and not on the principle of place-value.

If in Mahāvīra's rule we overlook the omission of the operation corresponding to the term b^3 in the expression $a^3+3a^2b+3ab^2+b^3$, the above three rules for finding the cube of a number differ only in the words italicised. The italicised word *sthānādhikyam* in Śrīdhara's rule means 'so as to produce increase in places.' The italicised word *utsārya* which occurs twice in Mahāvīra's rule means 'pushing one place to the right.' This very word has been used in this sense by the younger Āryabhaṭa in his rules for division and extraction of the square root,[4] by Śrīdhara in his rules for multiplication, squaring a number, and extraction of the square root, and by Bhāskara in his rules for multiplication and extraction of the square root. Hence the words *sthānādhikyam* and *utsārya* imply the same thing.

Mahāvīra's and Śrīdhara's rules may now be translated thus:

Set down the cube of *antya*; and then the product of the square of *antya* and three times the *śeṣa* (or *pūrva*) *after pushing it* (i.e. the product) *one place to the right or so as to produce increase in places.* Next set down the product of three times the *antya* and the square of *śeṣa* (or *ādya*) *after pushing it one place to the right* (or so as to produce increase in places). Then similarly set down the cube of *śeṣa* (or *ādya*). Such is the rule here.

I have purposely refrained from translating certain terms before finding their correct interpretations.

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[1] To complete the rule I would like to replace the words 'evam atra vidhiḥ' by the words 'atha śeṣa-ghanaḥ' (then the cube of *śeṣa*) which satisfy the metre equally well.

[2] See footnote 2 above.

[3] In the translation of *Triśatikā* published by Kaye in the *Bibliotheca Mathematica*, 3rd series, Vol. XIII, this word has been omitted. Śrīdhara has used the word *utsārya* five times as such and once in the past participle form *utsārita* (Rule 10). These words have also been omitted in the translation. As will be seen below, all these words indicate the use of the modern notation in India. The word *pada* meaning 'place' has been mistranslated wherever it occurs. Although the translation is said to be the work of Prof. Rāmānujāchāria M.A., the hand of Kaye is clearly visible behind these significant errors of omission and commission. I cannot think of an Indian scholar omitting or misinterpreting those very words which, when correctly interpreted, reflect great credit on his mother-land.

[4] In the latter rule the word occurs in the past participle form *utsārita*.

The question of increasing the number of places or of pushing one place to the right can arise only when the modern place-value notation is used. For example, let us find the cube of fifty-four.

In modern notation.		In any other notation.	
$5^3=$	125	$50^3=$	125000
$3.5^2.4=$	300	$3.50^2.4=$	30000
$3.5.4^2=$	240	$3.50.4^2=$	2400
$4^3=$	64	$4^3=$	64
$54^3=$	157464	$54^3=$	157464

It will thus be seen that the words *antya* and *śeṣa* (or *purva* or *ādya*) cannot mean the component parts of a number. They must mean the digits used to express a number in our modern notation. Therefore *antya* means the digit occurring in the last place and *ādya* or *pūrva* the digit in the first place. Relative to *antya*, *śeṣa* (or *uttara*) is the digit which comes after *antya* and is, therefore, the digit in the first place. For, as stated before, only two places are contemplated in the rules as they stand.

The reader's attention is particularly invited to the remarkably close resemblance between the wordings of the rules of Brahmagupta and Śrīdhara. The word *sthānādhikyam* of the latter takes the place of the word *tat-prathamāt* of the former. This word indicates how the continued product of 3, the *uttara*, and the square of *antya*, should be set down relatively to the cube of *antya*. If these two numbers were to be added as usual, the word *tat-prathamāt* would have been superfluous. But Indian framers of *sūtras* or rules, far from indulging in the use of superfluous expressions, are brief to a fault.[1] The fifth-case ending in the word *tat-prathamāt* is also very significant. It implies removal or motion from the thing denoted by the base-word to which it is affixed. The word *tat-prathamāt* is connected with the word *sthāpyā* understood as the latter word has already been used once (though in the masculine form). These two words together mean "should be set down being removed from the first place of that (i.e., the preceding number)." Hence, Brahmagupta's *tat-prathamāt*, Mahāvīra's *utsārya*, and Śrīdhara's *sthānādhikyam* imply the same thing. The three rules quoted above are, therefore, merely three different statements of the same rule. It, therefore, follows that, like Mahāvīra's and Śrīdhara's rules, Brahmagupta's rule also is not applicable to any other notation than the modern place-value notation.

Let us now examine the rules given by the elder Āryabhaṭa and Brahmagupta for the extraction of the square and cube roots. These rules are:

(i) For the extraction of the square root.

Āryabhaṭa:

Bhāgaṃ haret+avargāt+nityaṃ
dvi-guṇena vargamūlena

[1] Vincent Smith, *The Oxford Students' History of India*, p. 36.

vargāt+varge śuddhe
labdhaṃ sthānātare mūlam.

(ii) For the extraction of the cube root.
Āryabhaṭa:
Aghanāt+bhajet+dvitīyāt
tri-guṇena ghanasya mūla-vargeṇa
vargaḥ+tri-pūrva-guṇitaḥ
śodhyaḥ prathamāt+ghanaḥ+ca ghanāt.
Brahmagupta:
Chedaḥ+aghanāt+dvitīyāt+
ghanamūla-kṛtiḥ+tri-saṅguṇā+āpta-kṛtiḥ
śodhyā tri-pūrva-guṇitā
prathamāt+ghanataḥ+ghanaḥ+mūlam.

Āryabhaṭa and Brahmagupta have given the same rule for the extraction of the cube root. The two statements have some important expressions in common. All these rules have been explained by A. N. Singh.[1] Āryabhaṭa's rules have been explained also by Rodet[2] and the present writer.[3] Singh and the present writer have adversely criticised the translation of Āryabhaṭa's above-mentioned rules published by Kaye in the *Journal of the Asiatic Society of Bengal.*[4] Further explanation of these rules is, therefore, unnecessary. I shall here try to show that these rules imply a knowledge of the place-value notation. Kaye writes that these rules are perfectly general (i.e., algebraical) and apply to all arithmetical notations.[5] If Kaye were right, how is it that the Greeks could not find the cube root of a number? Sir Thomas Heath writes that "in no extant Greek writer do we find any description of the operation of extracting the cube root."[6] Heron who has been assigned to the third century A.D. by this authority[7] has given a method of finding $4\frac{9}{14}$ as the cube root of 100.[8] But there is nothing to show that he could find the cube root of any number in general. On taking a very liberal view of his method the utmost credit that we can give him is that, when the integral part in the cube root of a number is known, he can find the fractional part approximately.

Again, the above rules occur in the midst of other rules relating to arithmetic or mensuration but not to algebra. The authors of these rules have omitted many essential topics and cannot be accused of incorporating rules which they did not use. The above rules are, therefore, arithmetical and are meant to apply

[1] *Bulletin,* Calcutta Math. Society, Vol. XVIII, pp. 123–140.
[2] *Journal Asiatique,* Vol. XIII (1879), pp. 397, 405–408.
[3] *Journal,* Bihar & Orissa Research Society, March, 1926, pp. 78–82.
[4] July, 1907, p. 493; March, 1908, pp. 119 & 120.
[5] *Ibid.*
[6] *History of Greek Mathematics,* Vol. I, p. 63.
[7] *Ibid.,* Vol. II, p. 306.
[8] *Ibid.,* p. 341.

to the notation used by their authors. As the Greeks with all their ingenuity could not apply the corresponding algebraical formula to their "additive notation" which was similar to the Indian Brāhmī notation in principle, it is but reasonable to infer that the notation employed by Āryabhaṭa and Brahmagupta was based not on the additive principle but on the principle of local value. This view is further supported by the expression *labdhaṃ sthānātare mūlam* which means 'the quotient is the part or figure of the root in the next place.' Although Kaye knows quite well that *sthāna* means *place* and not *term*, he has conveniently translated the expression as 'the quotient is the root to the next term.'[1] In March, 1908, he replaces it by the meaningless sentence "The quotient in a place set apart is the root."

From an examination of the rules of Āryabhaṭa quoted above Rodet holds that Āryabhaṭa performed the operations indicated by them on numbers written in figures with the value of position and the zero.[2] As Rodet learns from a commentary of Bhāskara's *Līlāvatī* the meanings of the terms *vargāt*, *avargāt*, *ghanāt* and *aghanāt*, Kaye thinks him to be wrong.[3] But the commentator has interpreted the terms after the manner of the elder Āryabhaṭa, Mahāvīra, the younger Āryabhaṭa, Caturvedācārya, Śrīdhara, and Bhāskara.

The words *avarga* and *varga* occurring in the forms *avargāt* and *vargāt* in the elder Āryabhaṭa's rule for the extraction of the square root may refer to notational places or, as Kaye interprets, to the component parts of the number whose square root is required. But the occurrence of the word *sthānāntare* renders Kaye's interpretation inapplicable. In the verse giving his alphabetic notation the elder Āryabhaṭa uses these two words *avarga* and *varga* to mean places corresponding respectively to the non-square units 10, 10^3, 10^5, etc., and to the square units 1, 10^2, 10^4, etc. This is also the view of Fleet.[4]

Mahāvīra (850 A.D.) gives the same rule for the extraction of the cube root as Āryabhaṭa and Brahmagupta, but he states it in two different ways. In the first statement he very appropriately uses the terms *bhājya* and *śodhya* respectively for *dvitīya aghana* and *prathama* (*aghana*). In his own statement of the same rule the younger Āryabhaṭa (c. 950 A.D.) clearly explains the terms of Mahāvīra. He writes that notational places (*padāni*) have respectively *ghana*, *bhājya* and *śodhya* for their names.[5] Ancient Indian mathematicians use the term *pada* in three different senses, (i) a notational place, (ii) the square root, (iii) the number of terms in a series. The second and third meanings are evidently inapplicable. In the immediately preceding rule which the younger Āryabhaṭa gives for the extraction of the square root he similarly names the places (*sthāna*) of a number as *odd* and *even*.

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[1] J.A.S.B., July, 1907, p. 493.

[2] *Journal Asiatique*, XIII (1879), p. 408.

[3] J.A.S.B., July, 1907, p. 494.

[4] J.R.A.S., 1911, p. 115 and p. 116, footnote 1.

[5] *Mahāsiddhānta*, Dvivedī's edition, XV, 8.

Caturvedācārya (Pṛthudakasvāmī) who wrote his commentary on Brahmagupta's *Brāhma-sphuṭa-siddhānta* in the latter half of the tenth century A.D. begins the explanation of Brahmagupta's rule for the extraction of the cube root as follows:

"Here (*i.e.*, in this rule) the first place of the number whose cube root is required is called *ghana* and the next two places towards the left are called *aghana*; then again the next one (place) is called *ghana* and the next two (places) are called *aghana*; and so until all the places are exhausted."[1]

This is exactly what Mahāvira means when he says "Ghanamekaṃ aghane dve" (the first place is *ghana* and the next two are—*aghana*) in his second statement of the rule for the extraction of the cube root. Śrīdhara also distinctly says this when he writes "ghanapadaṃ aghanapade dve ghanapadataḥ apāsya ghanam etc" (for one *ghana* (cubic) place there are two *aghana* (non-cubic) places; subtracting the cube from the cubic place etc.).

It will appear from the above that the terms *dvitīya* (second) *aghana* and *prathama* (first) *aghana* respectively mean 'second non-cubic place' and 'first non-cubic place.' Whatever be the notation, one must begin with the largest units to extract the square or cube root of an arithmetical number. For the extraction of the cube root the largest *ghana* (cubic) element must, therefore, be taken first. According to the rules given by Āryabhaṭa and Brahmagupta its place is followed on the right by the second *aghana* (non-cubic) place which is followed by the first *aghana* place. Then comes the *ghana* (cubic) place of the next lower order. This is quite in agreement with the classification of places explained by Caturvedācārya. Hence Rodet is not wrong in holding that the elder Āryabhaṭa's rules for the extraction of the square and cube roots imply the use of the modern notation.

Brahmagupta's rule for the extraction of the cube root being verbally almost the same as that of Āryabhaṭa, remarks made on the one apply equally to the other. It has already been shown that Brahmagupta's rule for cubing is applicable only to the modern notation. It would be unnatural if his rule for the extraction of the cube root, which is an inverse operation to that of cubing, were not likewise applicable only to the modern place-value arithmetical notation.

A GRAPHICAL SOLUTION OF THE QUARTIC

By J. D. GRANT,[2] University of Illinois

Some years ago Professor Running showed[3] that the real roots of a cubic or quadratic could be found from graphs made up solely of straight lines. These lines were all tangents to the discriminant of the equation studied so that the

[1] *Brāhma-sphuṭa- siddhānta*, Dvivedī's edition, p. 175.

[2] Died July 9, 1932.

[3] This MONTHLY, Vol. 28, 1921, page 415, also text *Graphical Mathematics*, 1927, p. 34.

method is an application of the tangential or line equations of Projective Geometry. The purpose of this note is to show that the same method may be applied to the quartic. The general reduced quartic may be written

$$z^4 + az^2 + bz + c = 0.$$

If $a \neq 0$, one may divide by a^2 and write $z/|a|^{1/2} = m$; and if $a = 0$, simply write $z = m$. Then according as a is positive, zero, or negative,[1] the equation takes one of the three forms

$$(1) \qquad m^4 + m^2 + xm - y = 0, \quad \text{where} \quad x = b/a^{3/2},\ y = -c/a^2,$$

$$(2) \qquad m^4 + xm - y = 0, \quad \text{where} \quad x = b,\ y = -c,$$

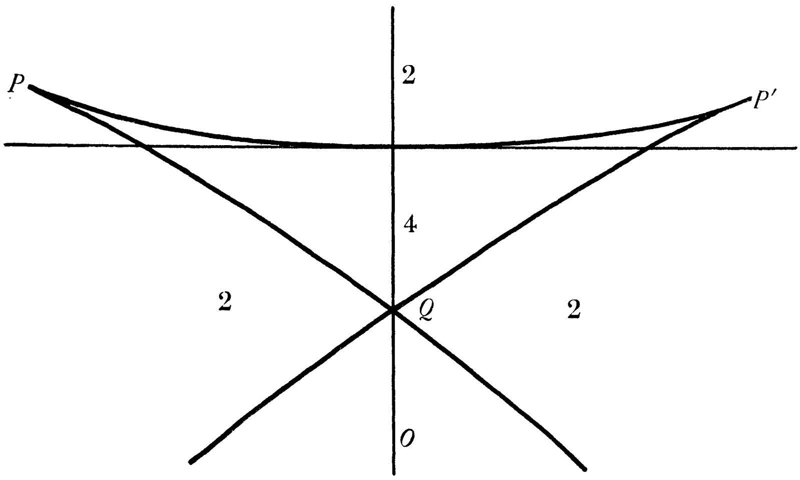
$$(3) \qquad m^4 - m^2 + xm - y = 0, \quad \text{where} \quad x = b/(-a)^{3/2},\ y = -c/a^2.$$

It is clear that if the a, b, c are given real the resulting x and y must be real. The discriminant curves to which these families of lines are tangent are as follows:

$$(1) \qquad 27x^4 + 4(36y + 1)x^2 + 16y(4y + 1)^2 = 0$$

$$(2) \qquad 27x^4 + 256y^3 = 0$$

$$(3) \qquad 27x^4 - 4(36y + 1)x^2 + 16y(4y + 1)^2 = 0.$$



Since $a < 0$ is a necessary condition for the existence of four real roots,[2] the third one of these curves is of particular interest. It is shown in the accompanying figure. The cusps, P, P', are at the points $(\pm 2\sqrt{6}/9,\ 1/12)$, and the node Q at $(0,\ -1/4)$. The curve divides the plane into three regions from the points of which 0, 2, or 4 tangents to the curve may be drawn as indicated in the figure.

[1] This is the method used by d'Ocagne in the study of the Quintic in *Traité de Nomographie*, 1899, page 341.

[2] Burnside and Panton, *Theory of Equations*, 1928, Vol. I, page 145.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

JACOBIANS OF THE SEMINVARIANTS S_i OF A BINARY P-IC

H. S. THURSTON, University of Alabama

A homogeneous isobaric polynomial in the coefficients of the binary form

$$f(x, y) \equiv \sum_{i=0}^{p} \binom{p}{i} a_i x^{p-i} y^i$$

is called a seminvariant of $f(x, y)$ if it is an invariant of $f(x, y)$ with respect to all transformations of the type

$$T_k\colon\ x = \xi + k\eta,\ y = \eta.$$

There exists a set of seminvariants

$$S_i = (-1)^{i-1}(i-1)a_1^i + \sum_{j=2}^{i}(-1)^{i-j}\binom{i}{j} a_0^{j-1} a_1^{i-j} a_j, \tag{1}$$

where i may take on the values $2, 3, \cdots, p$, such that every seminvariant of $f(x, y)$ is the quotient[1] of a polynomial in the seminvariants $a_0, S_2, \cdots, S_p$ by a power of a_0.

If each S_i be considered as a function of the independent variables $a_0, a_1, \cdots, a_p$, and J_m denote the Jacobian of $S_2, \cdots, S_{m+1}$ with respect to $a_0, a_1, \cdots, a_{m-1}$, it may be readily verified that

$$J_2 = a_0(4a_1a_3 - 3a_2^2)$$

$$J_3 = a_0^4(9a_2a_4 - 8a_3^2)$$

$$J_4 = a_0^8(16a_3a_5 - 15a_4^2)$$

$$J_5 = a_0^{13}(25a_4a_6 - 24a_5^2).$$

These results naturally suggest the general expression

$$J_m = a_0^{(m^2+m-4)/2}\left\{m^2 a_{m-1}a_{m+1} - (m^2 - 1)a_m^2\right\}. \tag{2}$$

It is the purpose of this note to prove (2) for $m = 2, 3, \cdots, p-1$. Incidentally, for $m = 1$, this formula yields a_2, in accordance with the natural definition of J_1 as $\partial S_2/\partial a_0$.

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[1] Dickson, *Modern Algebraic Theories*, pp. 14–16.

From (1), we have

$$\partial S_i/\partial a_0 = \sum_{j=2}^{i}(-1)^{i-j}\binom{i}{j}(j-1)a_0^{j-2}a_1^{i-j}a_j$$

$$\partial S_i/\partial a_1 = (-1)^{i-1}i(i-1)a_1^{i-1} + \sum_{j=2}^{i-1}(-1)^{i-j}(i-j)\binom{i}{j}a_0^{j-1}a_1^{i-j-1}a_j$$

$$\begin{aligned}\partial S_i/\partial a_j &= (-1)^{i-j}\binom{i}{j}a_0^{j-1}a_1^{i-j}, && \text{if } i \geqq j > 1\\ &= 0, && \text{if } i < j.\end{aligned}$$

On account of the simpler notation involved we shall consider the Jacobian J_{m-1}, and later obtain (2) by replacing m by $m+1$.

$$J_{m-1} = \begin{vmatrix} \dfrac{\partial S_2}{\partial a_0} & \dfrac{\partial S_2}{\partial a_1} & a_0 & 0 & \cdots 0 \\ \dfrac{\partial S_3}{\partial a_0} & \dfrac{\partial S_3}{\partial a_1} & -3a_0a_1 & a_0^2 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial S_i}{\partial a_0} & \dfrac{\partial S_i}{\partial a_1} & (-1)^i\dbinom{i}{2}a_0a_1^{i-2} & (-1)^{i-1}\dbinom{i}{3}a_0^2a_1^{i-3} & \cdots \dfrac{\partial S_i}{\partial a_{m-2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial S_m}{\partial a_0} & \dfrac{\partial S_m}{\partial a_1} & (-1)^m\dbinom{m}{2}a_0a_1^{m-2} & (-1)^{m-1}\dbinom{m}{3}a_0^2a_1^{m-3} & \cdots \dbinom{m}{2}a_0^{m-3}a_1^2 \end{vmatrix}$$

Removing the factors $a_0, a_0^2, \cdots, a_0^{m-3}$, from the third, fourth, $\cdots$, $(m-1)$th columns respectively, and subtracting from the elements of the first and second columns respectively, the products of the elements of the third column by a_2 and $-2a_1$, we obtain a determinant in which all except one of the elements of the first row are zero. Having reduced this to a determinant of order $m-2$, and having removed a_0 from the first and second columns, we subtract from the elements of the first and second columns respectively, the products of those of the third column by $2a_3$ and $-3a_2$. The order may now be reduced to $m-3$, and a_0 factored from the first and second columns. Subtracting from the elements of these columns the products of the elements of the third column by $3a_4$ and $-4a_3$, the order may again be reduced. In general, we may reduce a determinant of order $m-k$ to one of order $m-k-1$ by subtracting from the elements of the first and second columns, the products of those of the third column by ka_{k+1} and $-(k+1)a_k$ respectively. Proceeding in this way, we finally obtain

$$J_{m-1} = a_0^{(m^2-m-6)/2}\begin{vmatrix} (m-2)a_{m-1} & -(m-1)a_{m-2} \\ (m-1)a_0a_m - m(m-2)a_1a_{m-1} & -ma_0a_{m-1} + m(m+1)a_1a_{m-2} \end{vmatrix}$$

which on expansion yields

$$a_0^{(m^2-m-4)/2}\left\{(m-1)^2a_ma_{m-2} - m(m-2)a_{m-1}^2\right\}.$$

Replacing m by $m+1$, we obtain (2).

AN UPPER LIMIT FOR THE ROOTS OF AN EQUATION

By R. J. MYERS, Bethlehem, Pa.

The following generalization of a theorem found in Dickson's Theory of Equations[1] is believed to be new.

Theorem: If in an equation having real coefficients

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

the first negative term is preceded by k coefficients which are either positive or zero, and if G denotes the greatest of the numerical values of the negative coefficients, then each real root is equal to or less than $1+(G/a_p)^{1/(k-p)}$, where a_p ($\neq 0$) is the coefficient of the $(p+1)$th term, and p is less than k.

Proof: For positive values of x, $f(x)$ will be diminished in value or remain unchanged if we omit the terms a_0x^n, $a_1x^{n-1}, \cdots, a_{p-1}x^{n-p+1}$, $a_{p+1}x^{n-p-1}, \cdots,$ $a_{k-1}x^{n-k+1}$, all of which are positive or zero, and if we change each coefficient from a_k to a_n inclusive from its present value to that of the negative coefficient which has the greatest numerical value (i.e. to $-G$). Then,

$$\begin{aligned} f(x) &\geqq a_px^{n-p} - G(x^{n-k} + x^{n-k-1} + \cdots + x + 1), \\ &\geqq \left\{a_p(x-1)x^{n-p} - Gx^{n-k+1} + G\right\}/(x-1). \end{aligned}$$

Dropping G from the right side of the equation does not affect the inequality if we assume $x>1$.

Hence

$$f(x) \geqq \frac{x^{n-p}\left[a_p(x-1) - Gx^{p-k+1}\right]}{x-1};$$

or, since $p-k+1 \leqq 0$, and $x>1$,

$$f(x) \geqq \frac{x^{n-p}\left[a_p(x-1) - G(x-1)^{p-k+1}\right]}{x-1}.$$

If now $x>1+(G/a_p)^{1/(k-p)}>1$, we have $(x-1)^{k-p}>G/a_p$, $(x-1)^{p-k}<a_p/G$, and $a_p(x-1)-G(x-1)^{p-k+1}>0$; and hence $f(x)>0$. Thus no value of x greater than $1+(G/a_p)^{1/k-p}$ can be a root.

Dickson gives two methods for the determination of the upper limit to the roots of an equation. The first is a special case of my theorem corresponding

[1] Dickson, First Course in the Theory of Equations, pp. 21–22.

to $p=0$. Moreover in all cases where p equals k the second method in Dickson is superior to mine, so my theorem can only be of use when there are at least three positive terms preceding the first negative term.

Consider the equation

$$x^4 + 18x^3 + x^2 - 36x - 14 = 0$$

and let us investigate the upper limit to the roots determined by these three methods.

By Dickson's first method the upper limit is found to be 4.31,

by his second method the upper limit is 2.80, and

by the method of this paper the upper limit is 2.41.

The only positive root is 1.50.

On Polygons and Circles

J. M. Feld, Brooklyn College

In a recent note[1] by J. R. Musselman on a problem proposed by W. H. Echols, Professor Musselman presented some elegant demonstrations of theorems on sets of equilateral triangles. Similar theorems can be established for polygons by a generalization of Professor Musselman's method.

Let $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ be two directly similar polygons, A_i and B_i being homologous vertices. If A_i and B_i are represented by the complex numbers a_i and b_i respectively, there obtain the n relationships

$$a_i = \lambda b_i + \mu \qquad (i = 1, 2, \cdots, n) \tag{1}$$

where λ and μ are complex. Therefore all determinants of the third order in the matrix

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} 1 \\ a_i \\ b_i \end{pmatrix}$$

vanish. Conversely if all the third order determinants of M vanish, the a_i and b_i are related as in (1) and the polygons A and B are directly similar. Hereinafter we write $M=0$ to indicate that all third order determinants of the matrix M vanish.

Let the k polygons $A_1A_2 \cdots A_n$, $B_1B_2 \cdots B_n$, $\cdots$ and $P_1P_2 \cdots P_n$ all be directly similar to $Q_1Q_2 \cdots Q_n$, the vertices bearing the same subscript being homologous. Furthermore let a_i, b_i, $p_i \cdots$ and q_i be complex numbers representing A_i, B_i, $P_i \cdots$ and Q_i respectively. Consequently we have the k conditions

$$\begin{pmatrix} 1 \\ a_i \\ q_i \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ b_i \\ q_i \end{pmatrix} = 0, \cdots, \begin{pmatrix} 1 \\ p_i \\ q_i \end{pmatrix} = 0. \tag{2}$$

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[1] *On Equilateral Triangles.* This Monthly, vol. 39 (1932), p. 290.

Adding the k matrices appearing in (2) and dividing the resulting matrix by k we obtain

$$M_k = \begin{pmatrix} 1 \\ (a_i + b_i + \cdots + p_i)/k \\ q_i \end{pmatrix}.$$

Since by virtue of (2) every third-order determinant in M_k vanishes, $M_k = 0$. Therefore the centroids of the sets of homologous vertices of k directly similar n-gons are the vertices of another n-gon directly similar to the others.

If the n-gons are regular each may be regarded as the situs of a set of $t \leqq n$ distinct n-gons such that the homologous vertices of any two in the set can be brought into coincidence by rotating one about their common center. With a set of k regular n-gons it is therefore possible to associate numerous other regular n-gons by selecting sets of homologous vertices in various ways.

Let A_i^1 $(i=1, 2, \cdots, n)$ be the vertices of a positive regular n-gon A^1. Similarly let $A_i^2, \cdots, A_i^k, \cdots, A_i^n$ be the vertices of the positive regular n-gons $A^2, \cdots, A^k, \cdots, A^n$ respectively.[1] The complex number representing point A_i^k is a_i^k. The regular polygons $A^1, \cdots, A^n$ are so placed that the n points A_1^i $(i=1, 2, \cdots, n)$ are also vertices of a positive regular n-gon; and likewise the n points A_2^i are vertices of a positive regular n-gon. We therefore have

$$M^k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1^k & a_2^k & \cdots & a_n^k \\ 1 & \omega & \cdots & \omega^{n-1} \end{pmatrix} = 0, \qquad (k = 1, 2, \cdots, n) \tag{4}$$

and

$$M_k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_k^1 & a_k^2 & \cdots & a_k^n \\ 1 & \omega & \cdots & \omega^{n-1} \end{pmatrix} = 0, \qquad (k = 1, 2) \tag{5}$$

where ω is a primitive nth root of unity.

Since the n-gons are regular we may cyclically permute the a_i^k in the second row of M^k without invalidating the set of equalities expressed by $M^k = 0$. Thus we subject the a_i^k in each matrix M^k to a cyclic permutation so that in all the resulting matrices the sum of superscript and subscript of the a in the jth column shall be congruent to j (mod n). Therefore M^k becomes

$$M_*^k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{n-k+1}^k & a_{n-k+2}^k & \cdots & a_{n-k}^k \\ 1 & \omega & \cdots & \omega^{n-1} \end{pmatrix}.$$

Similarly we permute the a_1^i and the a_2^i in M_1 and M_2 producing

$$M_1^* = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1^n & a_1^1 & \cdots & a_1^{n-1} \\ 1 & \omega & \cdots & \omega^{n-1} \end{pmatrix}, \quad M_2^* = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_2^{n-1} & a_2^n & \cdots & a_2^{n-2} \\ 1 & \omega & \cdots & \omega^{n-1} \end{pmatrix}$$

[1] The superscripts are not exponents except when used with ω.

and the relationships expressed in (4) and (5) are replaced by

(4′) $$M_*^k = 0, \quad (k = 1, 2, \cdots, n)$$

and

(5′) $$M_k^* = 0, \quad (k = 1, 2)$$

respectively.

Adding the matrices M_*^k $(k=1, 2, \cdots, n)$ and subtracting M_1^* and M_2^* from this sum we obtain a matrix M. Dividing M by $n-2$ we obtain

$$M^* = \begin{pmatrix} 1 & 1 \cdots 1 & \cdots 1 \\ b_1 & b_2 \cdots b_j & \cdots b_n \\ 1 & \omega \cdots \omega^{j-1} & \cdots \omega^{n-1} \end{pmatrix}$$

where[1]

(6) $$b_j = \left(\sum_{k=1}^{n} a^k{}_{n-k+j} - a_1^{n+j-1} - a_2^{n+j-2}\right)/(n-2)$$

By virtue of (4′) and (5′) $M^*=0$. Therefore the points $B_j (j=1, 2, \cdots, n)$ represented by the complex numbers b_j are the vertices of a positive regular n-gon. Each vertex B_j is the centroid of a set of $n-2$ points selected in the manner indicated by (6). When $n=3$ this result reduces to Professor Musselman's theorem, viz., given two positively equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of any size or position in the plane, if we construct the positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$ then $A_3B_3C_3$ itself is a positively equilateral triangle.

A theorem on circles analogous to a previous one on polygons can be easily proved. A circle C is given by

$$z(t) = a + re^{i(\alpha+t)} (0 \leqq t < 2\pi)$$

where a (complex) represents its center, r (real), its radius and α (real), its phase angle.

Let the n circles C_k and their phase angles α_k be given by

$$z_k(t) = a_k + r_k e^{i(\alpha_k+t)}, \quad (k = 1, 2, \cdots, n).$$

The point $w(t_0) = (\sum_{k=1}^{n} z_k(t_0))/n$ is the centroid of the points on the n circles corresponding to $t=t_0$. As t runs from zero to 2π, $z_k(t)$ sweeps out circle C_k positively, starting from point $z_k(0)$, the phase angle of which is α_k, while $w(t)$ describes another positive circle $A+Re^{it}$ where

$$A = \left(\sum_{k=1}^{n} a_k\right)/n, \quad R = \left(\sum_{k=1}^{n} r_k e^{i\alpha_k}\right)/n.$$

The center of $w(t)$ is therefore the centroid of the centers of the n circles C_k, the

[1] In (6) subscripts and superscripts greater than n must be reduced by n.

radius of $w(t)$ is the modulus of R and the phase angle is the amplitude of R. We thus have the theorem: given n circles in the plane and given a point on each circle, let these points describe their respective circles positively at the same angular speed; then their centroid will describe a positive circle the center of which is the centroid of the centers of the given circles.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Linear Transformations in Hilbert Space and their Applications to Analysis. By Marshall H. Stone. American Mathematical Society Colloquium Publications, volume XV. New York, the American Mathematical Society, 1932. viii+622 pages.

Linguistic Analysis of Mathematics. By Arthur F. Bentley. Bloomington, Indiana, The Principia Press, 1932. xii+316 pages.

Symbolic Logic. By C. I. Lewis and C. H. Langford. New York, The Century Company, 1932. xii+506 pages. $5.00.

Reelle Funktionen. By Hans Hahn. Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, Band 13. Leipzig, Akademische Verlagsgesellschaft M.B.H., 1932. xii+416 pages. RM 28; bound, RM 30.

Physikalische Geodäsie. By F. Hopfner, Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, Band 14. Leipzig, Akademische Verlagsgesellschaft M.B.H., 1933. xii+434 pages.

Das Hauptproblem der Aeusseren Ballistik. By K. Popoff. Leipzig, Akad. Verlagsgesellschaft M.B.H., 1932. x+216 pages. 16.40 marks; bound, 18 marks.

Elementary Mathematics from a Higher Standpoint. By Felix Klein. Part I, Arithmetic, Algebra, Analysis. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, Macmillan, 1932. x+274 pages. $3.00.

Anschauliche Geometrie. By David Hilbert and S. Cohn-Vossen. Die Grundlagen der Mathematischen Wissenschaften, Band XXXVII. Berlin, Julius Springer, 1932. viii+310 pages.

An Attempted Proof of Fermat's Last Theorem by a New Method. By Correa M. Walsh. New York, G. E. Stechert, 1932. 42 pages. $1.00.

Elementary Differential Equations. By Lyman M. Kells, New York, McGraw-Hill Book Company, 1932. x+184 pages. $2.00.

Differential and Integral Calculus. By J. H. Neelley and J. I. Tracey. New York, Macmillan, 1932. x+496 pages. $4.00.

A Short Course in Trigonometry. By James G. Hardy. New York, Macmillan, 1932. x+182 pages; tables xxii+142 pages. $2.25.

The Principles of Financial and Statistical Mathematics. By Maximilian Philip. The College of the City of New York Series in Commerce, Civics and Technology. New York, Prentice-Hall, Inc., 1932. xx+406 pages.

REVIEWS

Studies in the Theory of Numbers. By Leonard E. Dickson. The University of Chicago Science Series, 1930. x+230 pages. $4.00.

This important volume has two claims to distinction: it contains an amazing number of new results in the theory of quadratic forms; and it represents a systematic treatment of the arithmetic theory of quadratic forms, starting from first principles. Either one of these accomplishments by itself would entitle the author and his students and collaborators (Arnold Ross, Gordon Pall, A. Oppenheim) to the lasting gratitude of all interested in the theory of numbers; the combination makes the book of quite outstanding value.

It would seem to call for some explanation why a systematic treatment *ab ovo* of an apparently well developed field such as the arithmetic theory of quadratic forms should be hailed as a noteworthy achievement. It will probably be to many readers, as it was to the reviewer, a shock to learn that in spite of the eminence of the mathematicians who have contributed to the theory (Gauss, Seeber, Smith, Zolatareff, Markoff, Frobenius, Minkowski, Eisenstein; to mention only some of those no longer living) and in spite of the very large number of textbooks on theory of numbers, we have no satisfactory exhaustive treatment of this field. In particular, the volumes of Bachmann "Die Arithmetik der Quadratischen Formen" are shown to be in important respects unreliable.

One can but admire the courage of an author who will undertake to rebuild the whole structure rather than to patch up the unsound portions. One can only guess at the amount of labor covered by the modest words of the preface; "It was no small task to write a satisfactory exposition." On the other hand, we know, in this country as well as in Europe, how much the theory of numbers owes to the insistence of Dickson on precision in the statement of theorems and to his uncanny ability to detect, and to mend, unsound arguments; it seems therefore only fair that to him and his students should belong the credit of writing the first reliable treatment of the arithmetic theory.

The work is divided into three parts containing eighteen chapters.

Part I. Pages 3–78. Arithmetic of ternary quadratic forms.

"The first six chapters deal with general aspects of quadratic forms, chiefly in three variables, with applications to quadratic Diophantine equations."

Chapter I. Quadratic forms in n variables.

Chapter II. Introduction to ternary quadratic forms; universal forms.

Chapter III. Representation of binary forms by ternary forms.

Chapter IV. Equivalence of indefinite, ternary, quadratic forms.

Chapter V. Genera and representations of numbers.

Chapter VI. Quadratic Diophantine equations.

Following a line of attack which he has inaugurated in other papers, Dickson investigates among other things in these chapters his "universal" forms, i.e., quadratic forms (with integral coefficients) which are capable of representing (for integral values of the variable) every integer, positive, negative, or zero. An outstanding new theorem states that "every universal, indefinite, ternary quadratic form is a zero form" that is, the number zero can be represented by values of the variables not all zero.

Of great elegance is the theorem: $f=ax^2+by^2+cz^2$, a, b, c integers $\neq 0$, is universal if and only if

(1). a, b, c not all of like sign.

(2). No two of a, b, c have a common odd factor >1.

(3). $abc \equiv 1 \bmod 2$ or $\equiv 2 \bmod 4$.

Chapter IV contains a correction or radical clarification of known results in Pell's equation, and a generalization of these results. Chapter V gives applications of the theory derived in Chapter IV to the representation of numbers by forms. Chapter VI contains the (first complete) proof that for every indefinite quadratic form f in five or more variables the equation $f=0$ has non-trivial solutions. Also, necessary and sufficient conditions for the existence of non-trivial solutions of $ax^2+by^2+cz^2+du^2=0$ are given, correcting and completing previous work by other authors; but still leaving, as is carefully stated, a desideratum (case when exactly two of a, b, c, d even, and $abcd/4 \equiv 5 \bmod 8$).

Part II., pages 77–151, is entitled "Minima of indefinite quadratic forms," and contains:

Chapter VII. Minima of indefinite, binary, quadratic forms.

Chapter VIII. Minima of indefinite, ternary, quadratic forms.

Chapter IX. Minima of indefinite, quaternary, quadratic forms.

Chapter X. Tabulation of reduced, integral, ternary, quadratic forms which are indefinite but not zero forms.

This part centers around a theory originated by Markoff, extended by Frobenius and Remak. Here, again, it was necessary to reorganize the whole work to fill lacunae. Credit is given to G. Pall and A. Oppenheim for sharing the work of this portion. The following theorem may serve as a sample of the carefully polished form in which results are given:

Theorem: Any integral solution (a, b, c) of $x^2+y^2+z^2=3xyz$ with $c \geqq a \geqq 1$, $c \geqq b \geqq 1$, generates two new solutions (a, c, b'), $b'=3ac-b>c$; (b, c, a'), $a'=3bc-a>c$, which are distinct unless $a=b$. Repetition of this generating process, starting with the solution (1, 1, 1), will yield all sets of solutions in positive integers. The numbers forming a set are relatively prime in pairs.

The main theorems in this section are too technical to quote briefly.

Part III, pages 155–225, is headed "Miscellaneous investigations of quadratic forms," "The final three chapters are independent of each other and of the earlier chapters. They represent recent researches of especial interest concerning the geometrical reduction of positive forms, the determination of all universal zero forms, and the representation as sums of squares."

Chapter XI. Reduced, positive, quadratic forms. Their minima.

Chapter XII. Universal, zero, quadratic forms.

Chapter XIII. Number of representations as a sum of 5, 6, 7, or 8 squares.

In Chapter XI, Seeber's theory of reduced forms (supplemented and completed by Gauss, Eisenstein, Dirichlet) is given in simplified form, based on Gauss' lattice point representation of binary and ternary positive quadratic forms. For ternary forms, the proof of the reduction, and particularly the proof of uniqueness, is by its nature very disagreeable owing to the large number of possible cases which have to be examined separately.

Chapter XII gives the reduction to certain normal forms of the author's universal, indefinite, ternary or quaternary forms.

In the last Chapter, XIII, which is the only one to deal with the analytic theory of numbers in a broad sense, the author acknowledges the assistance of Gordon Pall and A. Oppenheim. It deals with the newer results of Mordell (and Hardy) on the number of representations of a positive integer n as the sum of s integral squares. This chapter will, to many readers, be particularly valuable because it is (as far as I know) the only place in the literature where this very abstract and delicate work is presented in connected fashion, with all necessary information (in very condensed form) concerning the ϑ-functions involved in the method, the Hardy function-theoretic tools and singular series, the Gaussian sums, etc. These twenty-seven pages impress the reviewer as a marvel of intelligent condensation.

A. J. KEMPNER

Nomographie. By M. Fréchet and H. Roullet. Paris, Armand Colin, 1928. 208 pages. Boards 12f; paper 10.50f.

The subject of Nomography is mainly a product of French inspiration and genius. It is not yet fifty years since d'Ocagne developed the idea of collinear points for the graphical representation of three variables in a plane; it is barely thirty since he published his most excellent Traité de Nomographie. The practical importance of this subject in engineering and in scientific experimental work is increasingly being realized—so it is peculiarly fitting and appropriate that we examine this work by a French mathematician and a French engineer.

The purpose of the book is to present to workers in the various fields of science the theory and methods involved in the construction of nomographic charts. Examples taken from physics, engineering, architecture, meteorology,

etc., serve to show the practical importance and use of the subject. Eighty figures illustrate the ideas and words of the authors. The mathematics involved can easily be understood by one who has had a good course in analytical geometry. Canonical forms which are necessary and sufficient to characterize the particular forms of nomograms are introduced, so that the investigator can readily find the type of diagram which suits his problem.

The book is divided into three parts. The first section of sixteen pages, which deals with the relationships between two variables, serves as an introduction. Here the authors present the fundamental notions of graphs and scales. The ordinary scale, the logarithmic and the general homographic scale are discussed in particular; by transformation of the scales other relationships between two variables can be represented by straight lines on the diagrams.

The second part of the book, one hundred and eight pages, treats the relationships between three variables. The development follows along historical lines; thus the first method presented is the cartesian coordinate chart with contour lines and the transformation of variables which permits these topographical curves to be represented by straight lines. The methods of Lalanne based on transforming one scale, and those of M. Lafay which transform both scales, are given and the correspondence between the original nomogram and its transform is discussed geometrically. We come now to the work of d'Ocagne and we are shown how to represent the relationship between three variables on three parallel lines; on two parallel lines and a curve; on two parallel lines and an oblique one—"abacs like an N"; on three concurrent lines; and finally on a circle with the third variable on a curve or a straight line. The section ends with a representation of the simultaneous solution of two relations between three variables.

The third section of forty-six pages deals with the relationships (a) between four variables when they can be put into certain simple canonical forms, and (b) between n variables when they can be expressed as $\sum f_i = f_n$ or $\prod f_i = f_n$ $(i=1, 2, \cdots, n-1)$. It concludes with a discussion of the advantages possessed by certain types of nomograms.

Fifty-five exercises scattered throughout the text and at the back of the book are given with hints as to the method of attack. These should prove valuable to the worker who wishes to construct a nomogram, as they cover all types of formulas. A study of some particular one will help him solve his individual problem. Certain paragraphs which deal with the geometric relationships between various types of nomograms are marked with an asterisk—to be omitted by a hurried reader; to the mathematician they will be of interest. It is not the aim of the book to compete with the volumes of d'Ocagne or Soreau; it is the aim of the authors to explain the subject of Nomography in simple terms and to show the non-mathematical worker in science how to use this tool. In this they have succeeded admirably.

J. R. Musselman

Handbook of Statistical Nomographs, Tables and Formulas. By J. W. Dunlap and A. K. Kurtz. New York, World Book Company, 1932. vii+163 pp. $6.00.

While nomograms have been in use in engineering and certain scientific fields, the worker in statistics has steered clear of graphic calculations. This is doubtless due to the fact that, as so much time must be spent in the numerical calculation of the values of the statistical measures before using a nomogram, the worker feels he might as well perform the last step also on his calculating machine. While Karl Pearson in his *Tables for Statisticians and Biometricians* (1914) includes two abacs or nomograms the present work with its twenty-eight charts is really a pioneer in the field. It will be interesting to see how much graphic methods of calculation will be stimulated by the handbook. The celluloid strip used for reading the nomograms can be improved; the lines on it should be decreased in thickness and the advertisement along the A scale removed, as its presence makes an exact reading difficult.

The list of 434 formulas collected from 48 books or articles on statistics has been written in a uniform notation and reference made to the books in which they can be found. In addition, errors in formulas occurring in these 48 books are noted. This portion of the handbook should prove most valuable. Statistics suffers from a lack of uniformity in notation and symbols—perhaps all writers should adopt this system as the standard. The reviewer regrets the inclusion of a formula like $\sigma_M = \sigma/(N-3)^{1/2}$ when $N<10$, for it must be entirely empirical as the theory of standard errors depends upon a normal curve which can hardly exist for so small a number of observations.

J. R. MUSSELMAN

Theorie der Raumkurven und krummen Flächen. By V. Kommerell and K. Kommerell. (Göschens Lehrbücherei. I. Gruppe: Reine und angewandte Mathematik, Band 20 und 21.) Berlin, de Gruyter, 1931. Band I. Krümmung der Raumkurven und Flächen. 205 pages. Rm. 10. Band II. Kurven auf Flächen. Spezielle Flächen. Theorie der Strahlensysteme. 194 pages. Rm. 10.

In presenting the fourth edition of their important work on the essentials of differential geometry, the authors have completely revised and rewritten the preceding edition of their three books which appeared as volumes 29, 44, and 62 of the Sammlung Schubert in 1921. The purpose and general plan of the earlier treatments remain unchanged and in the present edition the authors seem to have been especially successful in producing an elementary but scientifically sound introduction to the more extensive treatises of Bianchi, Eisenhart, Forsyth, and others. Numerous references to source material are given and as is stated in the preface, the diligent student of these volumes should be able to proceed to original investigation.

The method is that of classical differential geometry and the early introduction of the parametric representation of surfaces contributes greatly to the directness and simplicity of the treatment. Although vector methods are not

developed in the text, the geometric content of the result is frequently made more evident by the use of the vector notation. Naturally such portions of the text will be less easily read by the person who is not familiar with the elements of vector analysis. In keeping with the spirit of the times much greater emphasis has been placed on rigor in the present edition. Exact analytical definitions involving infinitesimals and limiting processes have replaced discussions of "infinitely near" or " consecutive" elements, these latter terms being retained only as modes of speech.

New subject matter mentioned in the preface includes the following: the parallel displacement of Levi-Civita; the Riemann-Christoffel curvature tensor with brief mention of its importance in the theory of relativity; the Gauss-Bonnet integral formula. The paragraphs dealing with the transformation of parameters and with the differential parameters of Beltrami have been completely recast.

Condensing the paragraph headings, we may say that a third of the first volume is devoted to the study of the space curve and its associated developable surfaces while the remainder of the volume deals with the curved surface. Here a rather complete study of the common properties of a curved surface in the neighborhood of an arbitrary point leads to a discussion of the mapping of one surface on another. The types of mapping considered are classified according as they preserve 1: angles (conformal mapping), 2: area, or 3: both angles and area (deformation). The last chapter of the volume considers geodesic curves and coordinates on a given surface, geodesic curvature and torsion of an arbitrary curve, and finally the formula of Gauss for the total curvature of a geodesic triangle on a surface.

The second volume opens with a discussion of the fundamental (Mainardi-Codazzi and Gauss) equations of a surface and the first chapter introduces differential parameters of the first and second orders (Beltrami) and applies them to the study of systems of curves on a surface and to the deformation of surfaces. Chapter two deals with the parallel displacement of Levi-Civita and extends the ordinary curvature tensor to the Riemann-Christoffel curvature tensor for an n-dimensional manifold. The next two chapters discuss surfaces of Weingarten and, in particular, the special cases—minimal surfaces and surfaces of constant curvature. The fifth chapter deals interestingly with ruled surfaces and triply orthogonal systems of surfaces. A very clear discussion of congruences of straight lines is given in the last forty pages of the text. In its essential points the treatment rests on the work of Kummer and several special types are considered.

The list of problems covering the text of the two volumes has been completely rewritten. The number has been more than doubled and they vary greatly in difficulty. Suggestions are frequently given and there are numerous references to source material.

The usefulness of these volumes as reference books is enhanced by a table of contents at the beginning of each volume and, at the end, by a list of authors

referred to and an index of subjects treated. The figures have been redrawn and their number increased. The books are attractively printed and the very few typographical errors noted will not cause confusion.

C. H. YEATON

Recent Developments in the Teaching of Geometry. By Jabir Shibli. Published by the author, State College, Pa., 1932. viii+252 pages.

This attractive volume is a worthwhile contribution to the literature of the teaching of mathematics chiefly because of its clarity and directness, as well as its timeliness. The first two chapters sketch the history of the teaching of geometry from classical antiquity to the year 1928 and provide an adequate background for the rest of the book. This material is the more welcome because of the inaccessibility of A. W. Stamper's "A History of the Teaching of Elementary Geometry," published in 1907, and long since out of print. The next three chapters, dealing with "Intuitive Geometry," "Introduction to Geometry," and "Foundations of Geometry," are rather good, although perhaps they leave a little to be desired because of their brevity. Chapters 6 and 7, which in a sense constitute the major portion of the book, present an excellent account of recent trends with regard to the treatment of book propositions and original exercises. The presentation of the aims and values of teaching demonstrative geometry is also good; the only objection would appear to be that in the discussion of transfer of training in connection with geometry the author makes no allusion to Mr. Betz's illuminating article on this subject in the fifth Yearbook of the National Council of Teachers of Mathematics.

On the whole, the book is well written, with an attractive typographical set-up, adequate footnote references, and a fairly adequate bibliography. It is unquestionably a valuable addition to the literature; taken together with the National Council's Fifth Yearbook we have a comprehensive survey of the field of demonstrative geometry teaching, and a substantial foundation upon which to base further reorganization and improvement of the 10th year mathematics of the future.

W. L. SCHAAF

Mathematics of Finance. Revised Edition. By H. L. Rietz, A. R. Crathorne, and J. C. Rietz. New York, Henry Holt and Company, 1932. xvi+346 pages.

Eleven years ago it was a pleasure to the writer to review the first edition of this book in the Journal of Commerce, New York City (Nov. 19, 1921), because there was so much to commend, in the formulation of the material, in the character and variety of the problems, in its teachableness as a class-room text, and in the workmanship of its manufacture.

The present text has been improved in the light of experience gained by teaching the previous edition to classes in colleges and universities throughout the country. And the authors, who are eminent teachers themselves, can say

that "Most of the changes are in the nature of simplifications." That is what we might expect to hear from living teachers, for art is simple. Some of the simplifications have become practicable through the prevalence of more extensive tables of compound interest and annuities. The tables are more than twice as extensive as in the first edition. The number of exercises has been greatly increased and their quality improved.

Originally, we were taken through Annuities Certain in two chapters covering 61 pages. Now, the same material is put into four chapters, 79 pages. The last two chapters are for reference, and treat Logarithms and Progressions, pages 252 to 282 inclusive.

C. C. GROVE

Philosophy and Modern Science. By H. T. Davis. Bloomington, Indiana, Principia Press, 1931. xiv+355 pages. Price $3.50.

The brief title is possibly misleading. The philosophy and modern science discussed are restricted to such as bear upon the physical sciences in a narrow sense. The work treats effectively historic problems but chiefly offers to the reader an introduction to the latest views in their historical and philosophical setting. The ground covered is fairly indicated by the successive chapter headings: Science—the modern metaphysics; is there an ether?; does matter drag the ether?; time, space and the fourth dimension; what is gravitation?; is matter finite?; the atom concept; the laws of chance; the metaphysics of the quantum theory and wave mechanics; the discovery and interpretation of cosmic radiation; what shall we believe?

The author is not only professor of mathematics, but also of the philosophy of natural science. It is in the latter capacity that he speaks in this book. A review commensurate in length with the excellence of this work would be inappropriate in a mathematical magazine. The style is brisk and is rendered lively by numerous brief quotations, being largely significant source material from the writings of the great discoverers whose results are under discussion, but in part in lighter vein, humorous or poetic. Lewis Carroll's immortal Alice speaks not infrequently. One catches hints of the enthusiasm of the original searchers after truth. In an undertaking with so ambitious an outlook (some 300 persons are mentioned by name) many great movements are necessarily dismissed with a few well chosen sentences, and elaborate evaluation yields to a condensed recital of facts and theories. Hence the book despite simple language, animated style and well sustained continuity, is not adapted to rapid reading.

There are here presented no original results of the author, either scientific or philosophical, but the choice of material is excellent, whether in classic problems already treated by a long line of distinguished writers or in the newest fields. The gathering of data, scientific, philosophical and literary is unusually effective. Technical notation is restricted to essentials, and the incompleteness

of current investigations is acknowledged by a frank undogmatic attitude of inquiry. The trained physicist may object to both matter and manner, the historian may balk at facts stated so baldly, the philosopher may miss his favorite topics, but the average intelligent reader who has not previously indulged in extensive specialized reading in the lines suggested, should find the book stimulating, informing, reliable, and should appreciate the abundance of explicit references giving him encouragement to extend his reading profitably. The proof-reading seems not on a par with the care devoted to other features in the production of this book.

A. A. BENNETT

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Lehigh University

The Chapter at Lehigh devoted the entire year to the study of the history of mathematics. Meetings were monthly with numerous papers by both faculty and student members. One open address was sponsored during the year. This was delivered by Dr. T. C. Fry, of Bell Telephone Laboratories, on March 17. Dr. Fry spoke on "Mathematics comes into its own." He gave a historical treatment of mathematical theories in the physical sciences. He believes that mathematics comes into its own when it successfully handles problems in physics.

At the April meeting the following officers were elected for the ensuing year. L. L. Smail, Director; Melvin Dresher, First Vice Director; J. W. Langhaar, Second Vice Director; W. C. Bachman, Secretary; J. A. Tempest, Treasurer; and G. M. Dewees, Librarian.

New members were publicly pledged in the Lehigh Chapel and initiated at a banquet at Hotel Bethlehem in May.

The Chapter reports a successful year.

TOMLINSON FORT, *Director*

LOCAL MATHEMATICS CLUBS

Undergraduate Mathematics Club of the University of Iowa

The officers for the year 1931–1932 were: Deane Montgomery, President; Marian Frey, Secretary-Treasurer; Dr. Allan T. Craig, Faculty Advisor. The officers are elected annually by the members of the club.

We have twenty-seven active members in our club.

The meetings and programs were as follows:

December 3, 1931: "Multiplication" by Professor R. P. Baker.
January 14, 1932: "Hyperbolic and elliptic functions" by Professor J. F. Reilly.
February 18, 1932: "Application of inversion to conics" by J. W. Querry.
March 17, 1932: "Curves associated with two given curves" by C. H. Fischer.
April 28, 1932: "Special conics" by E. G. Harrell.

MARIAN FREY, *Secretary*

The Mathematics Club of the Case School of Applied Science

The officers for 1931–1932 were: Ray Boyer, President; S. Eisler, Vice President; Edwin L. Smith, Secretary and Treasurer; Carl A. Cotman, Recording Secretary.

The purpose of the "Case Mathematics Club" is to promote an interest in the study of mathematics and also to show students the unity in the field of mathematics which can not be so effectively shown in the class rooms.

Any undergraduate or post-graduate student, as well as any member of the faculty is eligible for membership. Interest in mathematics is the sole qualification. Nominal dues of fifty cents ($.50) per term are required of active members. Our active members during the past year numbered twenty-five.

As usual, most of our meetings were held on the campus. Those not held on the campus were: The meeting of May 14, 1931 which was held at the Case Club; that of October 16, 1931 was held at Dean Focke's home; on April 29, 1932 we met at the Case Observatory; and on May 10, 1932, we met at the Case Club.

The meeting and programs were as follows:

May 14, 1931: "The unity of numbers" by Dr. Simon, Professor of Mathematics at Western Reserve University.
October 16, 1931: "The lure of mathematics" by Dr. Cairns, Professor of Mathematics at Oberlin College.
October 30, 1931: "Mathematics as an exact science" by Mr. Oldenburger, Instructor of Mathematics.
November 20, 1931: "Non-Euclidean geometry" by L. G. Henyey.
December 4, 1931: "The theory of relativity" by W. N. Simon, Jr.
December 18, 1931: "Theories of parallelism" by J. R. Parker.
January 16, 1932: "The theory of numbers" by L. P. Tarasov.
February 26, 1932: "Application of mathematics to the building and designing of air vehicles" by Dr. Hemke, Professor of Aeronautical Engineering.
March 11, 1932: "The inversion of the sphere" by Ray Boyer.
March 28, 1932: "Mathematics as a vehicle of thought" by Professor Thomas, Professor of Mathematics.
April 15, 1932: "Some theorems of projective geometry" by B. Napier.
April 29, 1932: "Experiences at Göttington" by Dean Focke, Professor of Mathematics.
May 10, 1932: "New fields of mathematics" by Dr. Clarke.

CARL A. COTMAN, *Secretary*

The White Mathematics Club of the University of Kentucky

The White Mathematics Club has been an organization on the University of Kentucky Campus for several years. It previously functioned under the auspices of the faculty of the Department of Mathematics. This year it was placed under the management of the student body.

The first meeting was held on October 15, 1931 under the leadership of Pi Mu Epsilon. At this meeting, the following officers were elected by popular vote: Dr. H. H. Downing, Advisor; Mary Allison Threlkeld, President; Elizabeth Ragland, Vice President; Anna Bruce Gordon, Secretary; Raymond Voll, Publicity Chairman.

The aim of the club is to increase the appreciation and knowledge of mathematics.

Students who are majoring or minoring in mathematics, those who have a definite interest in the subject, and graduate assistants in the department are eligible for membership. There are at the present time twenty active members in the club.

The meeting and programs were as follows:

October 15, 1931: "The problem of coloring maps" by Dr. Leon Cohen.

November 2, 1931: "The breaking up of numbers into factors" by Mary Allison Threlkeld.

December 3, 1931: "The foundations of mathematics" by T. L. Smith.

January 4, 1932: "The application of a problem in physics to mathematics" by J. H. Kirk.

February 11, 1932: "How our number system originated" by Nancy Duke Lewis.

March 10, 1932: "One to one correspondence" by Margaret LeStourgeon.

April 7, 1932: "A mathematical parody on Annabel Lee" by D. Ferrell.

April 22, 1932: The Mathematics Club entertained with a party, having as their guests all members of the faculty and their wives.

MARY ALLISON THRELKELD, *Secretary*

THE KAPPA MU EPSILON MATHEMATICAL FRATERNITY

This fraternity originated at Northeastern Teachers College in 1931 and has for its purpose the development of an appreciation of the beauty in mathematics, a greater interest in the subject and a broader understanding of it. Membership is extended to students who are majoring or minoring in mathematics and are in the upper half of their classes scholastically.

The chief emblems chosen by the fraternity are: The five pointed star and the pentagon. The pin and the key are pentagons. The seal is a star and the major part of the crest is a star. The wild rose is the fraternity flower. The pink of the wild rose and the silver of the star are the fraternity colors. In making the crest, five emblems, each representative of one of the sciences using mathematics, were chosen and placed around the star on the shield.

At the first Founder's Day banquet, held at Northeastern on April 18, 1931, the following officers were elected: Dr. Kathryn Wyant, Northeastern, President Pythagoras; Professor Ira S. Condit, Iowa State Teachers College, Vice President Euclid; Miss Lorene Davis, Secretary Diophantus; Professor L. P. Woods, Northeastern, Treasurer Newton; Miss Bethel DeLay, Historian Hypatia.

Chapters which have been installed since the founding are: Iowa Alpha, May 27, 1931, Iowa State Teachers College, Cedar Falls, Iowa; Kansas Alpha, May 20, 1932, Kansas State Teachers College, Pittsburg, Kansas; Missouri Alpha, Southwest Missouri Teachers College, Springfield; and Mississippi Alpha, May 30, 1932, Mississippi State Teachers College for Women, Columbus, Mississippi.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 16. *Proposed by G. A. Yanosik, New York University.*

Prove that the envelope of the circles whose diameters run from points on a parabola to its focus, is the straight line tangent to the parabola at its vertex.

E 17. *Proposed by Morgan Ward, California Institute of Technology.*

In Bierans de Haan's "Nouvelles Tables d'Intégrales Définies," Table 113, formula 1, page 162, it is stated that

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} = \frac{2\pi^2}{27}$$

Show that this result is incorrect.

E 18. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Of all the right triangles whose areas exceed a million square units and whose three sides are integers without common factor, find that one whose perimeter is a minimum.

E 19. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Find the radius of a sphere whose surface area and volume are each numerically equal to π times a four-place integer, and show that the solution is unique.

E 20. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

The following letters represent the digits of a simple multiplication:

$$\begin{array}{rrrrr} & & A & B & C \\ & & & B & D \\ \hline & C & E & E & B \\ F & B & C & D & \\ \hline F & G & C & G & B \end{array}$$

Solve and show that the solution is unique.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3587. *Proposed by P. R. Rider, Washington University.*

The axes of two right circular cylinders of indefinite length intersect perpendicularly. Find the volume common to the two cylinders. (This problem actually arose in designing a tank car.)

3588. *Proposed by N. H. McCoy, Smith College.*

Find an explicit expression for f_n if $f_0=1$ and

$$f_{n+1} = (x^2 + y^2)f_n + 2ty\frac{\partial f_n}{\partial x} + t^2\frac{\partial^2 f_n}{\partial x^2},$$

where t is a parameter. Show that

$$(f_n)_{x=y=0} = K_n t^n, \ K_n = \left.\frac{d^n(\sec 2t)^{1/2}}{dt^n}\right|_{t=0}.$$

3589. *Proposed by R. E. Gaines, University of Richmond.*

If a tangent to the cardioid $\rho=a(1+\cos\theta)$ at the point P_1 cuts the curve again in P_2 and P_3, the area of the segment cut off by the chord P_2P_3 is $\frac{3}{4}a^2(\phi-\sin\phi)$, where ϕ is the angle which the chord subtends at the origin.

3590. *Proposed by G. E. Raynor, Lehigh University.*

Prove that Simpson's (one-third) rule gives the correct value of the integrals

$$\int_0^\pi \sin^{2m}\theta d\theta, \quad \int_0^\pi \cos^{2m}\theta d\theta,$$

where m is a positive integer, if the number of intervals used is greater than two.

3591. *Proposed by B. F. Kimball, Schenectady, N. Y.*

Let the nth difference of $\log x$, with difference interval one, be denoted by $\Delta^n \log x$. Show that

$$\lim_{n\to\infty} n^x \log n\Delta^n \log x = \Gamma(x).$$

3592. *Proposed by V. Thébault, Le Mans, France.*

Given the tetrahedron $ABCD$; construct five equal spheres so that one of them shall be tangent to the remaining four, and each of these shall touch three faces of the tetrahedron.

3593. *Proposed by J. B. Reynolds, Lehigh University.*

Find the locus of the center of gravity of the volume cut from a homogeneous cube by a plane cutting off a constant volume.

SOLUTIONS

3501. [1931, 341]. *Proposed by E. B. Seitz, from "The Mathematical Visitor."*

Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken anywhere within the circle, find the average area of the triangle formed by joining the three points.

Solution by F. L. Wilmer, Odebolt, Iowa.

In this solution, which follows a method developed in W. Woolsey Johnson's Integral Calculus (1907), two triangles are considered different when and only when they are not identical. Let a be the radius of the given circle; let the given diameter be along Ox; let $B(r, \theta)$ be any point in the upper semi-circle; $C(\rho, \phi)$, any point in the lower semi-circle; and $A(R, \psi)$, any point in the circle. If N denotes the "number of cases" where A is restricted to the upper semi-cricle, then $N = (\pi a^2/2)^3 = \pi^3 a^6/8$.

We also have

$$\text{area } ABC = \tfrac{1}{2}[r\rho \sin(\phi - \theta) + R\rho \sin(\psi - \phi) + Rr \sin(\theta - \psi)],$$

and the expression on the right is positive if A lies in the sector of the circle with the angle θ. We shall restrict A in this manner, and this gives just half of the cases corresponding to N when A, B, C take all possible positions indicated above. Let M denote the desired average, then

$$M \cdot N = \int_{\pi}^{2\pi} \int_0^{\pi} \int_0^{\theta} \int_0^{a} \int_0^{a} \int_0^{a} [r\rho \sin(\phi - \theta) + R\rho \sin(\psi - \phi) + Rr \sin(\theta - \psi)]$$
$$R r \rho \, dR \, dr \, d\rho \, d\psi \, d\theta \, d\phi$$
$$= a^8[\pi^2 + 8]/18.$$

Hence

$$M = 4a^2[\pi^2 + 8]/9\pi^3.$$

3502. [1931, 341]. *Proposed by B. F. Finkel, Drury College.*

The centroid of a triangle is joined to its vertices and a point is taken at random in each of the three parts. Find the average area of the triangle formed by joining the three random points.

$$
(6)\quad
\begin{aligned}
M_1 = \frac{1}{18} lm \sin\alpha_1 \sin\beta_1 \Big\{ & 4\sin C + \sin C \frac{(\csc^2\alpha_1 - \csc^2\alpha_2)(\csc^2\beta_1 - \csc^2\beta_2)}{(\cot\alpha_1 + \cot\alpha_2)(\cot\beta_1 + \cot\beta_2)} \\
& + 2\cos C \frac{\csc^2\beta_1 - \csc^2\beta_2}{\cot\beta_1 + \cot\beta_2} - 2\cos C \frac{\csc^2\alpha_1 - \csc^2\alpha_2}{\cot\alpha_1 + \cot\alpha_2} \Big\} .
\end{aligned}
$$

The remaining two parts, M_2 and M_3, of M are easily written by a cyclic interchange of the letters.

If ABC is equilateral with O at its centroid, the result is simple

$$(7)\qquad M = 4^{-1}3^{-1/2}l^2 = 4^{-1}3^{-3/2}a^2,$$

where a is the length of the equal sides.

3525 [1932, 46]. *Proposed by H. A. Campbell, Omaha, Nebraska.*

A semicircle with center O has AC for the diameter at its base. The radius OC is prolonged to D so that $CD = OC = r$. The semi-circumference is bisected at Z, and on the arc ZC the point Y is taken so that ZY subtends at O an angle of 60°. Let the chord ZY produced cut OD in X; and with X as center and radius XD describe a semicircle on the same side of AD as that of the first one. Let B be any point on the arc AZ, and draw OB. Construct on this side of AD the angle DXM equal to the angle AOB, where M is the intersection of the side XM with the second semicircle. Let N be the projection of M on OD, and draw NB cutting again in T the semicircle at O. Prove or disprove that $NT = r$.

Editorial Note. If NT and r were equal, the angle AOB would be three times the angle ONB; and we know that no such construction will give this result for *any* position of B on the arc AZ. Determine the maximum error in this approximate method for trisecting an angle less than 90°, and show that the error is zero only for 90° and a zero angle.

Solution by W. B. Campbell, Rangoon, Burma

From the given construction we find that $OX = 3^{1/2}r$, $XD = 2r - OX$, $XN = XD\cos\theta$, where $\theta = \angle AOB$. Then $\tan \angle ONB = r\sin\theta/(r\cos\theta + OX + XN)$. Setting $\angle ONB = \theta/3 + \epsilon$, we find

$$(1)\qquad \epsilon = \tan^{-1}\left[\frac{\sin\theta}{3\cos\theta + 3^{1/2}(1-\cos\theta)}\right] - \theta/3.$$

Differentiating we find that ϵ is a maximum when

$$(2)\qquad \cos\theta = \frac{3^{3/2}-1}{13}\cdot$$

Since $\epsilon = 0$ for $\theta = 0°$ or 90°, it does not equal zero for any angle in the first quadrant, as the function has only one turning point in this interval.

From (2) the value of θ for the first quadrant is $\theta=71°10'9''$, $\theta/3=23°43'23''$, $\angle ONB=23°50'45''$, $\epsilon=7'22''$.

A Note by Otto Dunkel. For very small angles the error is given approximately by $\epsilon=\theta^3/393$. The geometric fact stated in the first sentence of the editorial note on the problem can be utilized in a more interesting and successful manner. If $\angle ONB$ is a good first approximation to a third of $\angle AOB=\theta$, a better second approximation $\angle ON'B$ is obtained by finding N' on ON, or ON produced, so that $TN'=r$. If the first approximation is too large, the second is too small, and conversely. If BN' cuts the circle (O) in T', we find a third and still better approximation $\angle ON''B$ in the same manner, and so on. If the error in the initial approximation is ϵ, then the error in the second approximation is approximately $\epsilon'=-4\epsilon\theta^2/27$, if θ is small. Hence in this case the successive approximations converge rapidly. See the article by E. C. Kennedy, *Angle Division*, in this MONTHLY, vol. 39 (1932), p. 478.

Also solved by A. Pelletier and R. C. Staley. One unsigned solution was received from Carnegie Institute of Technology.

3535 [1932, 174]. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

The pedal curve p with respect to a point P of a given curve c is defined as the locus of the feet of the normals from P to the tangents of c, and c is known as the negative pedal of p. Prove that the negative pedals of circles and straight lines with respect to a point P, not on the circles or straight lines, are respectively central conics and parabolas having a focus at P. The parabolas are tangent to their pedals and the central conics are doubly tangent to theirs.

Solution by Ethel I. Moody, Sweet Briar College

Let the point P be taken as the origin. The equation of any straight line not passing through P may be written in the form

$$Ax+By+1=0. \tag{1}$$

The lines through P may be represented by the equation

$$x+\lambda y=0. \tag{2}$$

Each line of the system (2) intersects the straight line (1) in a point the coordinates of which are

$$[-\lambda/(A\lambda-B),\ 1/(A\lambda-B)]. \tag{3}$$

Each tangent to the negative pedal of (1) with respect to P passes through a point (3) and has slope λ, and, therefore, this system of tangents can be represented by the equation

$$\lambda^2(Ax+1)-\lambda(Bx+Ay)+By+1=0. \tag{4}$$

The negative pedal of the line (1) with respect to P is the envelope of the system

diameter, and therefore the circle (6) and its negative pedal with respect to P have common tangent lines at the two points in which the line (10) intersects the circle (6).

Also solved by E. M. Berry, W. B. Campbell, Marie M. Johnson, W. V. Parker, and C. A. Rupp.

A Note by Otto Dunkel. The geometry of this problem is rather simple. In the general case of any curve p let M, M' be two neighboring points of p, and let the perpendiculars to PM and PM' through M and M', respectively, meet in Q'. Since P, M, M', Q' lie on a circle with the diameter PQ', the limiting position Q of Q' on MQ', that is the point of tangency of MQ' with c, is easily found as follows: Let the perpendicular bisector of PM cut in C the normal at M to p, and produce PC to Q so that $CQ=PC=CM$.

If p is a straight line l, let K and K' be the feet of perpendiculars from P and Q to l; then $PK+QK'=2CM=2PC$. Produce PM to cut QK' extended in S. Then $QS=PQ$, since $K'S=PK$. Hence Q lies on a parabola with the focus P and with the directrix parallel to l so that K is the mid-point of the perpendicular to it from P. Obviously the parabola is tangent to l at K. The proof also shows that the tangent to the parabola at Q bisects the angle between the focal ray PQ and the line through Q parallel to the axis KP. We also see that the envelope of circles having the focal ray PQ as a diameter is the tangent at the vertex.

If p is a circle with center C_0 and radius r, the same construction gives Q, where in this case the normal at M is the radius C_0M. For convenience suppose that P is inside the circle p, and let PC_0 cut this circle in the ends of the diameter DD'. Let the parallel to MC_0 through Q cut DD' in P'; then $PC_0=C_0P'$, since $PC=CQ=CM$. Also $P'Q=2C_0C=2C_0M-2CM$, and $PQ=2CM$. Therefore $P'Q+PQ=2C_0M=2r$, and Q lies on the ellipse with foci P, P'. This ellipse is tangent to the circle at D and D'. This also shows that the envelope of circles with the focal ray PQ as diameter is the auxiliary circle p. The construction shows that the normal at Q to the ellipse is parallel to MP, and since MCP is isosceles, the normal bisects the angle between the focal rays PQ and $P'Q$. The case of P outside p is treated in a similar manner.

3537 [1932, 175]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The letters H, V, and P designate a system of mutually perpendicular coordinate planes, and the letters α, β and γ the angles which a plane Q makes with H, V, and P respectively. If none of these angles exceeds 90°, what are the extreme limits of the sum $\alpha+\beta+\gamma$?

Solution by E. M. Berry, Lynchburg College

Let H, V, and P be the zy, xz and xy planes respectively and let $x\cos\alpha+y\cos\beta+z\cos\gamma-p=0$ be the plane Q. Then α, β, and γ are the angles HQ, VQ, and PQ respectively.

Then we have

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{1}$$

The sum $S = \alpha + \beta + \gamma$ is to be a maximum or a minimum. Substituting the value of α from (1) we have

$$S = \beta + \gamma + \cos^{-1} (1 - \cos^2 \beta - \cos^2 \gamma)^{1/2}. \tag{2}$$

The partial derivatives $\partial S/\partial \beta$ and $\partial S/\partial \gamma$ must equal 0. This gives us the two conditions

$$\cos \beta \sin \beta = \cos \gamma \sin \gamma = (1 - \cos^2 \beta - \cos^2 \lambda)^{1/2}(\cos^2 \beta + \cos^2 \gamma)^{1/2}. \tag{3}$$

From these we have $\sin 2\beta = \sin 2\gamma$.

$$\therefore \quad \beta = \gamma \quad \text{or} \quad \beta + \gamma = 90^\circ.$$

Substituting $\gamma = \beta$ in equation (3) we get

$$3 \cos^2 \beta = 1 \text{ or } \cos \beta = 0$$

$$\therefore \quad \alpha = \beta = \gamma = 54^\circ 44', \text{ and } S = 164^\circ 12', \text{ or}$$

$$\alpha = 0, \beta = \gamma = 90^\circ, \text{ and } S = 180^\circ.$$

By trying values near by, we find the former is the minimum and the latter is the maximum for the sum S.

If $\beta + \gamma = 90^\circ$ we find from (1) that $\alpha = 90^\circ$ and the sum S is 180°. This is the maximum for the sum.

If the angles were allowed to exceed 90° then 180° would not be a maximum value for the sum.

Also solved by W. B. Campbell, F. Underwood and the Proposer.

A Note by Otto Dunkel. The determination of the extremes may be put in a form which shows more clearly the variation of S. Let the unit sphere with its center at the origin cut the rectangular axes in A, B, C, and let P be the point in which the sphere is cut by a ray from the origin in the first octant. In terms of arcs of great circles we have $AP = \alpha$, $BP = \beta$, $CP = \gamma$. If P is at A, $\alpha = 0$ and $\beta + \gamma = \pi$; if P is on the side BC of the spherical triangle ABC, $\alpha = \frac{1}{2}\pi$, $\beta + \gamma = \frac{1}{2}\pi$. For these positions of P, $S = \alpha + \beta + \gamma = \pi$. If P is at any point inside, then by geometry $\frac{1}{2}\pi < \beta + \gamma < \pi$. Consider the parametric curve $\beta + \gamma = t$ (constant), where $\frac{1}{2}\pi < t < \pi$, $0 < \alpha < \frac{1}{2}\pi$. For this curve, we have

$$\frac{d\alpha}{d\beta} = \frac{2 \cos t \sin (t - 2\beta)}{\sin 2\alpha} \cdot \tag{1}$$

Since $\cos t < 0$, $\sin 2\alpha > 0$, we see that α decreases from $\pi - t$ as β increases from $t - \frac{1}{2}\pi$ until $\beta = \frac{1}{2}t$. After this point α increases until $\beta = \frac{1}{2}\pi$ and then $\alpha = \pi - t$. Hence the maximum for S on this curve is π at the two points where it cuts BA and CA. This maximum $S = \pi$ is attained therefore at each point of the perimeter and at no point of the interior.

Also the minimum value of S for this curve occurs at the point where it cuts the median arc AM_a, where M_a is the mid-point of side BC. We now examine the values of S along this median, where $\beta=\gamma$. We have

$$\frac{dS}{d\beta}=\frac{4\cos(\alpha+\beta)\sin(\alpha-\beta)}{\sin 2\alpha}. \tag{2}$$

As before $\frac{1}{2}\pi\leqq\alpha+\beta<\pi$, $\cos(\alpha+\beta)\leqq 0$, $\sin 2\alpha\geqq 0$. For $\beta=\frac{1}{4}\pi$, $\alpha=\frac{1}{2}\pi$; for $\beta=\frac{1}{2}\pi$, $\alpha=0$; and at these points only is the denominator in (2) zero. As β increases α decreases. Hence S decreases as β increases from $\frac{1}{4}\pi$ until $\beta=\alpha$; after this point S increases. Therefore there is a minimum S at just one point of the triangle, $\alpha=\beta=\gamma$, the pole of the triangle ABC. Very simple theorems of spherical geometry have been used above. The proof could be made entirely geometrical by assuming certain properties of the spherical ellipse $\beta+\gamma=t$.

3539 (1932, 175). *Proposed by R. E. Gaines, University of Richmond.*

A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends, will project over the edge of the table?

Solution by C. H. Fischer, University of Iowa

Since the rod is resting on the table it is obvious that the center must also lie on the table, i.e. within the given circle of radius r. The axes may be chosen so that the x-axis is parallel to the rod, and so that the rod then lies in the first two quadrants. The equation of the given circle (table) is $x^2+y^2=r^2$. If the rod is moved so that it remains parallel to the x-axis and its left end continues to make contact with the circumference of the given circle, the center of the rod describes an arc of a circle of radius r and center at $(a, 0)$. A similar procedure with the right end of the rod making contact with the circumference causes the center to describe an arc of a circle of radius r and center at $(-a, 0)$. These auxiliary circles intersect on the y-axis at $[0, (r^2-a^2)^{1/2}]$.

Since $r>a$, it is seen that if the center of the rod lies within the portion of the semicircle bounded by the two auxiliary circles, $(x-a)^2+y^2=r^2$, $(x+a)^2+y^2=r^2$, and the x-axis, neither end of the rod can project over the edge of the table.

If the center of the rod lies to the left of the auxiliary circle, $(x-a)^2+y^2=r^2$, the left end of the rod projects over the edge of the table. Similarly, if the center of the rod lies to the right of the auxiliary circle $(x+a)^2+y^2=r^2$, the right end of the rod projects beyond the edge of the table. Hence, in the region bounded by the given circle and the two auxiliary circles, i.e., where the center of the rod lies to the left of the first auxiliary circle and to the right of the second auxiliary circle, both ends of the rod project over the edge of the table. If the center of the rod lies in either of the two portions of the region, each half bounded by the three circles and the x-axis, then only one end of the rod will project over the edge of the table.

The probabilities may be obtained by dividing the areas of the above regions by the area of the semicircle, $\frac{1}{2}\pi r^2$. After calculating these areas in the usual manner we find:

(1). The probability that neither end of the rod will project over the edge of the table is

$$\frac{2}{\pi}\arccos\frac{a}{r} - \frac{2a}{\pi r^2}(r^2 - a^2)^{1/2}.$$

(2). The probability that one end of the rod will project over the edge of the table is

$$\frac{4a}{\pi r^2}(r^2 - a^2)^{1/2} - \frac{a}{\pi r^2}(4r^2 - a^2)^{1/2} - \frac{4}{\pi}\arccos\frac{a}{r} + \frac{4}{\pi}\arccos\frac{a}{2r} \cdot$$

(3) The probability that both ends of the rod will project over the edge of the table is

$$\frac{a}{\pi r^2}(4r^2 - a^2)^{1/2} - \frac{2a}{\pi r^2}(r^2 - a^2)^{1/2} + \frac{4}{\pi}\arcsin\frac{a}{2r} - \frac{2}{\pi}\arcsin\frac{a}{r} \cdot$$

It is possible that another choice of independent variable might lead to a different result, in somewhat the same manner as the various results of Bertrand's paradox are obtained. Another quite different formulation of the problem which the writer has worked through leads, however, to the identical result above.

Also solved by Thomas Butler, V. F. Ivanoff, and W. M. Rust, Jr.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Dr. Niels Bohr, professor of physics at the University of Copenhagen, expects to visit the United States during the summer of 1933.

Assistant Professor W. L. Ayres has received the Henry Russel Award for 1932 at the University of Michigan. This prize is awarded annually to a young man of the faculty of that University for past achievement and future promise in research. The prize was established in 1925.

Dr. William Bowie, of the United States Coast and Geodetic Survey, has been elected a member of the State Russian Geographical Society.

Professor A. H. Compton, of the University of Chicago, has been elected a corresponding member of the Prussian Academy of Sciences.

Dr. Irving Langmuir, of the General Electric Company, has been elected a member of the Academy of Sciences of Halle.

Dr. Robert A. Milligan was presented with the distinguished service medal of the Roosevelt Memorial Association on October 27, 1932.

At the University of Chicago Professor H. E. Slaught has retired from active service in the department of mathematics with the title of Professor Emeritus, having been connected with the University since it opened its doors in the autumn of 1892. He was a Fellow in mathematics at the University of Chicago from 1892 to 1894, and was one of the first group of graduate students who received the Ph.D. degree in mathematics from the University. From 1894 on he has been a member of the staff of the department of mathematics. He was a central figure in the founding of the Mathematical Association of America, served a term as its president, and has been for many years a managing editor of this MONTHLY. He has also served the American Mathematical Society with distinction as secretary of the Chicago Section and in numerous other capacities. The inspiring significance of his influence in college and university mathematical instruction is widely recognized.

Professor Oswald Veblen, who, as previously announced, will join the staff of the Institute of Advanced Study at Princeton, has presented his resignation to the trustees of Princeton University. Professor Veblen has been connected with Princeton for 27 years.

Dr. O. E. Brown has been promoted to an assistant professorship at the Case School of Applied Science.

Dr. Laura Guggenbuhl of Hunter College has been promoted to an assistant professorship of mathematics.

Dr. H. L. Slobin, head of the department of mathematics at the University of New Hampshire, has been appointed dean of the Graduate School.

Mr. Paul A. Benitz and Miss Isabel C. McLaughlin have been appointed to instructorships at Hunter College.

Professor E. H. Moore, professor emeritus of the University of Chicago, died December 30, 1932, aged 70 years. Professor Moore was one of the outstanding mathematicians of the present generation. He began his teaching career as instructor at the Academy at Northwestern University in 1886. From 1887 to 1889 he held a position as tutor at Yale. From 1889 to 1892 he was associated with Northwestern University, first as assistant professor and later as associate professor. In 1892 he was made professor of mathematics at the University of Chicago, and he held this position until his retirement a few months ago. He was honored by various universities both in America and abroad, receiving honorary degrees from Yale, Wisconsin, and Clark Universities in the United States, and from Toronto University and the University of Göttingen. He served as vice-president of the American Mathematical Society, 1898–1900, and as president, 1900–1902. In addition to this, he served on the editorial staffs of various mathematical journals, both in the United States and abroad. He was interested in the founding of the Mathematical Association of America and was a charter member.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS.
INDIANA.
IOWA, April.
KANSAS.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., March.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 6.
PHILADELPHIA, Philadelphia, Pa., Dec. 2.
ROCKY MOUNTAIN, April.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA.
TEXAS, February.
WISCONSIN, Beloit, Apr. 8.

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE SEVENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The seventh annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Swarthmore College on Saturday, November 26, 1932, Professors Dresden and Kline presiding.

The attendance was seventy-four, including the following thirty-nine members of the Association: S. S. Cairns, P. A. Caris, J. W. Clawson, Mary L. Constable, E. S. Crawley, J. E. Davis, Arnold Dresden, Tomlinson Fort, Orrin Frink, Jr., J. S. Gold, Michael Goldberg, H. V. Gummere, J. R. Kline, P. A. Knedler, V. V. Latshaw, Marguerite Lehr, F. L. Manning, R. W. Marriott, D. L. McDonough, Edith McDougle, J. A. Miller, H. H. Mitchell, Richard Morris, C. A. Nelson, F. W. Owens, Helen B. Owens, G. E. Raynor, C. J. Rees, George Rosengarten, J. A. Roulton, J. A. Shohat, C. A. Shook, C. A. Short, L. L. Smail, W. M. Smith, Anna Pell Wheeler, A. H. Wilson, Clement Winston, W. L. Wright.

At the business meeting the following officers were chosen for next year: Chairman, J. R. Kline, University of Pennsylvania; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Kline, Caris, Owens, Raynor.

The next meeting will be held on Saturday, December 2, 1933, at Philadelphia.

The following papers were presented:

1. "Some boundary value problems in potential theory" by Professor G. E. Raynor, Lehigh University.

2. "The independent arcs of a continuous curve" by Professor J. R. Kline, University of Pennsylvania.

3. "On curves with assigned singularities" by Dr. Marguerite Lehr, Bryn Mawr College.

4. "The problem of measure" by Professor Orrin Frink, Jr., Pennsylvania State College.

5. "The life and work of Ramanujan" by Professor H. H. Mitchell, University of Pennsylvania.

Abstracts of the papers follow:

1. The first part of Professor Raynor's paper is a brief exposition of the methods which have proved most powerful in the recent progress made with the Dirichlet problem and gives some indication of the present status of the problem. The second part deals with the Dirichlet-Neumann problem, and presents necessary and sufficient conditions that the problem have a solution for the sphere with singular point at the center.

2. Professor Kline discussed the various definitions of a curve giving the short-comings of the earlier attempts to define this concept. He then discussed theorems concerning the Menger order of points of a curve and the independent arcs which join two points of a continuous curve. In particular he discussed the

work of Nöbeling, Ruth and Zippin in this connection. He also discussed regular curves, perfect continuous curves, and continuous curves which are the sum of a countable infinity of arcs. Various examples were given showing the interrelations of these types.

3. The relations between the numerical characteristics of a plane algebraic curve known as the Plücker equations were set up a century ago, but until recently no definitive answer had been given to the inverse problem: Given a set of non-negative integers satisfying the Plücker equations, does a plane algebraic curve exist having these integers as its Plücker characteristics? In the existing literature the intimation has been that the answer would be affirmative. Dr. Lehr presented a proof of non-existence of curves having certain non-negative Plücker numbers, based on a theorem of Zariski's:—If f be irreducible of order m with nodes and cusps only, and if β be the maximum integer such that $6\beta < m$, the complete systems of curves passing through the cusps of f are regular for order $m-3-\beta$ and greater. Two possibilities for development were commented on: first, the fact that though the theorem is one of plane geometry, the proof given is based on systems of algebraic surfaces; second, the theorems arising when the restriction that f have nodes and cusps only is removed.

4. In recent years interesting results have been obtained in the problem of extending the notions of length, area, volume, etc., to arbitrary point sets. Professor Frink reviewed the definitions of measure of Jordan, Borel, Lebesgue, Caratheodory, and Hausdorff, which do not apply to all point sets, and Banach's definition of one and two dimensional measure for all bounded linear and planar sets. He showed why we can not expect a universal measure to be infinitely additive, and discussed Hausdorff's paradox, which proves the non-existence of even finitely additive universal measure definitions in spaces of dimensionality three or greater. In connection with von Neumann's results it was shown how one might weaken the restrictions to be placed on the measure function so as to allow it to be defined for all sets. Some of the outstanding problems were discussed, such as the existence of a universal one dimensional definition of measure for sets in two-space or three-space.

5. The title of Professor Mitchell's paper sufficiently indicates the nature of the paper.

P. A. Caris, *Secretary*

ORGANIZATION MEETING OF THE WISCONSIN SECTION

About thirty-five of the members of the Mathematical Association living in Wisconsin met at the Extension Division of the University of Wisconsin in Milwaukee on November 3, 1932, to organize a Wisconsin Section. The meeting was called to order by Professor G. A. Parkinson who was then elected temporary chairman, Professor H. P. Pettit being elected temporary secretary.

Professor Parkinson gave a brief resumé of the steps which had already been taken looking toward the present organization, after which followed a discussion

of the proposed By-Laws, which were adopted subject to the approval of the national organization.

The first regular meeting of the section was set for April 8, 1933, and an invitation for that time from Beloit College, proffered by Professor H. H. Conwell, was accepted.

It was decided that the temporary officers should function until the meeting in April, and that the temporary chairman should appoint a nominating committee to propose a list of officers which would be sent out by mail to the members shortly before the spring meeting.

Professor Conwell and Mr. Henry Ericson were elected to the program committee.

A vote of thanks was offered to Professor Parkinson and the Extension Division for their work and hospitality.

H. P. PETTIT, *Temporary Secretary*

THE EXPANSION OF DETERMINANTS OF COMPOSITE ORDER

By JOHN WILLIAMSON, Johns Hopkins University

It is often convenient, in discussing determinants of order mn, to consider the n^2 square arrays of order m contained in the determinant[1] as matrices of order m. In what follows we use this method to reduce[2] a determinant of order mn to a determinant of order $m(n-1)$. We first consider the special case in which $n=2$ and then pass on to the more general case. The final theorem proved is simply an extension of Horner's theorem[3] in which single elements of a determinant are replaced by matrices of order m.

Let Δ be the determinant of the square array

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

where A, B, C, D are all square matrices of order m whose elements are complex numbers, so that Δ is a determinant of order $2m$. We may denote this by

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

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[1] H. W. Turnbull, *Determinants, Matrices and Invariants*, pp. 39 sq.

[2] A method of reducing a determinant of order mn to one of order $m(n-1)$, without the use of matrices, is given in Muir and Metzler, Theory of Determinants, p. 150.

[3] Muir, *History of the theory of determinants*, vol. 3, pp. 15–16.

With this notation

$$(1) \qquad \Delta = (-1)^m \begin{vmatrix} B & A \\ D & C \end{vmatrix} = (-1)^m \begin{vmatrix} C & D \\ A & B \end{vmatrix} = \begin{vmatrix} D & C \\ B & A \end{vmatrix},$$

for

$$\begin{vmatrix} B & A \\ D & C \end{vmatrix}$$

is obtained from Δ by interchanging the first set of m columns of Δ with the second set of m columns, while

$$\begin{vmatrix} C & D \\ A & B \end{vmatrix}$$

is obtained from Δ by a similar process applied to the rows of Δ. Further if E denotes the unit matrix of order m,

$$(2) \qquad \begin{vmatrix} E & B \\ C & D \end{vmatrix} = | D - CB |,$$

a result which expresses a determinant of order $2m$ as a determinant of order m. To prove formula (2) we note that

$$\begin{vmatrix} E & B \\ C & D \end{vmatrix} = \begin{vmatrix} E & B \\ C - CE & D - CB \end{vmatrix},$$

since on the right the $(m+i)$-th row $(i = 1, 2, \cdots, m)$ is obtained by subtracting a linear combination of the first m rows from the corresponding row on the left. But

$$\begin{vmatrix} E & B \\ C - CE & D - CB \end{vmatrix} = \begin{vmatrix} E & B \\ 0 & D - CB \end{vmatrix}$$
$$= | E | \, | D - CB | = | D - CB |,$$

where 0 denotes the zero matrix.

We are now in a position to prove the theorem:

THEOREM I. *If A is non-singular, then*

$$(3) \qquad \Delta = | A | \, | D - CA^{-1}B |$$

where A^{-1} is the matrix reciprocal to A.

By the multiplication theorem on matrices we have

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & A^{-1}B \\ C & D \end{pmatrix}$$

and by taking determinants of both sides of this matrix equation, we obtain

$$|A^{-1}|\,\Delta = \begin{vmatrix} E & A^{-1}B \\ C & D \end{vmatrix},$$
$$= |D - CA^{-1}B| \text{ by (2)},$$

and this final result is equivalent to (3). Formula (3) may also be written in either of the forms

$$(4) \qquad \Delta = |AD - ACA^{-1}B|$$

or

$$(5) \qquad \Delta = |DA - CA^{-1}BA|.$$

From (4) it follows that, if A and C are commutative,

$$(6) \qquad \Delta = |AD - CB|$$

and from (5) that

$$(7) \qquad \Delta = |DA - CB|,$$

if A and B are commutative. If B is non-singular

$$\Delta = (-1)^m \begin{vmatrix} B & A \\ D & C \end{vmatrix} \text{ by (1)},$$
$$= (-1)^m |B|\,|C - DB^{-1}A| \text{ by (3)},$$
$$(8) \qquad = |B|\,|DB^{-1}A - C|.$$

Similarly, if C is non-singular,

$$(9) \qquad \Delta = (-1)^m |C|\,|B - AC^{-1}D| = |C|\,|AC^{-1}D - B|,$$

and, if D is non-singular,

$$(10) \qquad \Delta = |D|\,|A - BD^{-1}C|.$$

If B and D are commutative, from (8)

$$(11) \qquad \Delta = |DA - BC|$$

and from (9), if C and D are commutative,

$$(12) \qquad \Delta = |AD - BC|.$$

Formula (3) is not in its most convenient form for the calculation of numerical determinants, as the adjugate matrix is usually more convenient for calculation than the reciprocal matrix. If a denotes the determinant of A, so that $|A| = a$, then the matrix $A^{-1} = a^{-1}A^*$, where A^* is the adjugate matrix of A. Accordingly (3) becomes

$$(13) \qquad \Delta = a\,|D - CA^*Ba^{-1}| = a^{1-m}\,|Da - CA^*B|.$$

Let Δ now be the determinant of the square array A_{rs} $(r, s, = 1, 2, \cdots, n)$, where A_{rs} is a square matrix of order m, so that Δ is a determinant of order mn. We may denote the matrix of the array by (A_{rs}) and the determinant Δ by $|A_{rs}|$. If A_{ij} is non-singular, that is if the determinant of the matrix A_{ij} which we may denote by a is different from zero, the inverse matrix A_{ij}^{-1} exists. If (K_{rs}) is a matrix of the same type as (A_{rs}), where $K_{rs}=0$, $r\neq s$; $K_{rr}=E$, $r\neq i$; $K_{ii}=A_{ij}^{-1}$, by multiplication

$$(K_{rs})(A_{rs}) = (B_{rs}),$$

where $B_{rs}=A_{rs}$, $r\neq i$, $B_{ik}=A_{ij}^{-1}A_{ik}$. Moreover $|K_{rs}|=a^{-1}$ and therefore $\Delta=a|B_{rs}|$. But by reasoning similar to that used in the proof of formula (2),

$$| B_{rs} | = | C_{rs} | ,$$

where $C_{rs}=A_{rs}-A_{rj}A_{ij}^{-1}A_{is}$, $r\neq i$,

$$C_{is} = A_{ij}^{-1}A_{is}.$$

But $C_{rj}=0$, $r\neq i$, and $C_{ij}=E$. Therefore by a formula similar to (1), we have the following theorem:

THEOREM 2. *If the m-rowed square matrix A_{ij} is non-singular and if Δ is the determinant of the nm-rowed square matrix (A_{rs}),*

$$\Delta = (-1)^{m(i+j)}a\,|C_{rs}| \qquad \begin{array}{l} r = 1, 2, \cdots, i-1, \;\; i+1, \cdots, n \\ s = 1, 2, \cdots, j-1, \;\; j+1, \cdots, n, \end{array} \tag{14}$$

where

$$C_{rs} = A_{rs} - A_{rj}A_{ij}^{-1}A_{is}$$

and a is the determinant of the matrix A_{ij}.

Theorem 2 expresses a determinant of order mn as a determinant of order $m(n-1)$. It is a direct extension of Horner's theorem on the expansion of a determinant. In fact if $m=1$, that is, if all the matrices A_{rs} are matrices of one row and column, theorem 2 is exactly Horner's theorem.

Formula (14) may also be expressed in terms of the matrix adjugate to A_{ij} instead of in terms of the reciprocal of A_{ij}. If this is done formula (14) becomes

$$\Delta = (-1)^{m(i+j)}a^{1-m(n-1)}\,|D_{rs}| ,$$

where $D_{rs}=aA_{rs}-A_{rj}A^*_{ij}A_{is}$ $r\neq i$, $s\neq j$, and A^*_{ij} is the matrix adjugate to A_{ij}.

So far we have been dealing with determinants, whose elements are complex numbers, but it is interesting to note the connection between this paper and one by A. R. Richardson, in which determinants whose elements are numbers of a division algebra are discussed.[1] This is best illustrated by a simple example. If A, B, C, D are elements of a division algebra the left hand determinant of

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[1] A. R. Richardson, *Simultaneous Linear Equations Over a Division Algebra*, Proceedings of the London Mathematical Society, Series 2, Volume 28, pp. 395–430, (1928).

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

is defined as

$$DA - DBD^{-1}C \tag{15}$$

where D is not zero. If in this definition we regard A, B, C, D as matrices of order m and impose the restriction that D be non-singular, (15) may be regarded as the matrix-determinant of

$$\begin{matrix} A & B \\ C & D \end{matrix}.$$

Equation (10) is thus equivalent to the statement that the determinant of the matrix-determinant of the square array

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

is equal to the ordinary determinant of this array. A similar connection exists between theorem 2 and Richardson's definition of determinants of order n. It is obvious that, since the algebra of matrices is not a division algebra, the matrix-determinant defined by (15) is in itself of no practical importance.

A SIXTEENTH CENTURY CHINESE APPROXIMATION FOR π

By J. M. BARBOUR, Ithaca College

The standard work on Chinese mathematics is Yoshio Mikami's "The Development of Mathematics in China and Japan."[1] In it two chapters are devoted to circle-measurements. Recently Professor David Eugene Smith[2] has raised doubts concerning the authenticity of some of the supposedly ancient Chinese texts. He would ask us perhaps to accept with reservations the history of the earlier Chinese attempts to determine π, as he has excellently summarized them on page 309 of Volume 2 of his *History of Mathematics:*

"The value 3 was used probably as early as the 12th century B.C. and is given in the *Chôu-peï* and the *Nine Sections.* Ch' ang Höng [Hêng] (c. 125) used $10^{1/2}$, and Wang Fan (c. 265) used 142/45, which is equivalent to 3.155 . . . ." "Among other early Chinese values of no high degree of accuracy are those of Men (c. 575), who gave 3.14, and Wu (c. 450), whose value was 3.1432+." "Tsu Ch' ung-chih (c. 470) was able, by starting with a circle of diameter 10 feet, to obtain 3.1415927 and 3.1415926 for the limits of π, and from these, by

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[1] Published as Vol. 30 of *Abhandlungen zur Geschichte der Mathematischen Wissenschaften,* 1913.

[2] *Unsettled Questions Concerning the Mathematics of China,* The Scientific Monthly, vol. 33, 1931, pp. 244–250.

Note by the Author. After the above article was in type, my attention was directed—through the courtesy of Dr. Yuen-ren Chao, director of the Chinese Educational Mission—to an article by Liu Fu in the special volume (commemorative of the 65th birthday anniversary of Ts'ai Yüan-p'ei) of the *Bulletin of the National Research Institute of History and Philology, The Academia Sinica*, Peiping, 1932, pp. 279–310.

Mr. Shao-wen Ling, a graduate student at Cornell, kindly assisted me in obtaining some idea of the contents of this article, the title of which may be translated "Chu Tsai-yü, Inventor of Equal Temperament." The first part of the article carefully explains the whole theory of acoustics and temperament, with copious references to such European authorities as Helmholtz, Ellis, et al. The history of temperament in Europe is taken from secondary sources, principally from Hugo Riemann.

Tsai-yü, his family, his life, and his work are then discussed exhaustively. He was born in 1536, being 60 years of age when his book on temperament appeared. He died after 1610, the year of publication of the last of his 18 volumes. Amiot and Courant are mentioned as having given a correct account of Tsai-yü's researches into equal temperament. Furthermore, a lengthy paragraph is devoted to Tsai-yü's value for π, expressed in decimal form (3.1426968) and compared with the similar value 22/7 and the closer approximation 355/113. There the parallelism with my article ends, for Tsai-yü's value is not given in its equivalent radical form $(20\ 2^{1/2})/9$.

BI-AFFINE GEOMETRY IN THE PLANE

By C. E. CLARK, Brown University

1. In the customary treatment of plane projective geometry[1] various other geometries are obtained by specializing certain elements. Specializing a "line-at-infinity," the sub-group of projective transformations for which this line is self-corresponding yields affine geometry, which contains Euclidean geometry as a special case. Affine and Euclidean geometries, however, lack a distinctive property of projective geometry, namely of being self-dual. In this paper we consider another specialization of affine geometry in which not only a "line-at-infinity" but also a point (the "origin") not incident to the given line is fixed.[2] This bi-affine geometry is self-dual. The transformations are those used in a homogeneous system such as a vector space, in which there is a unique zero vector that can not be transformed into any other finite vector.[3]

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[1] As for instance in Graustein, *Higher Geometry;* Veblen and Young, *Projective Geometry*; Forder, *Higher Course Geometry*; and Heffter and Koehler, *Lehrbuch der Analytischen Geometrie.*

[2] Although this is a type of transformation very frequently used in analytic applications, apparently the only discussion of the geometry of these transformations is that given by A. A. Bennett in the Annals of Mathematics, Vol. 27, 1925–26, p. 84.

[3] See Courant and Hilbert, *Methoden der Mathematischen Physik*, p. 1.

Consider as our plane the points with homogeneous coordinates (x_1, x_2, x_3),[4] where x_1, x_2, and x_3 are in a given field F[5]. As usual one defines a line, conic, etc., as the set of these points satisfying respectively a linear equation, a quadratic equation, etc., the coefficients of the equation in every case being in the given field F. The origin $(0, 0, 1)$ and the points at infinity $(x_1, x_2, 0)$ are called *improper points*, the remaining points being *finite points*. By definition, a bi-affine point-point transformation is one that transforms finite points into finite points.

In section 2 we classify the bi-affine linear transformations, in section 3 reduce the second degree form to normal form, and in section 4 classify the conics, giving their normal forms in section 5.

2. A bi-affine linear transformation is of the form[6] $x_1=aX_1+bX_2$, $x_2=cX_1+dX_2$, $x_3=X_3$, with a, b, c, and d in F, and $ad-bc\neq 0$. If $px_1+qx_2=0$ is a self-corresponding[7] beam[8] for this transformation, $p(ax_1+bx_2)+q(cx_1+dx_2)\equiv \rho(px_1+qx_2)$ for a fixed ρ in F, and $p(a-\rho)+qc=0$, $pb+q(d-\rho)=0$. It follows that the self-corresponding beams of the given transformation are in (1,1) correspondence with the values of ρ satisfying the quadratic equation

$$\begin{vmatrix} a-\rho & b \\ c & d-\rho \end{vmatrix} = 0$$

(unless $a=d$ and $b=c=0$, in which case every beam is self-corresponding). Hence, in a field such as the complex field where every quadratic equation is reducible, every transformation has one or more self-corresponding beams. Furthermore, a transformation without self-corresponding beams in a given field F_1 may have self-corresponding beams in some extended field F_2 which contains F_1 as a sub-field. We now classify the bi-affine linear transformations, and give normal forms in which they may be expressed by proper choice of coordinate system.[9]

[4] Although in most of our discussion non-homogeneous coordinates will suffice, we shall use homogeneous coordinates such as are customarily employed in analytic projective geometry.

[5] For a definition and the elementary properties of a number field see G. Scorza, *Corpi numerici e algebre*, Messina, 1921; or L. E. Dickson, *Algebras and their arithmetics*, Chicago, 1923.

If, in particular, F is the real field, our plane is the one customarily employed in elementary geometry; if F is the field of rational numbers, we consider only points whose coordinates are rational.

The question of whether or not certain numbers are squares is important in the following theory. This property of a number being a square depends upon the field; in the real field all positive numbers are squares and all negative numbers are not; in the rational field not all positive numbers are squares, e.g. 2 is not the square of a rational number; in the complex field every number is a square; in a Galois field $GF(p)$, half the non-zero numbers are squares and half are not.

[6] See footnote 1.

[7] Following the notation of Hudson's *Cremona Transformations* we say that a locus is *self-corresponding* if the transformation leaves the locus as a whole invariant, and that a locus is *invariant* if every point of the locus is invariant.

[8] A beam is a line that contains the origin.

[9] We may choose any coordinate system such that one vertex of the triangle of reference is the origin and the corresponding opposite side $(x_3=0)$ is the line at infinity. This means that we may make a bi-affine transformation of coordinates.

(i) Every beam self-corresponding; called a *magnification* or *scalar transformation*; normal form $x_1=a\ X_1,\ x_2=a\ X_2,\ x_3=X_3\ a\neq 0$. When $a=1$ this is the identical transformation.

(ii) Exactly two self-corresponding beams; called a *stretch*; normal form $x_1=a\ X_1,\ x_2=b\ X_2,\ x_3=X_3,\ a\neq b,\ ab\neq 0$. If $a=1$ the stretch is a *simple stretch*, otherwise a *magnified stretch*.

(iii) Exactly one self-corresponding beam; called a *shear*; normal form $x_1=a\ X_1,\ x_2=b\ X_1+a\ X_2,\ x_3=X_3,\ a\neq 0$. If $a=1$ the shear is a *simple shear*, otherwise a *magnified shear*.

(iv) No self-corresponding beam; called a *twist*; normal form $x_1=X_2$, $x_2=a\ X_1+b\ X_2,\ x_3=X_3,\ b^2+4a$ not a square.

3. By means of bi-affine linear transformations we now reduce to normal form the general second degree form $f(x_1, x_2, x_3) =\sum a_{ij}x_ix_j,\ i, j=1, 2, 3,\ a_{ij}$ in F.

If $a_{11}=a_{12}=a_{22}=0$, f is reducible either to $a_{33}\ x_3^2$, or to $x_3(x_2+a_{33}\ x_3)$. Obviously a_{33} is an absolute bi-affine invariant.[1]

If at least one of $a_{11},\ a_{12},\ a_{22}$ is different from zero, then we can assume $a_{13}=0$ and either $a_{11}=0$ and $a_{23}\neq 0$ or $a_{11}=1$. In the first case, when $a_{11}=0$ and $a_{23}\neq 0$, if $\Delta=|a_{ij}|=0$, f is reducible to either

$$\frac{2a_{22}}{a_{23}^2}\,x_2^2+2x_2x_3+a_{33}x_3^2 \text{ or } x_1x_2+x_2x_3,$$

while if $\Delta\neq 0$, f is reducible to

$$2x_1x_2+2x_2x_3+a_{33}x_3^2,\ a_{33}\neq 0.$$

Finally, if $a_{13}=0$ and $a_{11}=1$, f is reducible to either

$$x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2,\ \lambda=\frac{a_{12}^2-a_{22}}{a_{23}^2},$$

$$\text{or } x_1^2-\mu x_2^2+a_{33}x_3^2,\ \mu=a_{12}^2-a_{22}.$$

λ is an absolute bi-affine invariant of the class of forms considered in this paragraph (i.e. forms of the type $x_1^2+a_{22}\ x_2^2+a_{33}\ x_3^2+2a_{23}\ x_2x_3+2a_{12}\ x_1x_2$) and μ is a bi-affine invariant of weight two.

4. We now classify the non-singular conics.[2] We define a non-singular conic to be a hyperbola, parabola, or ellipse according as the locus contains two, one, or no points at infinity, and dually the non-singular conic is said to be hyper-

[1] For a complete system of bi-affine invariants of a set of forms see R. Weitzenböck, *Invariantentheorie*, Groningen, 1923, p. 232.

[2] For a Euclidean classification of conics see C. C. MacDuffee, *Euclidean invariants of second degree curves*, American Mathematical Monthly, vol. 33, 1926, p. 243.

bolic, parabolic, or elliptic according as two, one, or no tangents can be drawn to the curve from the origin. An origin-centered conic is one for which the origin is the pole of the line at infinity. It should be noted that according to this definition the same equation may give different types of conics in different fields. For instance, the equation of an ellipse in the real field will represent a hyperbola in the complex field, which latter field contains no ellipses.

Let $f(x_1, x_2, x_3)=0$ and $F(u_1, u_2, u_3)=0$ be the equations of the non-singular conic in homogeneous point coordinates and homogeneous line coordinates respectively. We recognize the following classes:

	$f(x_1, x_2, 0)$ having non-proportional factors	$f(x_1, x_2, 0)$ having proportional factors	$f(x_1, x_2, 0)$ irreducible
$F(u_1, u_2, 0)$ having non-proportional factors	hyperbolic hyperbola	hyperbolic parabola	hyperbolic ellipse
$F(u_1, u_2, 0)$ having proportional factors	parabolic hyperbola	parabolic parabola	parabolic ellipse
$F(u_1, u_2, 0)$ irreducible	elliptic hyperbola	elliptic parabola	elliptic ellipse

The class of hyperbolic hyperbolas falls into origin-centered and non-origin-centered loci; similarly for elliptic ellipses.

5. By means of the results of section 3, we can now give normal forms of the equations of the conics considered in section 4. Let λ, μ, and a_{33} have the values assigned them in section 3, and let $\Lambda=a_{33}(1+\lambda a_{33})$. Λ is an absolute invariant when λ is (see section 3). The normal forms are then:

hyperbolic hyperbola (non-origin-centered) (two distinct types) (i) $2x_1x_2+2x_2x_3+a_{33}x_3^2=0$, $a_{33}\neq 0$, and
(ii) $x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ and Λ both squares $\neq 0$
parabolic hyperbola. . . . . $x_1^2-\lambda x_2^2+2x_2x_3=0$, λ a square $\neq 0$
elliptic hyperbola. $x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ a square $\neq 0$, Λ not a square
hyperbolic ellipse. $x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ not a square, Λ a square $\neq 0$
parabolic ellipse. $x_1^2-\lambda x_2^2+2x_2x_3=0$, λ not a square
elliptic ellipse (non-origin-centered). . . . $x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ not a square, Λ not a square
hyperbolic hyperbola (origin-centered). . $x_1^2-x_2^2+a_{33}x_3^2=0$, $a_{33}\neq 0$
elliptic ellipse (origin-centered). . . . . $x_1^2-\mu x_2^2+a_{33}x_3^2=0$, μ not a square
hyperbolic parabola. . . . . $x_1^2+2x_2x_3+a_{33}x_3^2=0$, a_{33} a square $\neq 0$
parabolic parabola. . . . . $x_1^2+2x_2x_3=0$
elliptic parabola. $x_1^2+2x_2x_3+a_{33}x_3^2=0$, a_{33} not a square.

A READING LIST IN THE ELEMENTARY THEORY OF EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

The following reading list is the result of a rather careful study of the five principal mathematical journals published in this country. It is hoped that it may serve two purposes. First, it may be used in several ways to supplement the usual work in a class in the theory of equations. The simpler articles (those of the list which are unstarred) may be put on an optional or required supplementary reading list, or made the basis of class reports, and in many cases several related articles can easily lead to a longer paper to be read before a club or society. It would probably be better, in most cases, not to assign articles for reading or study without a preliminary examination, since it is true that many of those which are not starred are actually not simple as to methods used. Several of the papers concerned with limits for roots of equations employ rather complicated proofs, for example, but are left unstarred because at least some of their results can be understood and applied by the average student.

The more difficult articles, which appear in the list with a star, are included principally for the instructors; but it is believed that they, too, may be of some value in class work, since the subjects are all related to the elementary theory of equations. Thus, the teacher who wishes to explain something of the nature of the various proofs of the fundamental theorem of algebra will find three papers on this topic, while a careful reading of two other starred articles will certainly give him a broader knowledge of symmetric functions.

The second main purpose of this bibliography is to acquaint the more enterprising students with the American mathematical periodicals, and to help them see how the first course in the theory of equations leads naturally, if not always immediately, into a number of advanced fields of study. Such a student who had found, let us say, the Gauss-Lucas polygon theorem of interest might be led to a study of Walsh's valuable work, and that of others, on Jensen's theorem, Pellet's theorem and more or less related topics. (References will be found at the end of Van Vleck's paper, which is No. 8 in the following list.) Another might be attracted to Curtiss's papers on extensions of Descartes' rule of signs, in volume 16 of the Transactions and volume 19 of the Annals; or to the recent work in elimination by Morley and Coble in volume 49 of the American Journal, and by Moore in volumes 30 and 31 of the Annals. Cramlet's article in volume 27 of the Annals, on determinants, might conceivably give a curious student the desire to know something about tensor analysis. And these are, of course, only examples. No one would claim that an undergraduate could read these more difficult papers at once or easily, but he might certainly get ideas which would help him outline his graduate work intelligently.

Lest possible misunderstanding arise, it should be stressed that the articles mentioned in the last paragraph are not given in this bibliography; they are merely samples of what a student who is familiar with the journals will find for

himself. The present list is restricted to papers whose subject matter is connected with a first course in the theory of equations. Only about half a dozen such papers have been omitted, and these seemed either misleading or of no possible value.

To save space, the different periodicals are represented by numerals, the American Journal of Mathematics by 1, the first and second series of the Annals of Mathematics by 2 and 3 respectively, the first and second series of the Bulletin of the American Mathematical Society by 4 and 5 respectively,[1] the Transactions of the American Mathematical Society by 6, and the American Mathematical Monthly by 7. Immediately after each title in the list appear, in order, the periodical number, the volume, the year of publication, and the paging. The brief comments which then follow, in certain cases, are designed merely to tell something of the nature of the article, when the title does not do this sufficiently, or to point out that part of the article may be difficult. They are not at all to indicate the more valuable papers. Finally, the first 19 volumes of the Monthly, through 1912, were not available to the writer of this paper. However, it is probably true that many libraries will not have these first volumes.

I. *Historical and General*[2]

1. Pierpont, Early history of Galois' theory of equations, 5, 4, 1898, 332–40. Largely a life of Galois.
2. White, Bézout's theory of resultants and its influence on geometry, 5, 15, 1909, 325–38.
3. Cajori, Horner's method of approximation anticipated by Ruffini, 5, 17, 1911, 409–14.
4. Miller, Definitions of the discriminant of a rational integral function of one variable, 7, 25, 1918, 287–90.
5. McClenon, A contribution of Leibniz to the history of complex numbers, 7, 30, 1923, 369–74. Remarks on Cardan's formulas for the cubic.
6. Miller, On the history of determinants, 7, 37, 1930, 216–19.
7. Funkhouser, A short account of the history of symmetric functions of the roots of equations, 7, 37, 1930, 357–65.

II. *Properties of Roots of Equations*

8. Van Vleck, On the location of roots of polynomials and entire functions, 5, 35, 1929, 643–83. Has a bibliography of 144 titles, of which 113 are on polynomials. Of these, 13 are listed in this paper, and 8 others appearing in American journals were examined but not included on account of their difficulty.

9.* Bôcher, Gauss's third proof of the fundamental theorem of algebra, 5, 1, 1895, 205–09.

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[1] Series 1 (1892–4) was the Bulletin of the New York Mathematical Society.

[2] Under the main headings the grouping is partly chronological and partly by similar topics. If an article could appear under more than one heading a more or less arbitrary choice is made.

10.* Bôcher, Simplification of Gauss's third proof that every algebraic equation has a root, 1, 17, 1895, 266–68.
11.* Sheffer, A proof of the fundamental theorem of algebra, 5, 35, 1929, 227–30.
12. Coolidge, The continuity of the roots of an algebraic equation, 3, 9, 1908, 116–18.
13. Van Vleck, A sufficient condition for the maximum number of imaginary roots of an equation of the nth degree, 3, 4, 1903, 191–2. Condition is that a set of $n+1$ determinants are to be positive.
14. Kellogg, A necessary condition that all the roots of an algebraic equation be real, 3, 9, 1908, 97–98. Certain determinants must be negative.
15. Dunkel, Sufficient conditions for imaginary roots of algebraic equations, 3, 10, 1908, 46–54. A number of tests follow from a fundamental theorem
16. Dunkel, Generalized geometric means and algebraic equations, 3, 11, 1909, 21–32. An extension of paper 15 on conditions for imaginary roots. Also has a section on approximating roots.
17. Johnston, Real roots of a class of reciprocal equations, 7, 39, 1932, 415–18.
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[1] Owing to incorrect paging, this is the second page 64 in vol. 6.

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VII. *Miscellaneous*

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117.* Sayre, The solution of algebraic equations by partial differential equations, 3, 10, 1909, 116–22.
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I may also add that papers of my own in the Monthly, vols. 34–39, and in the Annals, vol. 29, may be of some interest. Of the 121 papers listed, 65 are from the Monthly, 29 from the Annals, 20 from the Bulletin, 6 from the American Journal and 1 from the Transactions. A number of solved problems in the Monthly are also worth consulting, among them vol. 22, pages 132 and 230, vol. 23, page 254, 24–131 and 232, 25–124 and 218, 26–291, 32–266, 33–229, 36–106, 37–317 and 554, 39–432.

CURVES DETERMINED BY A ONE-PARAMETER FAMILY OF TRIANGLES

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Some years ago the author published a paper[1] setting forth certain loci associated with a triangle which had its circumcircle and nine-point circle fixed. The results were obtained by the methods of elementary synthetic and analytic geometry. Later Prof. Murnaghan showed[2] that absolute coordinates could be used to good advantage in the proofs of these same properties. In the present paper absolute coordinates are used to prove some of the results in the paper cited above, and also to develop some new theorems relating to the same configuration.

In this discussion $A_i (i=1, 2, 3)$ will denote the vertices of the triangle, H the orthocenter and O the circumcenter. Let O be the origin. Then if the circumcircle is taken as the unit circle, the points A_i may be represented by $z_i = t_i$, where t_i is a turn, i.e., $|t_i| = 1$. Let $\bar{z}$ be the conjugate of z. Then without any difficulty it may be shown that the map equation of a straight line through the extremities of the vectors z_1 and z_2 may be written[3]

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) = z_2\bar{z}_1 - z_1\bar{z}_2. \tag{1}$$

Using this notation the following equations may be written. The side A_iA_j of the triangle

$$z + \bar{z}t_it_j = t_i + t_j. \tag{2}$$

In all the equations that follow, $i=1, 2, 3, j=1, 2, 3, k=1, 2, 3, i \neq j \neq k$. Altitude from the vertex A_k,

$$z - \bar{z}t_it_j = (t_k^2 - t_it_j)/t_k. \tag{3}$$

Tangent to the circumcircle at A_i

$$z + \bar{z}t_i^2 = 2t_i. \tag{4}$$

For a proper choice of the square root, the equation of the internal bisector of the angle A_i

$$z + \bar{z}t_i\sqrt{t_jt_k} = t_i + \sqrt{t_jt_k}. \tag{5}$$

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[1] *A System of Triangles Related to a Poristic System*, American Mathematical Monthly, vol. 31, 1924, p. 337.

[2] *Notes on Mr. Weaver's Paper*, American Mathematical Monthly, vol. 32, 1925, p. 37.

[3] In this connection see F. Morley, *Metric Geometry of the Plane n-line*, Transactions of the American Mathematical Society, vol. 1, 1900, p. 97: F. D. Murnaghan, loc. cit.: O. J. Ramler, *Loci of Some One-Parameter Systems of Lines*, American Mathematical Monthly, vol. 37, 1930, p. 130: H. A. DoBell, *On The Geometry of the Triangle*, American Mathematical Monthly, vol. 39, 1932, p. 71: A. R. Forsythe, *Complex Variables in Plane Geometry*, Quarterly Journal of Mathematics, vol. 42, 1911, p. 1.

For the same choice of square root as in (5) the equation of the external bisector of A_i

$$z - \bar{z}t_i\sqrt{t_jt_k} = t_i - \sqrt{t_jt_k}. \tag{6}$$

The polar of a point Z_1 with respect to the circumcircle

$$z\bar{z}_1 + \bar{z}z_1 = 2. \tag{7}$$

Side of the medial triangle parallel to A_iA_j[4]

$$z + \bar{z}t_it_j = (t_k^2 + \sigma_2)/(2t_k). \tag{8}$$

The line in which the external bisectors of the angles of the tangential triangle meet the opposite sides

$$z\sigma_2/\sigma_3 + \bar{z}\sigma_1 = 6. \tag{9}$$

Polar line of the orthocenter of the tangential triangle with respect to that triangle

$$z(\sigma_1^2 - 2\sigma_2) + \bar{z}(\sigma_2^2 - 2\sigma_1\sigma_3) = \sigma_1\sigma_2 - 3\sigma_3. \tag{10}$$

Points may be represented by vectors as follows:
Orthocenter

$$z = \sigma_1. \tag{11}$$

Foot of the altitude on A_iA_j

$$z = \tfrac{1}{2}(\sigma_1 - \sigma_3/t_k^2). \tag{12}$$

Mid-point of A_iA_j

$$z = \tfrac{1}{2}(t_i + t_j). \tag{13}$$

Intersection of the tangents to the circumcircle at the points A_i and A_j

$$z = 2t_it_j/(t_i + t_j). \tag{14}$$

Reflection of 0 in A_iA_j

$$z = t_i + t_j. \tag{15}$$

Symmedian point

$$z = \frac{2(\sigma_2^2 - 3\sigma_1\sigma_3)}{\sigma_1\sigma_2 - 9\sigma_3}. \tag{16}$$

Circumcenter of the tangential triangle

$$z = \frac{2\sigma_1\sigma_3}{\sigma_1\sigma_2 - \sigma_3}. \tag{17}$$

[4] The σ_i are the ordinary symmetric functions of the three t's, i.e., the t's are the roots of the equation $t^3 - \sigma_1t^2 + \sigma_2t - \sigma_3 = 0$.

Orthocenter of the tangential triangle

$$z = \frac{2(\sigma_2^2 - \sigma_1\sigma_3)}{\sigma_1\sigma_2 - \sigma_3}. \tag{18}$$

Nine-point center of the tangential triangle

$$z = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3}. \tag{19}$$

Incenter with proper choice of square roots

$$z = \sqrt{t_1 t_2} + \sqrt{t_2 t_3} + \sqrt{t_1 t_3}. \tag{20}$$

Gergonne point of the tangential triangle

$$z = \frac{2(\sigma_2^2 - 2\sigma_1\sigma_3)}{\sigma_1\sigma_2 - 9\sigma_3}. \tag{21}$$

Nagel point of the tangential triangle

$$z = \frac{2(\sigma_2^2 + \sigma_1\sigma_3)}{\sigma_1\sigma_2 - \sigma_3}. \tag{22}$$

Certain Loci. Let us now assume that the orthocenter as well as the circumcircle remains fixed. We then have from (11) $\sigma_1 = c$ (c a constant) or $t_i + t_j = c - t_k$. If the line OH is chosen as the axis of reals then $c = \bar{c}$. The equation $\sigma_1 = c$ implies the conjugate equation $\sigma_2/\sigma_3 = c$. Hence we have the two equations

$$t_i + t_j = c - t_k \text{ and } t_i t_j = \frac{t_k(c - t_k)}{ct_k - 1}. \tag{23}$$

Equations (23) show that t_i and t_j depend on t_k and are uniquely determined when t_k is chosen. It should also be noted that σ_3 is a variable turn since the product of any number of turns is another turn.

Equation (12) may now be written, by virtue of (23)

$$z = c/2 - \sigma_3/2t_k^2$$

which shows that the feet of the altitudes traverse a circle, namely the nine-point circle which is fixed in this system.

Equation (13) becomes

$$z = c/2 - t_k/2$$

which also represents the nine-point circle.

It may be shown in a similar fashion that the midpoint of the segment from the orthocenter to a vertex traverses the nine-point circle. These results are well known and are merely set down here to illustrate the method.

Equation (14) by virtue of (23) becomes

$$z = t_i/(ct_k - 1)$$

which is a circle unless $c = \pm 1$, when the locus is the tangent to the circumcircle at the orthocenter, which is on the circumcircle, together with the line at infinity. Hence in this system the tangential triangle is inscribed in a circle unless the orthocenter lies on the circumcircle, in which case the vertices of the tangential triangle lie on the tangent to the circumcircle at the orthocenter.

It should be noted that the case $c = \pm 1$ can occur only when the triangle $A_1A_2A_3$ is a right triangle, with $t_i = -t_j$ and $t_k = c = \pm 1$.

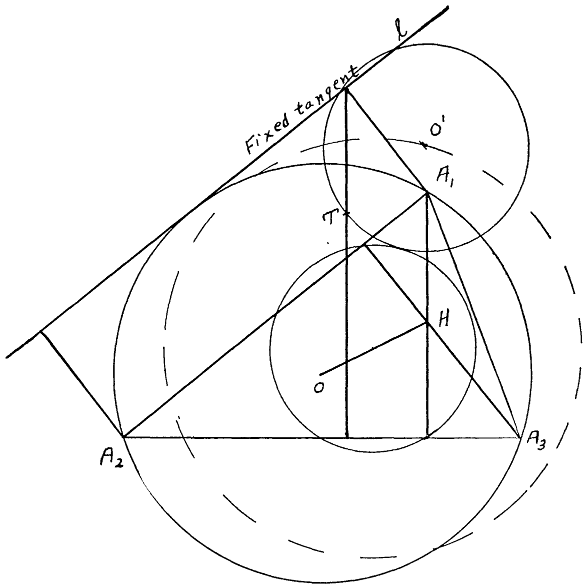

FIG. 1

The orthopole of a fixed tangent to the circumcircle is given by the equation[5]

$$z = \tfrac{1}{2}(\sigma_1 + 2t + \sigma_3/t^2) \quad \text{(where } t \text{ is a fixed vector).}$$

This is of the form $z = a + bt$ and is therefore a circle with the center at the point $(c+2t)/2$ and with radius $1/2$. Hence the locus of the orthopole of a fixed tangent to the circumcircle of the system is a circle equal to the nine-point circle. The center of this circle lies on a circle concentric with the nine-point circle and which has a radius equal to the radius of the circumcircle. We therefore conclude that the locus is also tangent to the nine-point circle (See Fig. 1, where l is the fixed tangent and the circle O' is the locus of the orthopole T of l).

Consider the point $P = t_0$. The equation of the line OP is

$$z - \bar{z}t_0^2 = 0.$$

[5] O. J. Ramler, loc. cit.

The equation of a tangent to the nine-point circle perpendicular to OP is

$$z + \bar{z}t_0^2 = (c + ct_0^2 - 2t_0)/2.$$

The intersection of these two lines is the point

$$z = (c + ct_0^2 - 2t_0)/4. \tag{24}$$

If t_0 is allowed to vary, this point traverses a limacon.[6] Let us now write

$$z = (c + ctt_0 - t - t_0)/4 \tag{25}$$

and let t be fixed. Equation (25) then represents a circle. For $t = t_0$ this circle has a point in common with the limacon (24). Moreover the limacon and circle are tangent at this point. For if we differentiate (24) and (25) with respect to t_0 we obtain from (24)

$$dz/dt_0 = (ct_0 - 1)/2$$

and from (25)

$$dz/dt_0 = (ct_0 - 1)/4.$$

These two derivatives represent vectors parallel to the two curves at the point $t = t_0$, and since these vectors are parallel the tangents must coincide. If in (25) t is considered as a parameter, we have a family of circles. The centers of the circles of this family will lie on the circle $z_1 = (c - t)/4$ and the radii of the circles of the family will be equal to the absolute value of $(ct - 1)/4$ which is the same as the absolute value of $(c - t)/4$. Hence all the circles of the family pass through the circumcenter of the family of triangles, and the limacon (24) is the envelope of the family of circles (25). The common point of the family of circles is the node of the limacon.

Equation (16) tells us that the locus of the symmedian point is a circle unless $c = \pm 3$ when the locus is at infinity. Equation (15) shows that the locus of the reflection of O in A_iA_j is a circle equal to the circumcircle and having its center at the orthocenter. Equation (17) states that the circumcenter of the tangential triangle is a fixed point which is at infinity if $c = \pm 1$. Equations (18), (19), (21), and (22) show that the respective loci are circles, while (20) leads us to a quartic. The proofs for these statements are simple and follow the same lines as those given above.[7]

Certain envelopes. Equation (2) by virtue of (23) becomes

$$z + \frac{\bar{z}t_k(c - t_k)}{ct_k - 1} = c - t_k. \tag{26}$$

Partial differentiation of (26) with respect to t_k gives for its envelope

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[6] F. Morley, loc. cit.

[7] Equations 17–22 were developed by Miss Olive Givin, a student of the author.

$$\bar{z} = \frac{(ct_k - 1)^2}{ct_k^2 + c - 2t_k} \tag{27}$$

which is in general a conic, an ellipse if $c < 1$, a hyperbola if $c > 1$, and indeterminate if $c = 1$. The foci are the circumcenter and the orthocenter.

Now write

$$\bar{z} = \frac{(ct_k - 1)(ct - 1)}{ctt_k + c - t_k - t} \quad \text{(where } t \text{ is fixed)}. \tag{28}$$

This is a straight line and considerations such as those applied to equations (24) and (25) show that (28) represents, for t as a parameter, the family of tangents to the conic (27). It may be shown in a similar fashion that the envelope of the medial triangle of the system is a conic.

Consider now a fixed point, $P = t$, on the circumcircle. The Simson Line of this point is

$$tz + \bar{z}\sigma_3 = \frac{t^3 + t^2\sigma_1 - t\sigma_2 - \sigma_3}{2t}.$$

The partial derivative of this with respect to t_k gives

$$z = (c - t)/2.$$

Hence the envelope of the Simson Lines of a fixed point on the circumcircle is a fixed point on the nine-point circle, and the radius of the nine-point circle to this point is parallel to the radius of the circumcircle to the given point.

From (13) we have that the midpoint of A_iA_j is

$$z = (c - t_k)/2.$$

The polar of this point with respect to the circumcircle is, because of (7),

$$\frac{z(ct_k - 1)}{t_k(c - t_k)} + \bar{z} = \frac{4}{c - t_k}.$$

The partial derivative of this equation with respect to t_k will determine the envelope of polars which is

$$z = \frac{4t_k^2}{ct_k^2 + c - 2t_k}.$$

This is the equation of an ellipse if $c < 1$, a parabola if $c = 1$, and a hyperbola if $c > 1$.

The vertices of the conic, for $c \neq 1$, are $2/(c+1)$ and $2/(c-1)$. The foci are the circumcenter and the point $4c/(c^2 - 1)$. In the case of the parabola the vertex coincides with the orthocenter, which coincides with the vertex t_k, of the triangle, and the focus is the circumcenter.

Consider the external bisectors of the angles of the tangential triangle. These meet the opposite sides of this triangle in the points of the line

$$cz + c\bar{z} = 6$$

which is a fixed line. Hence the locus of the intersections of the external bisectors of the angles of the tangential triangle with the opposite sides is a fixed line.

Numerous other loci and envelopes may be determined by this method. For example, no mention is made here of the points and lines of the anti-medial triangle, and the Brocard configuration has been left untouched. However, it is hoped that enough examples have been given to show the beauty and power of the method.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A GEOMETRIC PARADOX—PROPOSED BY C. H. ROWE[1]

REPLIES BY J. L. COOLIDGE AND B. Z. LINFIELD

The Paradox

Consider the quadric surface that is represented by the general equation of the second degree in four homogeneous coordinates. It is well known that we must impose three independent conditions on the coefficients in order that the quadric should reduce to a pair of planes. However the following argument seems to show that two conditions are sufficient.

By imposing the condition that the discriminant of the equation should vanish we ensure that the quadric is degenerate, and that we can transform to a new system of homogeneous coordinates so that the transformed equation contains only three of the four variables. One further condition is sufficient to ensure that this transformed equation represents a pair of planes. The total number of conditions that we need is therefore two.

1. *A Reply by J. L. Coolidge, Harvard University.*

Professor Rowe's geometric paradox may be explained as follows. A quadric surface consists of two planes if the rank of the discriminant matrix is exactly two. That imposes on the coefficients three distinct equations of condition and certain inequalities besides. If the discriminant determinant is 0, the problem

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[1] This paradox was proposed by Professor Rowe in the June-July issue of this MONTHLY, vol. 39 (1932), p. 352.

of finding the vertex of the supposed cone is determinate only if you rule out the case desired, by assuming the rank to be three and not two or else assume some extraneous condition. The total number of conditions is three as before.

2. *A Reply by B. Z. Linfield, University of Virginia.*

Professor Rowe's recent geometric paradox can be explained by the elementary fact that

The n^2 conditions for the square matrix A of order n to be zero, i.e. for $A_{ij}=0$ $(i, j=1, 2, \cdots, n)$, are equivalent to the single condition that the quadratic function in x

$$x\cdot A\cdot x \equiv \sum_{i,j=1}^{n} A_{ij}x_i x_j$$

be identically zero for all values of x.[1]

For, let the homogeneous equation of the quadric be

$$x\cdot A\cdot x \equiv \sum_{i,j=1}^{4} A_{ij}x_i x_j = 0 \quad (A_{ij} = A_{ji}, \text{ i.e. } A = trnA).$$

If its discriminant be zero, i.e. det $A=0$, then (on expressing Professor Rowe's geometric argument in vector-matric equations) there exists a square matrix B of order 4, (det $B\neq 0$) such that under the linear transformation $x=y\cdot B$, we get

$$x\cdot A\cdot x = y\cdot BAB'\cdot y \quad (B' \equiv trnB),$$

and the symmetric matrix BAB' has its 4th row and column all zeros, i.e.

$$BAB' = \begin{Vmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Therefore the quadratic function $y\cdot BAB'\cdot y$ will then not contain y_4, and it will be factorable (and therefore $x\cdot A\cdot x$ will be factorable) if only

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[1] Just as the n conditions that the vector $u=(u_1, u_2, \cdots, u_n)$ be zero, i.e. that $u_1=u_2=\cdots=u_n=0$, are equivalent to the single condition that the scalar product

$$u\cdot x \equiv u_1x_1 + u_2x_2 + \cdots + u_nx_n$$

be identically zero for all values of the vector x.

The notation used here is that if A is a rectangular matrix of n rows and m columns, then $A_1, A_2, A_3, \cdots$ designate its rows (vectors), and

$$\begin{aligned} x\cdot A &\equiv A\cdot x = x_1A_1 + x_2A_2 + \cdots + x_nA_n, \\ x\cdot A_k &\equiv A_k\cdot x = A_{k1}x_1 + A_{k2}x_2 + \cdots + A_{km}x_m, \\ x_iA_j &\equiv A_jx_i = (A_{j1}x_i, A_{j2}x_i, \cdots, A_{jm}x_i), \end{aligned}$$

and thus $x\cdot A$ is a vector linearly dependent on the vectors (rows) of the matrix A.

$$\det C \equiv \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = 0.$$

Hence the *two* conditions det $A=0$ and det $C=0$ are sufficient conditions that the quadric $x \cdot A \cdot x = 0$ degenerate into planes, while, as Professor Rowe points out, "it is well known that we must impose three independent conditions on the coefficients in order that the quadric should reduce to a pair of planes."

In fact, in order that det $C=0$ we must have rank $BAB' \leqq 2$. And, since the ranks of the matrices BAB' and A are equal, we must have rank $A \leqq 2$, i.e. adj $A=0$, and this alone involves three independent conditions (implying however det $A=0$).

The explanation of this difference in the number of conditions required that the quadric should degenerate into planes lies in the fact that det C is not a function of A alone. That, on the contrary,

$$\det C = z \cdot adjA \cdot z,$$

where the vector z is arbitrary. And that therefore the condition det $C=0$ im plies the identity $z \cdot \text{adj } A \cdot z = 0$ for all values of z which we have already seen means adj $A=0$.

For, to find the matrix B in the transformation $x = y \cdot B$, we have, since det $A=0$, that the four linear homogeneous equations in $x \cdot A = 0$ have a solution $b \cdot A = 0$ and $b \neq 0$. Now for *any* matrix D of the form

$$D \equiv \begin{Vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \end{Vmatrix},$$

and

$$B \equiv \begin{Vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ b_1 & b_2 & b_3 & b_4 \end{Vmatrix},$$

we have, no matter what be the values of the terms in D, that the product

$$BAB' = \begin{Vmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \quad DAD' \equiv C = \begin{Vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{Vmatrix},$$

and

$$\det C = | DAD' | \equiv | D | \cdot adjA \cdot | D | ,$$

where $|D|$ is the vector orthogonal to each of the vectors (rows) D_1, D_2, D_3 of D, and of magnitude $|DD'|^{1/2}$. Hence, since the terms in the matrix D are arbitrary, (except that det $B \neq 0$), the value of the vector $|D|$ is perfectly arbitrary. And the only way det C can be zero for all values of $|D|$ is for adj $A = 0$.

The identity $|DAD'| \equiv |D| \cdot \text{adj } A \cdot |D|$ is independent of the order of the square matrix A, and can be used to explain analogous paradoxes for conics or hyperconicoids.

Thus, for the conic

$$x \cdot A \cdot x \equiv \sum_{ij=1}^{3} A_{ij} x_i x_j = 0$$

it is well known that its coefficients must satisfy three independent conditions in order that it shall represent one line, i.e. in order that $x \cdot A \cdot x$ should be a perfect square. However, if we impose the one condition that det $A \equiv |A| = 0$, we can find a transformation

$$x = y \cdot B, \text{ where } B_3 \cdot A = 0, \text{ and } | B | \neq 0,$$

such that

$$x \cdot A \cdot x \equiv y \cdot BAB' \cdot y,$$

and the quadratic function $y \cdot BAB' \cdot y$ will not contain y_3, i.e. the matrix

$$BAB' = \left\| \begin{matrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 0 \end{matrix} \right\| .$$

Then the second condition,

$$| C | \equiv \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} = 0,$$

is sufficient to make $y \cdot BAB' \cdot y$ a perfect square, i.e. sufficient to insure the existence of a vector d such that $y \cdot BAB' \cdot y = (d \cdot y)^2$, and therefore $x \cdot A \cdot x = [(d \cdot B'^{-1}) \cdot x]^2$. Hence two conditions (det $A = 0$ and det $C = 0$) are sufficient to make $x \cdot A \cdot x$ a perfect square!

Obviously the explanation lies in the fact that the matrix D of the first two vectors (rows) of B is arbitrary (except that $|B| \neq 0$), that

$$C = DAD' \text{ and } | C | = | D | \cdot adjA \cdot | D | ,$$

and therefore $|C| = 0$ requires adj $A = 0$. (This vector $|D|$ is sometimes called the "vector product" of D_1 by D_2 and designated by $D_1 \times D_2$, which is here the same as $B_1 \times B_2$ because $D_1 = B_1$ and $D_2 = B_2$.)

On the other hand, here is a paradox à-la-Coolidge about conics requiring a different explanation.

It is well known that *two* independent conditions must be imposed on a (real) central conic

$$F(x) \equiv A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2 + 2fx_1 + 2gx_2 + c = 0$$

in order that it should become a circle (namely $A_{12}=0$, $A_{11}=A_{22}$). Still, we can find a linear transformation $x=u+y\cdot B$ such that

$$F(x) \equiv t_1y_1^2 + t_2y_2^2 + F(u),$$

where t_1 and t_2 are the roots of the characteristic equation

$$\left| A - t\delta \right| \equiv \begin{vmatrix} A_{11} - t & A_{12} \\ A_{21} & A_{22} - t \end{vmatrix} = 0, \text{ trn } A = A,$$

of the matrix A. Then the *single* condition $t_1=t_2$ is sufficient to make this conic a circle!

Moreover, t_1 and t_2 here are clearly functions of the coefficients of $F(x)$ only; and therefore a single condition upon t_1 and t_2 is a single condition upon the coefficients of $F(x)$!

The explanation here lies in the fact that the condition that the roots of the characteristic equation

$$\left| A - t\delta \right| \equiv t^2 - (A_{11} + A_{22})t + \left| A \right| = 0$$

be equal is

$$(A_{11} + A_{22})^2 - 4\left| A \right| \equiv (A_{11} - A_{22})^2 + 4A_{12}^2 = 0 \quad (A_{21} = A_{12}),$$

which, because of the reality of the coefficients of $F(x)$, amounts to the two conditions $A_{11}-A_{22}=A_{12}=0$.

Similarly, it is well known that *five* independent conditions must be imposed on a (real) central quadric

$$F(x) \equiv x\cdot A\cdot x + 2b\cdot x + c = 0$$

in order that it should become a sphere, namely

$$A_{12} = A_{23} = A_{31} = 0 \text{ and } A_{11} = A_{22} = A_{33}.$$

Still, we can find a transformation (translation and rotation) $x=u+y\cdot B$ such that

$$F(x) \equiv t_1y_1^2 + t_2y_2^2 + t_3y_3^2 + F(u),$$

where t_1, t_2, t_3, are the roots of the characteristic equation

$$\left|A - t\delta\right| \equiv -\left(t^3 - t^2 \operatorname{div} A + t \operatorname{div}^2 A - \left| A \right|\right) = 0,$$

and

$$\operatorname{div} A = A_{11} + A_{22} + A_{33}, \quad \operatorname{div}^2 A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}.$$

And then the *two* conditions $t_1 = t_2 = t_3$ are sufficient to make this quadric a sphere!

In fact, the *single* condition $(\operatorname{div} A)^2 - 3 \operatorname{div}^2 A = 0$ is sufficient to make this quadric a sphere, for

$$(\operatorname{div} A)^2 - 3\operatorname{div}^2 A = \frac{(A_{11} - A_{22})^2 + (A_{22} - A_{33})^2 + (A_{33} - A_{11})^2}{2} + 3(A_{12}^2 + A_{23}^2 + A_{31}^2).$$

More generally, the n-dimensional (real) central hyperquadric

$$F(x) \equiv x \cdot A \cdot x + 2b \cdot x + c = 0 \quad (A = \operatorname{trn} A)$$

is a hypersphere if the coefficients of quadratic terms satisfy the *single* condition

$$(n - 1)(\operatorname{div} A)^2 - 2n \operatorname{div}^2 A = 0, \ (|A| \neq 0),$$

where

$$|t\delta - A| \equiv t^n - t^{n-1} \operatorname{div} A + t^{n-2} \operatorname{div}^2 A - \cdots + (-1)^n |A|,$$ [1]

i.e.

$$\operatorname{div} A = A_{11} + A_{22} + \cdots + A_{nn}, \ \operatorname{div}^2 A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \cdots + \begin{vmatrix} A_{n-1,n-1} & A_{n-1,n} \\ A_{n,n-1} & A_{nn} \end{vmatrix}.$$

Obviously it is equivalent to $(n+2)(n-1)/2$ independent conditions, as

$$(n - 1)(\operatorname{div} A)^2 - 2n \operatorname{div}^2 A = (A_{11} - A_{22})^2 + \cdots + (A_{n-1,n-1} - A_{nn})^2 + + 2n(A_{12}^2 + \cdots + A_{n-1,n}^2).$$

To use only the simplest properties of matrices on the original paradox by Prof. Coolidge[2] we have first that the three points

$$u = (u_1, u_2, u_3) \quad v = (v_1, v_2, v_3) \quad w = (w_1, w_2, w_3)$$

are collinear if, and only if, the rank of the matrix

$$M \equiv (u - v, u - w) \equiv \begin{Vmatrix} u_1 - v_1 & u_2 - v_2 & u_3 - v_3 \\ u_1 - w_1 & u_2 - w_2 & u_3 - w_3 \end{Vmatrix}$$

is one, which imposes two independent conditions on u, v, w. On the other hand, argues Prof. Coolidge (geometrically) the single condition

[1] Each coefficient of t is a Euclidean invariant of $F(x) = 0$.

[2] This MONTHLY, vol. 38, page 222.

$$\sqrt{(u-v)\cdot(u-v)}+\sqrt{(u-w)\cdot(u-w)}=\sqrt{(v-w)\cdot(v-w)}$$

is sufficient to make these three points collinear. But, squaring both sides of the last equation, we get

$$\begin{aligned}&2\sqrt{(u-v)\cdot(u-v)}\sqrt{(u-w)\cdot(u-w)}=\\&(v-w)\cdot(v-w)-(u-v)\cdot(u-v)-(u-w)\cdot(u-w)\equiv\\&\qquad-2(u-v)\cdot(u-w),\end{aligned}$$

by the law of cosines for the triangle u, v, w. Squaring again, we get

$$\begin{vmatrix}(u-v)\cdot(u-v) & (u-v)\cdot(u-w)\\(u-v)\cdot(u-w) & (u-w)\cdot(u-w)\end{vmatrix}$$
$$\equiv |MM'| \equiv |M|\cdot|M| = 0,^1 \quad (M' = \operatorname{trn} M),$$

where $|M|$ is the vector orthogonal to $u-v$ and $u-w$ and of magnitude $|MM'|^{1/2}$ [sometimes spoken of as the vector product of $u-v$ by $u-w$ and designated by $(u-v)\times(u-w)$]. The matrix M being real, the condition $|M|\cdot|M|=0$ (being on the left a sum of squares) requires that each term in $|M|$ be zero, which is exactly the condition that rank $M=1$.

In n-dimensional space $(n>3)$ Professor Coolidge's single condition yields, as before, $|MM'|=0$ $(M_1=u-v,\ M_2=u-w)$. And, since

$$|MM'| \equiv \begin{vmatrix}u_1-v_1 & u_2-v_2\\u_1-w_1 & u_2-w_2\end{vmatrix}^2+\begin{vmatrix}u_1-v_1 & u_3-v_3\\u_1-w_1 & u_3-w_3\end{vmatrix}^2$$
$$+\cdots+\begin{vmatrix}u_{n-1}-v_{n-1} & u_n-v_n\\u_{n-1}-w_{n-1} & u_n-w_n\end{vmatrix}^2,$$

the condition $|MM'|=0$ is equivalent to rank $M=1$.

A Remark on Fourier Series of Continuous Functions

By William Randels, Brown University

In studying Fourier series Riemann proved that the Fourier coefficients of an integrable function converged to zero as n increased. This theorem was later proved for Lebesgue integrable functions by Lebesgue. The next question along this line is: can anything be said about the rate of decrease of the coefficients? Hobson[2] refers to Lebesgue[3] for a proof that the theorem of Riemann-Lebesgue cannot be made more precise if one only knows that the function is continuous. Lebesgue's method is, to show that, given any function $u(n)$ such that $\lim_{n\to\infty} u(n)=0$, one may construct a step function with an infinite number of

[1] $|MM'|=|M\delta M'|=|M|\cdot adj\ \delta\cdot|M|=|M|\cdot|M|$, as $adj\ \delta=\delta$ and $x\cdot\delta=x$.

[2] Hobson, *Theory of Functions of a Real Variable*, 2nd ed., vol. 2, p. 514.

[3] Bulletin de la Société Mathématique de France, vol. 38, 1910, p. 189.

steps, and taking on only the values $+1$ and -1, whose Fourier coefficients are not of order greater than that of $u(n)$. Then he says: "By an arbitrarily small change . . . one may make $f(x)$ continuous and one will have shown that the theorem of Riemann on the decrease to zero of the Fourier coefficients of a continuous function may not be made more precise if one only knows that the function is continuous." However as the step function has an infinite number of steps, the points of discontinuity will have a limit point and the function will have a point of discontinuity of the second kind which cannot be removed by an arbitrarily small change in the function. The purpose of this note is to give a method of construction which will produce a continuous function with the desired properties.

We need the following lemma which will not be proven.

$$\lim_{n\to\infty}\int_a^b |\cos nx|\,dx = \frac{2}{\pi}(b-a). \tag{1}$$

A set of quantities l_i is defined by the relations

$$l_i = 4\pi\sum_{n=1}^{i}5^{-n}.$$

It is obvious that $\lim_{i\to\infty} l_i=\pi$ and that $\pi-l_i=\pi/5^i$. A function $f(x)$ is assumed to be defined and continuous on the interval $(0, l_{i-1})$ and is to be defined on the interval (l_{i-1}, l_i). Then n_i is chosen so large that:

$$\left|\int_0^{l_{i-1}} f(x)\cos n_i x\,dx\right| \leqq 2\cdot 5^{-2i} \tag{2}$$

$$\int_{l_{i-1}}^{l_i} |\cos n_i x|\,dx > \frac{7}{4\pi}(l_i - l_{i-1}) = 7\cdot 5^{-i} \tag{3}$$

$$|u(n_i)| \leqq \tfrac{1}{3}\cdot 5^{-2i}. \tag{4}$$

The zeros of $\cos n_i x$ in the open interval (l_{i-1}, l_i) are denoted by α_{ij} $(j=1, 2, \cdots, m_i)$. Let $\delta_i=\min\,(5^{-i}/2(m_i+1),\ \alpha_{i1}-l_{i-1}/2,\ |\alpha_{im_i}-l_i|/2)$. Then $f(x)$ is defined as follows

$$f(x)=\begin{cases}\dfrac{5^{-i}}{\delta_i}\,\mathrm{sign}\cos n_i(l_{i-1}+\delta_i)(x-l_{i-1}) & (l_{i-1}\leqq x\leqq l_{i-1}+\delta_i)\\[2ex] \dfrac{5^{-i}}{\delta_i}\,\mathrm{sign}\cos n_i(l_i-\delta_i)(l_i-x) & (l_i-\delta_i\leqq x\leqq l_i)\\[2ex] \dfrac{5^{-i}}{\delta_i}\,\mathrm{sign}\cos n_i(\alpha_{ij}+\delta_i)(x-\alpha_{ij}) & (\alpha_{ij}-\delta_i\leqq x\leqq \alpha_{ij}+\delta_i)\\[2ex] 5^{-i}\,\mathrm{sign}\cos n_i x & \text{elsewhere on } (l_{i-1}, l_i).\end{cases}$$

As $f(x)$ is continuous on each one of the intervals (l_{i-1}, l_i) the only possible

point of discontinuity is the point π; but for every ϵ a neighborhood of the point π may be chosen such that, if x is contained in this neighborhood, $|f(x)| \leqq \epsilon$. So $f(x)$ is continuous everywhere on $(0, \pi)$. Then $f(x)$ is defined on $(-\pi, 0)$ by the relation $f(-x) = f(x)$. The n_ith Fourier coefficient of $f(x)$ is then

$$a_{n_i} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos n_i x dx = \frac{2}{\pi}\int_0^{\pi} f(x) \cos n_i x dx$$

$$= \frac{2}{\pi}\left(\int_0^{l_{i-1}} f(x) \cos n_i x dx + \int_{l_{i-1}}^{l_i} f(x) \cos n_i x dx + \int_{l_i}^{\pi} f(x) \cos n_i x dx\right);$$

but we know that

$$\left|\int_0^{l_{i-1}} f(x) \cos n_i x dx\right| \leqq 2\cdot 5^{-2i}$$

and

$$\left|\int_{l_{i-1}}^{l_i} f(x) \cos n_i x dx\right| \geqq 5^i \int_{l_{i-1}}^{l_i} |\cos n_i x|\, dx - 2\cdot 5^{-i}\delta_i(m_i + 1)$$

since $f(x) = 5^{-i}$ sign $\cos n_i x$ on (l_{i-1}, l_i) except when x is within a distance δ_i of l_{i+1}, l_i or α_{ij}, where it differs from 5^{-i} sign $\cos n_i x$ by at most 5^{-i}. Also, since $|f(x)| < 5^{-i}$ on (l_i, π)

$$\left|\int_{l_i}^{\pi} f(x) \cos n_i x dx\right| < 5^{-i}(\pi - l_i) = \pi 5^{-2i}.$$

Hence

$$|a_{n_i}| > \frac{2}{\pi}(7\cdot 5^{-2i} - 5^{-2i} - 2\cdot 5^{-2i} - \pi 5^{-2i}) > \frac{5^{-2i}}{3},$$

and, by (4),

$$|a_{n_i}| > |u(n_i)| .$$

This shows that for an infinite number of values of n, a_n is greater than $u(n)$ and as $u(n)$ was perfectly arbitrary except for the condition $\lim_{n\to\infty} u(n) = 0$, this shows that the Riemann-Lebesgue theorem furnishes the best possible estimate of the order of Fourier coefficients if we only know that a function is continuous.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Foundations of Point Set Theory. (American Mathematical Society Colloquium Publications, Volume XIII.) By R. L. Moore. New York, The American Mathematical Society, 1932. vii+486 pages. \$5.00 (\$4.50 to members).

The book is based on a set of axioms for a space S, which make that space topologically equivalent to an ordinary two-dimensional sphere or to a plane according as S is or is not compact. As the axioms are introduced, one or two at a time, the consequences of the new axiom or axioms together with some of the old ones are developed. As a result, theorems in the early chapters apply to very wide classes of spaces, including, for example, Hilbert-space in Chapter II.

The thorough-going investigation of the properties considered is due largely to Moore and to his students. This thoroughness is evident in the present volume, which contains many results hitherto unpublished. A brief outline of the contents follows.

Chapter I. Consequences of Axioms 0 and 1. Axiom 0: *Every region is a point set.* (Point and region are undefined terms.) Axiom 1 states the existence of a sequence of collections of regions covering S, having properties related to the property of compactness. Topics treated include connectedness and separation, compactness, "Borel property," "Borel-Lebesgue property," irreducible continua, separability, properties of arcs.

Chapter II. Consequences of Axioms 1 and 2. Axiom 2: *If P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R.* The development leads up to necessary and sufficient conditions that a compact continuum be a "continuous curve."

Chapter III. Consequences of Axioms 1, 2, 3 and 4; and Chapter IV, Consequences of Axioms 1–5. Axiom 3: *If O is a point, $S-O$ is connected.* Axiom 4: *If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.* Axiom 5: *If A is a point of a region R and B is a point distinct from A there exists, in R, a simple closed curve separating A from B.* The theorems include results in nature similar to Axioms 4 and 5; necessary and sufficient conditions that a point set be a simple closed curve, in terms of accessibility, and in other ways, and further conditions that a compact continuum be a continuous curve.

Chapter V. Upper Semi-Continuous Collections. Semi-continuity of a collection of point sets is a property defined in terms of limit points. An example of the kind of results obtained in this chapter is the following: *If K is a compact and countable point set, $S-K$ is arcwise connected.*

Chapter VI. Consequences of Axioms 1, 2, 4, 5_1, 5_2, 6 and 7; and Chapter

VII. Concerning Topological Equivalences and the Introduction of Distance. Axioms 5_1 and 5_2 are similar in nature to Axiom 5. Axiom 6: *No compact continuum separates two non-compact point sets from each other.* Axiom 7: *The set of all points is completely separable.* In Chapters VI and VII is proved what is stated in the preface, regarding the axioms named in the title of Chapter VI, ". . . every compact space that satisfies all of them is topologically equivalent to a sphere, while every non-compact one is topologically equivalent to a plane."

Of course, we have by no means referred to all the important results of the book.

The very convenient glossary on pages 485–486, giving the locations of the definitions of the technical terms of the subject, should also be mentioned.

A. B. BROWN

Plane and Spherical Trigonometry, With tables. Revised Edition. By George N. Bauer and W. E. Brooke. New York, D. C. Heath and Company, 1932. xiv+236+iv+139 pages. $2.00.

This book was first published in 1907, then revised in 1917, and again revised in 1932. When you have taught the book you realize why it has survived all these years, while other trigonometries have come and gone. It is a very teachable text book and yet it contains also a wide range of material that is useful in more advanced mathematics. This new revision of the book improves it both in content and in teachableness. However, an index would have made it handier for both student and teacher as a reference book. Also, when the authors revised the text, why did they not replace the exercises by new ones?

In this latest edition, the text has been expanded from 164 pages in length to 226 pages. No change has been made in the chapters on spherical trigonometry. A chapter on logarithms has been added, which is very necessary for the present-day student. There is a new article on reducing the expression $A \sin \theta + B \cos \theta$ to the form $C \sin (\theta+\phi)$. The graphical representation of the four fundamental operations as applied to complex numbers has been included in the chapter on De Moivre's Theorem. A four-place table of the natural trigonometric functions is now embedded in the chapter on the right triangle. More space is given to the generation of angles and to radians, short sets of examples are added at frequent intervals. Several identities are worked out in full in two ways. More space is given to graphs. More time is spent on expressing any function in terms of any other function. In a few cases alternative proofs are given of the fundamental relations between the trigonometric functions.

All this additional material improves the book without cluttering it up. It is still easy to find your way about in the text. The publishers have again produced a well-printed book in which the important topics and formulas stand out clearly.

A. D. CAMPBELL

Elementary Differential Equations. By Lyman M. Kells. New York, McGraw-Hill Book Company, 1932. ix+184 pages. $2.00.

A recital of the chapter headings and the lengths of the chapters in this book will give a good idea of the scope and nature of the text, and will show its elements of strength and of weakness. The eleven chapters are entitled as follows: Definitions and Elementary Problems (6 pages), Particular Solutions and Simple Applications (18 pages), Differential Equations of the First Order and the First Degree (21 pages), Simultaneous Equations and Problems Involving First-Order Differential Equations (3 pages), First-Order Equations of Degree Higher than the First, Singular Solutions (15 pages), Linear Differential Equations with Constant Coefficients (23 pages), Applications of Linear Equations with Constant Coefficients (16 pages), Miscellaneous Differential Equations of Order Higher than the First, Integration in Series (15 pages), Applications (9 pages), Partial Differential Equations of the First Order (18 pages), Partial Differential Equations of Order Higher than the First (17 pages). These chapters are followed by an index and a collection of answers to most of the problems. We note the small number of pages allowed each of the above subjects. In fact integration in series is summarily disposed of in two pages.

This book gives more applications of differential equations than do most similar books. We note that such topics are treated as compound-interest law problems, rates, the hodograph, harmonic motion and damping, Kirchhoff's current law and electromotive-force law and applications to networks, equation of elastic curve and deflection, along with the old standbys such as orthogonal trajectories in rectangular and in polar coordinates, envelopes, and other geometrical topics. The up-to-date character of the book is well illustrated by the section on the use of limits in place of determining the constant of integration and other constants, and also by the sections on simultaneous equations. A strong point of the book is its careful drill in setting up differential equations from given data.

The book is designed to appeal to physicists, engineers, and liberal arts students in general. For this reason the author follows a chapter on theory immediately by a chapter on applications, for he has probably often heard the question "What is the use of all this, anyhow?" Therefore the book should please the practical-minded student.

However, the brevity of the book is against it. Having fewer pages than other similar books, it covers more applications and therefore slights much theory. Even the applications are given rather too briefly. Chapter I is too short to give a student a clear idea of what the course is all about. No regular logical development is given to the subject, but rather a multitude of special types of problems are solved by ingenious but diverse methods. The discussions of the topics are not rounded off in all their details, and no wide selection of topics is offered for the teacher to choose from. No connecting link runs through the text. An intelligent student might well ask "Has every differential equation a solu-

tion? If not, why not? Just what kinds of equations can I solve in terms of well-known functions? How can I handle the other kinds of equations? What general methods of solution are there? How can I be sure that I have found all the possible solutions of any given equation?"

If the teacher is careful to supplement the text and to keep the work from becoming merely formal manipulation, he may be able to give a well-rounded elementary course in differential equations with this book as his text-book.

A. D. CAMPBELL

Georg Cantor, Gesammelte Abhandlungen. Berlin, Julius Springer, 1932. viii+486 pages. RM 48.

The scientific works of Georg Cantor are here collected, arranged and annotated by Ernst Zermelo. The mathematical papers with a few additional items—in particular appropriate extracts from the Cantor-Dedekind personal correspondence—are supplemented by a biographical sketch of 31 pages prepared by Adolf Fraenkel. A portrait serves as frontispiece. A list of seven articles on number theory and algebra is followed by nine upon trigonometric series, together filling 114 pages. The major part of the volume is of course devoted to Cantor's contributions to the theory of aggregates including the philosophy of transfinite numbers. There has been some judicious gathering under single titles of papers first appearing scattered at different times. Editorial comments are unobtrusive but highly significant, consisting of brief critical remarks, of cross-references, and of references to investigations of later mathematicians dealing with problems touched upon by Cantor.

The volume has no general index but is equipped with a serviceable index to technical terms used by Cantor in the theory of aggregates.

This book avoids giving the fragmentary impression characteristic of most editions of collected works. Neither is there the quaintness of obsolete notation. Cantor was methodical and thorough. He had few immediate pupils. He pursued so effectively the many lines of inquiry that suggested themselves in his own work that there was little left in the form of vague suggestions for pupils to pick up and develop. Such was his taste and good judgment that even in minor matters of notation, few changes have been proposed by later writers. One sees here the subject of transfinite numbers during its steady shaping at the hands of its discoverer, to a stage of well rounded symmetry. The basic concepts of Cantor are today the common property of all investigators in the theory of point sets. The more specialized notions have continued their development in the work of Hausdorff, Fraenkel, Hahn, Sierpinski, and others that require no listing here.

An elaborate appreciation of the rich material of this volume is inappropriate in this place. A trifling observation may be of interest to teachers of immature students. On looking over these pages one is impressed anew by the classical mathematical background, well-tried analytic power, and scholarly contacts of this genius whose bold originality is so startlingly conspicuous. The

hosts of undaunted circle-squarers and amateur investigators of Fermat's "last theorem," who insist that originality is to be anticipated on the part of those the soil of whose uncultivated minds is unsown to age-old prejudices, would find nothing here to support their clamor.

This book will serve a useful purpose. It is more than a gesture of homage, but the real monument to Georg Cantor is of course to be found in the subject of point sets, which he so significantly enriched.

A. A. BENNETT

Fastperiodische Funktionen. By H. Bohr. Berlin, Julius Springer, 1932. 96 pages. RM 11.40.

Almost Periodic Functions. By A. S. Besicovitch. Cambridge, University Press, 1932. xiii+180 pages. $3.75.

The monograph by Professor Bohr is one of a set of five which constitute the first volume of a new series which is being published under the direction of the editors of the "Zentralblatt für Mathematik." The new series bears the title of "Ergebnisse der Mathematik und ihrer Grenzgebiete," and is designed to supplement the efforts of the "Zentralblatt" by making available, in self-contained and concise form, expository accounts of some of the recent significant advances in the different branches of modern pure and applied mathematics.

One of the important recent developments in modern analysis has to do with the generalization of the concept of pure periodicity as a structural property of a function, and with the corresponding extensions in the theory of general trigonometric and exponential (Dirichlet) series. The class of functions to which these considerations lead, has been called "almost periodic" by Professor Bohr who initiated this line of research in 1925 in a series of memoirs published in the Acta Mathematica. The monograph under review gives a clear, self-contained exposition of some of these researches and will serve admirably as an introduction to these and subsequent investigations. It is the outgrowth of a series of lectures given by the author in 1930–31 when he was in this country as visiting professor at Stanford, the University of California and Princeton.

In order to keep the character of the work as elementary and simple as possible, the author has restricted his discussion to continuous functions of a single real variable. In the first chapter is to be found, as a necessary background for the generalizations which follow, a clear and straightforward presentation of that portion of the classical theory of periodic functions and their corresponding Fourier series, which culminates in the theorem which establishes the equivalence of the "uniqueness" theorem and the equation of Parseval. The theorem of Fejér on the summability of Fourier series and the Weierstrass approximation theorem are also given. The interrelation between all these results is shown in a direct and simple manner.

In the second chapter, almost periodicity is defined and the principal properties of almost periodic functions are established. Next, the Fourier development for such functions is defined and it is then shown that the equation of Parseval

also holds in this case, as well as its equivalence to the "uniqueness" and multiplication theorems. The chapter concludes with a proof of the important theorem which gives the fact that almost periodicity is a fundamental structural property which completely characterizes the class of functions which may be approximated uniformly by "exponential polynomials" of a certain type. Throughout these investigations the equation of Parseval plays an all important role. It is interesting to note that this same equation is also fundamental in the applications of the classical theory of Fourier series to geometry which were initiated by Adolf Hurwitz.

The monograph closes with two appendices, one giving a sketch of the generalizations of almost periodic functions which have been studied by Bohr, Stepanoff, Besicovitch, Weyl and others; the other is concerned with Bohr's own theory of analytic almost periodic functions and its connection with the theory of Dirichlet series.

The work of Dr. Besicovitch is more extensive and more formal in its treatment of the theory; as a whole there is relatively little overlapping with the monograph by Professor Bohr. The first chapter presents the theory of uniformly almost periodic functions in one and two real variables and follows, in the main, the methods of Bohr, de la Vallée Poussin, Weyl and Bochner. Next, there is a chapter devoted to a systematic investigation of the generalizations of the theory which have been indicated by Stepanoff, Wiener, Weyl, Bohr, Besicovitch and others. An appendix to this chapter contains some further results along these lines. The third and last chapter is given to the development of the theory of analytic almost periodic functions and their generalized Dirichlet series. Here, Bohr's work is followed in its essentials.

This authoritative work, giving the fundamental results of the theory as developed to date, will prove indispensable to any one undertaking a serious study of this interesting chapter of modern analysis.

It is to be hoped that in the not too distant future a monograph may appear dealing with the extant applications of the theory to linear differential and difference equations and with the theory of harmonic almost periodic functions developed recently (1927) by Favard in his Paris thesis.

M. A. Basoco

Modular Invariants. By D. E. Rutherford, Cambridge Mathematical Tracts, No. 27. Cambridge, University Press, 1932. vi+84 pages. $2.00.

The author of this tract has acquitted himself of a difficult task in a masterly fashion. L. E. Dickson's Madison Colloquium Lectures carried the subject of modular invariants as far as the year 1914, before the modular symbolical theory was instituted. Since then there have appeared numerous papers by Dickson, Glenn, Hazlett and others, but no complete account of the theory. The author had to select material of general import from all these papers, many of which are on special topics. He had to adopt a uniform terminology and notation. Such words as fundamental and modular, for example, vary in their

meaning from paper to paper. In Appendix I he summarizes the later papers not summarized in Dickson's History of the Theory of numbers. In Appendix II he gives a list of references. He follows the third appendix by an excellent index. The author achieves his results by giving in full some sample discussions of theory and of problems, referring to the papers for the more difficult proofs. The book is remarkably complete for so short a tract. For anyone unfamiliar with this theory of invariants we should recommend the reading of such a book as Dickson's Algebraic Invariants, followed next by Dickson's Madison Colloquium Lectures, and finally by this tract. Those who are familiar with this theory will find the tract useful as a source of information and of hints about problems for further research. For instance, the author calls attention to the fact that it has not yet been proved that all congruent covariants can be represented symbolically nor has it been shown that Miss Sanderson's theorem about invariants can be extended also to covariants.

The tract is divided into two parts. Part I deals with the general subject of modular covariants and invariants. Part II, which is taken mostly from lectures by Professor R. Weitzenbock, is given up to a fine discussion of rings and fields leading up to the proof of a theorem by E. Noether and its application to the task of proving that all the members of any system of modular covariants can be expressed in terms of a finite number of covariants.

Any mathematician, whether acquainted with the theory of modular invariants or not, will find this tract very interesting. As the author brings out so well, here we have some algebras, namely those with just a prime modulus p and also the Galois Fields, where the usual methods of attack on algebraic invariants and covariants collapse. Nothing daunted, Dickson, Glenn, Hazlett and the rest invent new and exceedingly ingenious methods. It is all a beautiful example of mathematical creative activity. Thus they cannot represent the binary cubic symbolically as α_x^3 where $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$ when the modulus $p = 3$, so they represent this cubic as $\alpha_x \beta_x \gamma_x$ where $\beta_x = \beta_1 x_1 + \beta_2 x_2$ and $\gamma_x = \gamma_1 x_1 + \gamma_2 x_2$.

Again, the operators and annihilators used in the study of algebraic invariants and covariants cannot be used for the modular types, so new operators and annihilators are invented. Dickson is especially resourceful and original in such discussions as his theory of classes and characteristic invariants and his treatment of universal covariants.

The author separates invariants and covariants into five types: algebraic, congruent where a modulus p enters in, formal where the coefficients of the transformations belong to a Galois Field, nonformal where the coefficients of the ground form belong to a Galois Field, residual where the coefficients both of the form and of the transformations belong to a Galois Field. He introduces the notation $\|$ to mean "residually congruent to" as $a^p \| a$ modulo p if a is a positive integer, and $|\,|\,|$ to mean "identically congruent to" as $6 |\,|\,| 3$ modulo 3. He gives Capelli's discussion of the generators of linear transformations. He shows the relation between congruent and algebraic covariants and invariants. He discusses the pseudo-isobarism of covariants in the Galois Fields where be-

cause of Fermat's Theorem we have $T^n \| t$ (a residue in the field) and so isobarism has no meaning. He distinguishes carefully between full systems and fundamental systems and he spends much time on the question of the finiteness of a full system. We need not list any more topics to show how thoroughly he treats modular invariants in such a short space by a judicious choice of proofs and by a thrifty use of words.

A. D. CAMPBELL

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Missouri

The officers for the academic year 1931–1932 were: Frank Cockerill, Director; R. P. Edwards, Vice Director; Robert Wier, Secretary; Mary Manley, Treasurer; Finis O. Duncan, Corresponding Secretary; Charles Clark, Librarian.

This year we had 45 active members. We awarded two prizes. The winner of the Calculus Prize was Donald Starrett Nutter and the winner of the Analytics Prize was Morris J. Gottlieb.

The meetings and programs were as follows:

September 29, 1931: At this meeting Dr. Louis Ingold gave a report of the meetings of the Mathematical Association of America and the American Mathematical Society.
October 13, 1931: "The growth of numbers" by Dr. W. D. A. Westfall.
October 27, 1931: Picnic.
November 1 , 1931: "The Phi function" by Dr. G. E. Wahlin.
November 24, 1931: "Various aids in computation" by Norman Beers and Ralph Traber.
December 1, 1931: Ciphering match (Numbers to base twelve).
December 14, 1931: Pledge examination for 10 candidates.
December 18, 1931: Formal initiation and party.
January 12, 1932: "The Mammoth cave" (Illustrated) by Dr. R. T. Dufford.
February 23, 1932: "Nomography" by John Waugh; "Vectors" by Richard Emberson.
March 8, 1932: "The algebra of logic" by Dr. Louis Ingold.
April 12, 1932: "Inequalities" by Dr. Hobart C. Carter.
April 26, 1932: Picnic.
May 10, 1932: Pledge examination for 8 candidates.
May 13, 1932: Formal initiation and banquet.

FINIS O. DUNCAN, *Corresponding Secretary*

Pi Mu Epsilon of the University of Montana

The Montana Chapter of Pi Mu Epsilon has been enjoying a very successful year. Meetings have been held regularly during the year. The officers for the year 1931–1932 were elected on November 11, 1931. They are: Franklin Long, Director; Robert Bóden, Vice Director; Kathryn Coe, Secretary; Professor G. D. Shallenberger, Treasurer.

Initiation for the new members was held April 13th at the annual banquet. The following took the pledge of membership: Juanita Armour, '34; Mary Castles, '34; Roderick Chisholm, '33; Herman Dickel, '33; Lavira Hart, '33; Ruben Lewon, '33; Charlie Krebs, '34; Ellen Shields, '34; Edward Skoog, '33; Mary Rose, '32; H. A. Veeder, '32.

There are now thirty-one active members. As one of the requirements for membership, certain papers have to be prepared and presented before Pi Mu Epsilon.

The meetings and programs were as follows:

November 4, 1931: "Infinite series" by Craig Smith.
November 11, 1931: "Bell Telephone Laboratories" by Professor E. M. Little.
December 3, 1931: Applications of Calculus to Physical Chemistry" by Franklin Long.
December 10, 1931: "Temperature values" by Professor G. D. Shallenberger.
January 27, 1932: "Vector analysis" by Robert Boden.
February 17, 1932: "The quadratrix" by Robert Sullivan.
March 2, 1932: "The installment plan" by Lavira Hart.
March 16, 1932: "Rosettes" by A. L. Craig.
April 20, 1932: "The table of integrals" by William Hensen.
April 27, 1932: "Higher plane curves" by Ellen Shields and Mary Castles.
May 4, 1932: "The curve whose equation is $x^2+y^2+z^2-2xyz=0$" by John Clark.
May 18, 1932: "Hyperbolic functions" by Mary Rose.

A joint picnic with the Mathematics Club ended another successful year for the Montana Chapter of Pi Mu Epsilon.

KATHRYN COE, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematical Society of New Jersey College for Women

The officers for the year 1931–1932 were: Helen Carpenter, President; Irene Stevens, Vice President; Elizabeth Phillips, Secretary; Edith Rapp, Treasurer.

The officers are elected by ballot in the Spring usually at the last or next to last meeting of the Society for that year.

There are about twenty-four active members, that is, those who attend the meetings regularly and also take an active part in those meetings.

The primary aim of the Society is to promote a deeper interest in mathematics and to cultivate a wider knowledge in the field. To be eligible for membership one must have taken or be taking Calculus.

The meetings and the various topics discussed during the year were as follows:

October 13, 1931: "Inversion of a circle and a line" by Marian Bruen; "Inversion of angles" by Alice Maier.
October 27, 1931: "Inversion of two intersecting lines and of two intersecting planes" by Mr. Robert Walter.
November 10, 1931: "Proof that the trisection of an angle is impossible with the straight edge and compass alone" by Lucille Bourath and Anne Rosenberg.
January 12, 1932: "Construction of a seventeen sided polygon" by Mabel Joslin and Margaret McCallum.
February 16, 1932: "Constructions with a compass alone" by Armie Apamian and Edna Schnitzler.
March 1, 1932: "Brianchon's theorem" by Verna Wilson; "Kasner's pentagon theorem" by Mabel Joslin.

March 14, 1932: "The average number of sides of a polygon" by Mr. Nelson.
April 1, 1932: "A peculiar function 'A piece of Pi'" by Ethel Petrie.
April 11, 1932: "A curve that fills an area" by May Blydenburgh.
May 1, 1932: "Paradox on integration" by Molly Bruce.
May 10, 1932: "Fourier series" by Irene Stevens; "Nomography" by Marian Bruen.

At a joint meeting of the Society and the Mathematics Club of Rutgers University on December 4, 1931, Professor Alonzo Church of Princeton University lectured on "The finite geometry."

At a second joint meeting of the two organizations on April 27, 1932, Professor Thomas Fiske of Columbia University lectured on his experiences in establishing the American Mathematical Society.

Student representatives of the Society attended the meetings of the Philadelphia Section of the Mathematical Association of America at Lehigh University on Nobember 28, 1931 and the meeting of the New Jersey State Teachers Association at Atlantic City on November 28, 1931.

A bridge was given at the beginning of the year to welcome new members. A banquet was given in Professor Church's honor on December 4, 1931. A Christmas party was held just before the Christmas vacation. The annual Society dance was held on May 7, 1932.

ELIZABETH PHILLIPS, *Secretary*

Delta Chi of the University of New Hampshire

The officers for the year 1931–1932 were: William J. Volkman, President; John Grady, Vice President; Alice M. Roe, Secretary; H. Leslie Curtis, Treasurer. The officers were elected April 23, 1931 by a majority vote of the members. Alice Roe was elected secretary October 23, 1931 to serve for the year 1931–1932 because of the resignation of Margaret W. Durgin.

There are 40 active members including sophomores, juniors and seniors. Active members are those in good standing who have paid their dues. Every member of the department of mathematics is an honorary member.

The program committee which has charge of the social activities for the year consisted of Gordon Ayer, John Grady and Alice Rowe.

The objects of Delta Chi are to promote interest in the subject of mathematics and to bind the members with lasting friendship.

A student in the University of New Hampshire whose major is mathematics, or who is pursuing the Electrical Engineering, Industrial Engineering, Mechanical Engineering, Civil Engineering, or Chemical Engineering course, and who has attained an average of at least 80% or its equivalent in Mathematics 1*a* (advanced algebra), 2*b* (trigonometry), 3*c* (analytic geometry) and 7*a* (differential calculus) or their equivalent (said average to be computed by summing the four term grades and dividing by four), and a miminum of 70% each term, shall be eligible for membership.

A meetings was held January 14, 1932 with the following program:

"The theory of the planimeter" by Mr. John J. Uicker; "The rational solutions of $x^y = y^x$ (not including $x = y$, $x < 0$, or $y < 0$)" by Mr. Donald Perkins.

"Delta Chi, honorary mathematics society, will present at the end of each academic year, a silver cup to that member of the sophomore class, eligible for membership in the society, who during two year's courses in mathematics has demonstrated valuable mathematical ability, by ranking as one of the five high students in mathematics. General scholastic standing, and personality shall also figure in determining the award. A committee consisting of the Dean of the College of Technology, the Dean of the College of Liberal Arts, the head of the department of mathematics, the President of Delta Chi, and one other student member of the society shall determine the winner each year."

The trophy was won by Henry J. Joyal for the year 1930–1931. The winner for the year 1931–1932 has as yet not been determined.

Delta Chi holds two banquets each year:—the initiation banquet and the Spring banquet. The initiation banquet was held February 17, 1932 at the President's Dining Hall at the University Commons. Twenty new student members were initiated, and two new members of the mathe-

matics department, Mr. William Kichline and Mr. Milthiades Demos received honorary membership. Dr. Slobin, head of the department of mathematics, was the speaker of the evening. Mr. Moran, from the department of physics, presented a series of campus movies.

The Spring banquet was held May 18, 1932 at the Highland House in Durham. The banquet speakers were Mr. Wilbur and Mr. Solt, both members of the department of mathematics.

ALICE M. ROE, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 21. *Proposed by V. F. Ivanoff, San Francisco.*

Prove that the differences of squares of two consecutive numbers in the third diagonal of the Pascal Triangle:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \text{etc.}$$

is always a perfect cube.

E 22. *Proposed by R. M. Winger, University of Washington.*

As a western version of problem E 7, find the digits represented by the various letters in the following problem in addition, and determine whether or not the solution is unique. (Except that obviously R and L and S and G are interchangeable.) No two different letters represent the same digit.

$$\begin{array}{ccccc} & S & E & N & D \\ & M & O & R & E \\ & G & O & L & D \\ \hline M & O & N & E & Y \end{array}$$

E 23. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In the following sum and product, the digits from 1 through 9 are represented in some order by the letters A through I. Determine the representation and prove that it is unique.

$$AB = CD + EF, \quad EF = G \times HI.$$

E 24. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

There are just three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately canceling a digit. One of these is

$$\frac{26}{65}=\frac{2\cancel{6}}{\cancel{6}5}=\frac{2}{5}\cdot$$

Find the other two and confirm the statement that there are no others.

E 25. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Find the only ten-place integer which is both a square and a triangular number.

SOLUTIONS

E 1. [1932, 489] *Proposed by H. E. Slaught, University of Chicago.*

Find all the unknown digits, represented by x's, in this exact division.

$$\begin{array}{l}
x\ x\ x)x\ x\ x\ x\ x\ x\ x\ x(x\ 7\ x\ x\ x\\
\quad\ \ \underline{x\ x\ x\ x}\\
\qquad\quad x\ x\ x\\
\qquad\quad \underline{x\ x\ x}\\
\qquad\quad x\ x\ x\ x\\
\qquad\quad\ \ \underline{x\ x\ x}\\
\qquad\qquad x\ x\ x\ x\\
\qquad\qquad \underline{x\ x\ x\ x}
\end{array}$$

Solution by C. A. Rupp, Pennsylvania State College.

Let D denote the divisor, and a, 7, b, c, d the digits of the quotient.

1] Obviously, $c=0$.

2] Since $xx<xxx$, $xxxx-bD<xxx-7D$, and $7<b$.

3] Since $xxx<xxxx$, $bD<aD$ and $bD<dD$.

4] From [2] and [3], $7<b<a$ and $7<b<d$, and since these letters represent digits, $b=8$, $a=d=9$, and the quotient is 97809.

5] Since $8D=xxx$, $D<125$, and $9D<1125<1200$.

6] Since $xxxx-8D=xx$ (the first two digits of $9D$), it follows that $1000-8D<12$, or $123<D$. Therefore $D=124$.

7] The dividend is $124\times 97809=12128316$ and the other digits of the problem follow immediately.

It is perhaps interesting to note that the problem is still soluble if the base of the number system is some integer greater than ten, the digits being 0, 1, 2, 3, $\cdots$, p, q, r, and the second digit of the quotient being p. Proceeding as

before, the quotient is $rpq0r$, $1000-12<qD<1000$, or $123+4/q<D<124+8/q$, and $D=124$ again in the new number system.

Editor's Note: Since publishing this problem, information has been received that it appeared previously in Le Sphinx.

Also solved by T. A. Bickerstaff, Annabel S. Boyce, W. E. Buker, H. E. H. Greenleaf, Arthur Haas, H. R. Leifer, A. C. Maddox, M. Markowitz, W. R. Ransom, F. C. Smith, S. Vatriquant, and B. C. Zimmerman.

E 2. [1932, 489] *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In finding the volume cut from the sphere $x^2+y^2+z^2=9$ by the cylinder $x^2+y^2=3x$, one may use cylindrical coordinates and choose as element of volume a column of cross-section dr by $rd\theta$ and of height $z=(9-r^2)^{1/2}$. The volume appears computable either by

$$V_1 = 2\int_{-\pi/2}^{+\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta,$$

where symmetry with respect to the xy-plane only is used, or by

$$V_2 = 4\int_0^{\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta,$$

where symmetry with respect to the xz-plane is also used. But upon integration it appears that V_1 exceeds V_2 by twenty-four cubic units. Which, if either, is correct, and why?

Solution by M. Markowitz, Brooklyn, New York.

It is obvious that the volume V_1 is incorrect, since the result obtained, 18π, is the volume of a hemisphere.

To understand why

$$\int_{-\pi/2}^{0}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta = 9\pi/2+6,$$

while

$$\int_0^{+\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta = 9\pi/2-6,$$

let us follow the formal steps of integrating the first integral and substituting the limits.

$$(1)\qquad \int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta = \left[-(9-r^2)^{3/2}/3\right]_0^{3\cos\theta}$$

$$(2)\qquad = -(9-9\cos^2\theta)^{3/2}/3+(9)^{3/2}/3$$

$$(3)\qquad = 9-9(1-\cos^2\theta)^{3/2}$$

(4) $$= 9 - 9 \sin^3 \theta.$$

Now the discrepancy arises due to the transition from step (3) to step (4), for the expression $(1-\cos^2 \theta)^{3/2}$ can not be negative, while its "equivalent," $\sin^3 \theta$, is negative when θ is in the third or fourth quadrant. The reason that V_2 gives the correct result (as may be ascertained by using rectangular coordinates) is that the limits from 0 to $\pi/2$ do not go outside of the first quadrant. A correct result would still be obtained if the limits taken were from 0 to π.

Also solved by W. O. Cook, W. C. Janes, W. R. Ransom and D. H. Richert.

E 3. [1932, 489] *Proposed by W. R. Ransom, Tufts College.*

Has the locus, $y=x^x$, a highest point in the second quadrant?

Solution by the Proposer.

This question turns largely upon how x^x is defined. For such values as $x=2/5$ and $x=2/7$, rational fractions with even numerators and odd denominators, there certainly are points in the second quadrant, dense along a definite curve. If the definition of x^x is completed by adopting all the limiting points, we may treat x^x as a continuous curve, put $x=-z$, and locate the maximum by differentiation. This leads to $x=-1/e=-.367879$, which gives the maximum $+1.444668$ in the second quadrant. But it is questionable whether this point may be regarded as present on the curve according to the usual convention as to powers. For the device by which it is made one of the points of the curve, demands that we acknowledge $(-3)^{-3}$ to have the value $+1/27$ as well as the value $-1/27$.

Also solved by W. C. Janes.

E 4. [1932, 489] *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

What is the simplest way to cut a wooden block 1 ft.$\times$1 ft.$\times$2 ft. into pieces which may be reassembled into a solid cube?

Solution by W. R. Ransom, Tufts College.

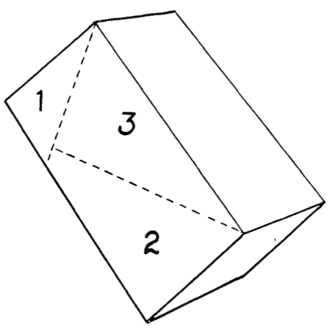

FIG. I.

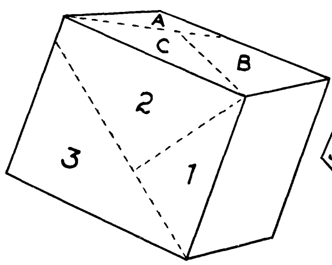

FIG. II.

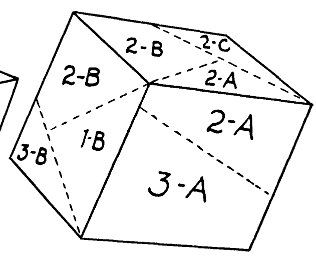

FIG. III.

A wooden block 1 ft.$\times$1 ft.$\times$2 ft. may be cut into eight pieces by four plane cuts and reassembled into a solid cube. The original block and the first two cuts are shown in figure I. Piece 1 is cut off by a plane cut through one of the shorter edges, the length of the cut being $2^{1/3}$. The other cut in this first figure is made by a plane perpendicular to the first cut, and through the edge farthest from the first cutting plane. The resulting pieces, 1, 2 and 3, are then reassembled as shown in figure II, the dimensions of which block are 1 ft.$\times 2^{1/3}$ ft.$\times 2^{2/3}$ ft. Two plane cuts are now made perpendicular to the two opposite faces of medium area. The first of these, cutting off piece A, goes through one edge of the reassembled block and is of length $2^{1/3}$. This plane dissects pieces 2 and 3, but does not touch piece 1. The final cutting plane is perpendicular to this last previous plane, and again through an edge of the reassembled block. It dissects pieces 1, 2 and 3. Thus we have piece 1 separated into two pieces, while pieces 2 and 3 are separated into three pieces each, giving us eight pieces in all. Finally, if the pieces from figure II are rearranged as shown in figure III, the result is a cube, each dimension being $2^{1/3}$.

Note: Since this solution was sent to the printer, the author has learned of a solution of the problem using *only seven* pieces, by A. H. Wheeler of Worcester, Mass.

Also solved by the proposer.

E 5. [1932, 489] *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

How may the total surface of a sphere be divided into the largest possible number of congruent pieces, if each side of each piece is an arc of a great circle less than a quadrant?

Solution by W. R. Ransom, Tufts College.

Inscribe a regular dodecahedron or a regular icosahedron in the sphere and drop perpendiculars from the center to each face. Form isosceles triangles, sixty in number, with the feet of these perpendiculars as vertices and the sides of the respective faces as bases, and centrally project these sixty triangles onto the surface of the sphere as isosceles spherical triangles.

Also solved by the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3594. *Proposed by H. T. R. Aude, Colgate University.*

Find sets of integers for rational right triangles which, as the numbers increase, approach a 30°–60° right triangle.

3595. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through the edges a, b, c, of a trihedral angle planes are drawn perpendicular to the faces bc, ca, ab and cutting these faces along the lines e, f, g, respectively. Show that the faces of the given trihedral angle bisect the dihedral angles of the trihedral angle formed by the lines e, f, g.

3596. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$ {}_sH_r = \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad {}_sC_r = \frac{s(s-1)(s-2)\cdots(s-r+1)}{r!}, $$

where s and r are any positive integers; and

$$ {}_sH_0 = {}_sC_0 = 1 $$

where s is any positive integer: prove that

$$ \phi(s, p) = \sum_{k=0}^{p}(-1)^k {}_sH_{p-k}\, {}_sC_k = 0 $$

for all positive integral values of s and p.

3597. *Proposed by J. E. Trevor, Cornell University.*

Continuous functions $f(x, y)$ and $F(x, y)$ of the independent variables x, y have continuous and non-vanishing first and second derivatives p, q, r, s, t and P, Q, R, S, T respectively. The surface $z = f(x, y)$ is tangent to the *developable surface* $Z = F(x, y)$ along a line whose projection on the x, y plane is a curve $\phi(x, y) = 0$. On the line of contact x and y are functions of q. Writing dx/dq and dy/dq for the first derivatives of these functions, and putting $\Delta = rt - s^2$, show that, at points on the line of contact,

$$ r - R = +\, T\Delta\left(\frac{dy}{dq}\right)^2 $$

$$ s - S = -\, T\Delta\frac{dx}{dq}\,\frac{dy}{dq} $$

$$ t - T = +\, T\Delta\left(\frac{dx}{dq}\right)^2. $$

This problem is the mathematical aspect of a problem in thermodynamics.

3598. *Proposed by V. Thébault, Le Mans, France.*

On the lines PA, PB, PC, PD, joining a point P in space to the vertices of the *orthocentric* tetrahedron $ABCD$ are marked the inverse points A_1, B_1, C_1, D_1, of the vertices in the inversion (P, k). The planes perpendicular at these points to the lines PA_1, PB_1, PC_1, PD_1, form a tetrahedron $\alpha\beta\gamma\delta$. Show that the planes perpendicular at P to the lines $P\alpha$, $P\beta$, $P\gamma$, $P\delta$, cut the planes $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, $\alpha\beta\gamma$ along four coplanar lines, and that the plane of these lines is perpendicular to the line joining P to the orthocenter of the tetrahedron $ABCD$.

3599. *Proposed by Roy MacKay, Albuquerque, New Mexico.*

If $f(0)=1$ and

$$f(x) = \left(\frac{x}{e^x - 1}\right)^k \quad \text{for} \quad x \neq 0,$$

where k is a positive integer $\geqq 2$; prove that

$$\left[\frac{d^r}{dx^r} f(x)\right]_{x=0} = \frac{(-1)^r \sigma_r}{\binom{k-1}{r}},$$

where r is any positive integer $\leqq k-1$; and σ_r is the elementary symmetric function of the numbers $1, 2, 3, \cdots, (k-1)$, that is σ_r is the sum of the products of these numbers taken r at a time.

3600. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

If α and β are the angles which the axis s of a cone of revolution makes with the H and V planes of projection respectively; ϵ the apex angle of the cone; a_1 and b_1 the major and minor semi-axes of the elliptical H-section of the cone; a_2 and b_2 the major and minor semi-axes of the elliptical V-section; θ the angle which the H-projection of s makes with the line HV of intersection of H and V: show that

$$\cos\frac{\epsilon}{2} = \frac{a_1 a_2}{\{a_1^2(a_2^2 - b_2^2) + a_2^2(a_1^2 - b_1^2)\sin^2\theta\}^{1/2}}$$

$$\cos\beta = \frac{a_1(a_2^2 - b_2^2)^{1/2}}{\{a_1^2(a_2^2 - b_2^2) + a_2^2(a_1^2 - b_1^2)\sin^2\theta\}^{1/2}}$$

$$\cos\alpha = \frac{a_2(a_1^2 - b_1^2)^{1/2}}{\{a_1^2(a_2^2 - b_2^2) + a_2^2(a_1^2 - b_1^2)\sin^2\theta\}^{1/2}}$$

SOLUTIONS

3513. [1931, 461]. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$f(s, p) = \sum_{k=1}^{p}(-1)^{p+k}(2k-1)_sH_{p-k}$$

where

$$_sH_r \equiv \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad _sH_0 \equiv 1.$$

Prove: (a) If p is odd, then $f(s, p)$ is positive for all values of s;

(b) If p is even, then $f(s, p)$ is positive, equal to zero, or negative according as s is less than, equal to, or greater than 3.

Solution by Frank Ayres, Jr., Dickinson College

The function $f(s, p)$ may be given a different form by writing $2k-1=k-1+k$ and then using the relation $_sH_n - {}_sH_{n-1} = {}_{s-1}H_n$. There results

$$(1) \qquad f(s, p) = (-1)^{p-1}\sum_{k=1}^{p}(-1)^{k-1}k\,{}_{s-1}H_{p-k}.$$

Now using a similar reduction after separating the even and odd terms, we find that

$$(2) \qquad f(s, p) = (-1)^{p-1}\sum_{i=1}^{q} i\,{}_{s-3}H_{p-2i+1},$$

where q is the greatest integer in $(p+1)/2$.

If p is odd, $2m+1$, $f(s, p)$ is positive for all real values of s except possibly in the intervals

$$(3) \qquad 4-2j < s < 5-2j, \quad j = 1, 2, \cdots, m.$$

If p is even, $2m$, $f(s, p)$ is negative for all real values of $s>3$; it is zero for $s=3$; it is positive for all other real values of s except possibly in the intervals

$$(4) \qquad 3-2j < s < 4-2j, \quad j = 1, 2, \cdots, m-1.$$

Further investigation is needed to determine the sign of $f(s, p)$ in the intervals (3) and (4).

A Note by the Editors. Since the printing of this problem it has been learned from the proposer that s is restricted to integral values, positive, negative, or zero. With this restriction the above solution is complete, as no integral values of s lie in the intervals (3) and (4).

Also solved by the proposer.

3527. [1932, 46]. *Proposed by R. E. Gaines, The University of Richmond.*

If a variable chord QR of a conic subtends a right angle at a fixed point P on the conic, the chord QR passes through a fixed point S. If P describes the

conic, the locus of S is a conic whose equation differs from the equation of the given conic only in the constant term.

Solution by Roscoe Woods, University of Iowa.

The fixed point S in the above problem is known as Frégier's point and is situated upon the normal to the conic at P. See Sommerville, *Analytical Conics* (1924), p. 104. This problem for the ellipse is stated and proved in Casey, *A Treatise on Analytical Geometry*, etc. (1893), p. 227.

The analytical solution of this problem can be put in very compact form as follows: By the proper choice of the axes the equation of the general conic takes the form

$$ax^2 + by^2 - 2gx = 0, \quad bg \neq 0. \tag{1}$$

Let $P(x_1, y_1)$ be any point on the conic in equation (1). Since every chord QR which subtends a right angle at P cuts the normal at P in a fixed point, S, we choose the two perpendicular chords $x=x_1$ and $y=y_1$, and find (if $a\neq 0$) the coordinates of Q and R to be $[(2g-ax_1)/a, y_1]$ and $(x_1, -y_1)$, respectively. Solving the equation of the chord QR with the normal at P simultaneously, we find the coordinates of S to be

$$\left[\frac{(b-a)x_1+2g}{a+b}, \frac{(a-b)y_1}{a+b}\right].$$

It is easily verified that this result holds also for $a=0$. The elimination of x_1 and y_1 from the coordinates of S by means of the equation (1) gives the locus of S to be

$$ax^2 + by^2 - 2gx + 4g^2b/(a+b)^2 = 0$$

which demonstrates the theorem.

A Note by the Editors. A proof of the fact that QR passes through a fixed point S is given by Goormaghtigh [1929, 113] in his solution of a generalization of problem 3302 [1928, 41]. The point S must lie on the normal to the conic at P, for, when PQ is tangent to the conic, QR coincides with the normal at P. An exception must be made in the case of an equilateral hyperbola, $a+b=0$. In this case all the chords QR are parallel.

Also solved by E. F. Allen, Theodore Bennett, J. H. Butchart, R. Goormaghtigh, V. F. Murray, and Paul Wernicke.

3529. [1932, 115]. *Proposed by V. F. Ivanoff, San Francisco, California.*

A ship is sailing with a speed and direction, v_1; the wind blows apparently (judging by the vane on the mast) in the direction of a vector, $\boldsymbol{a}$; on changing the direction and the speed of the ship from v_1 to v_2, the apparent wind is in the direction of a vector, $\boldsymbol{b}$.

Find the vector velocity of the wind.

Solution by T. C. Esty, Amherst College

The actual velocity, $\boldsymbol{v}$, of the wind is the sum of the ship's velocity and the apparent velocity of the wind. Without loss of generality we may assume $\boldsymbol{a}$ and $\boldsymbol{b}$ to be unit vectors, and may write

$$\boldsymbol{v} = \boldsymbol{v}_1 + s\boldsymbol{a} = \boldsymbol{v}_2 + t\boldsymbol{b}, \tag{1}$$

where s and t are undetermined scalars. Multiply (1) scalarly first by $\boldsymbol{a}$ and then by $\boldsymbol{b}$, and obtain

$$s - t\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1), \tag{2}$$

$$s\boldsymbol{a}\cdot\boldsymbol{b} - t = \boldsymbol{b}\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1). \tag{3}$$

Multiply (3) by $\boldsymbol{a}\cdot\boldsymbol{b}$, subtract the result from (2), and find

$$s = \frac{[\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}]\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1)}{1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2}.$$

Substituting in (1) we get

$$\boldsymbol{v} = \boldsymbol{v}_1 + \frac{[\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}]\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1)\boldsymbol{a}}{1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2}.$$

Also solved by R. P. Agnew, D. F. Gunder, A. Pelletier, H. D. Ruderman, H. L. Schug, and Paul Wernicke.

3530. [1932, 116] *Proposed by A. A. Shaw, University of Arizona.*

Prove that the product of any three consecutive integers is a multiple of 504, if the middle integer is a perfect cube.

Solution by Lester R. Ford, The Rice Institute

The product has the form $(n^3-1)n^3(n^3+1)$, or n^9-n^3, where n is an integer- We show that this is divisible by the relatively prime integers 7, 8, and 9, whose product is 504.

We have

$$n \equiv 0, \pm 1, \pm 2, \pm 3 \pmod 7.$$

Cubing and reducing modulo 7,

$$n^3 \equiv 0, \pm 1 \pmod 7.$$

Cubing again,

$$n^9 \equiv 0, \pm 1 \pmod 7.$$

Hence

$$n^9 - n^3 \equiv 0 \pmod 7.$$

In a similar manner

$$n \equiv 0, \pm 1, \pm 2, \pm 3, 4 \pmod 8,$$
$$n^3 \equiv 0, \pm 1, \pm 3 \pmod 8,$$
$$n^9 \equiv 0, \pm 1, \pm 3 \pmod 8,$$
$$n^9 - n^3 \equiv 0, \pmod 8;$$

and

$$n \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4 \pmod 9,$$
$$n^3 \equiv 0, \pm 1 \pmod 9,$$
$$n^9 \equiv 0, \pm 1 \pmod 9,$$
$$n^9 - n^3 \equiv 0 \pmod 9;$$

and the proposition is established.

Also solved by R. P. Agnew, B. R. Allen, E. Alliot, H. T. R. Aude, Brother Aurelius, Frank Ayres, W. E. Buker, Thomas Bulter, S. S. Cairns, Mannis Charosh, J. A. Clarkson, O. C. Collins, G. F. Cramer, J. W. Foust, H. M. Gehman, R. Goormaghtigh, D. F. Gunder, J. D. Hill, V. F. Ivanoff, Sister Mary Bonaventure, A. S. Merrill, Dave Montgomery, R. E. Moritz, A. Pelletier, W. M. Rust, M. A. Scheier, L. S. Shively, H. W. Smith, R. C. Staley, Mildred E. Taylor, F. Underwood, Paul Wernicke, T. R. C. Wilson.

3531. [1932, 116]. *Proposed by Eugene M. Berry, Lynchburg College.*

We have given a horizontal bar of length b, with a (vertical) leg at each end of the bar. The legs are short and of equal length. The front leg comes to a point at the bottom while the foot of the rear leg is a knife edge which is parallel to the bar. If this is placed with the legs on paper find the curve traced by the knife edge when the front leg traces a given circle.

Solution by D. F. Gunder, Colorado Agricultural College

From the given conditions we deduce the following relations, in which x, y and x_1, y_1 are respectively the coordinates of points on the required curve and the given circle.

$$(y - y_1)^2 + (x - x_1)^2 = b^2$$
$$(y - y_1) - \frac{dy}{dx}(x - x_1) = 0$$
$$x_1^2 + y_1^2 = a^2.$$

By elimination of x_1 and y_1 from these, the differential equation of the curve is found to be:

$$(x^2 + y^2 + b^2 - a^2)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 4b^2\left[y\frac{dy}{dx} + x\right]^2.$$

In polar coordinates this gives the first order equation,

$$d\rho\sqrt{4b^2\rho^2 - (\rho^2 + b^2 - a^2)^2} = \rho(\rho^2 + b^2 - a^2)d\theta,$$

which by quadrature yields the solution:

$$\theta + c = \frac{-b}{\sqrt{b^2 - a^2}} \tan^{-1}\left\{\frac{\sqrt{b^2 - a^2}}{b}\cdot f(\rho)\right\} + \tan^{-1} f(\rho),$$

where

$$f(\rho) = \frac{\sqrt{4b^2\rho^2 - (\rho^2 + b^2 - a^2)^2}}{\rho^2 - b^2 + a^2}.$$

Applying the initial condition that $\rho = b+a$, $\theta = 0$, gives $c = 0$, $\rho > 0$. The latter condition, $\rho > 0$, eliminates the initial condition $\theta = 0$, $\rho = -b-a$ or $-b+a$, which is otherwise contained in the solution when $c = 0$. The other initial condition, $\theta = 0$, $\rho = b-a$, which also gives $c = 0$ must also be excluded.

With these restrictions due to the initial conditions, which are general, the curve is briefly discussed as follows:

If $b > a$, the equation is found to represent a saw-toothed curve whose farthest points from the pole lie on a circle of radius $b+a$, the angle subtended at the pole by the arc determined by two consecutive extreme points being $[b(b^2-a^2)^{-1/2}-1]2\pi$. The nearest points of the curve lie on a circle of radius $b-a$ at the same angular intervals as above. All extreme points are at cusps of the curve. The curve repeats itself after p revolutions if $b(b^2-a^2)^{-1/2}-1 = p/q$, where p and q are relatively prime integers.

If $b < a$, the first term of the right hand member of the solution reduces to

$$\frac{b}{\sqrt{a^2 - b^2}} \tanh^{-1} \frac{\sqrt{a^2 - b^2}}{b} f(\rho),$$

and the solution represents a spiral, winding about and approaching a circle of radius $a-b$. If $b = a$, a spiral about the pole is obtained.

A Note by the Editors. The differential equation in polar coordinates may be obtained more directly from the formula $\tan\phi = \rho d\theta/d\rho$, where ϕ is the angle opposite a in the triangle whose sides are a, b, ρ. By the use of the law of cosines ϕ may be eliminated and we obtain the desired equation. It is easily seen that the center of curvature is the intersection of the normal to the curve with the corresponding radius of the circle. The solution and initial conditions given above are not quite general. For the initial conditions $\theta = 0$, $\rho = \sqrt{a^2-b^2}$, it is readily seen that the solution is the circle $\rho = \sqrt{a^2-b^2}$.

3532 [1932, 116]. *Proposed by R. E. Gaines, University of Richmond.*

If the triangle PQR is inscribed in an ellipse so as to cut off three segments of equal area, and tangents are drawn at P, Q, R, the triangle thus formed will be inscribed in a second ellipse which will be divided in the same way.

Solution by Roscoe Woods, University of Iowa

Consider a circle of radius a with center at the origin. Let pqr be any equilateral triangle inscribed in the circle. It cuts off three segments of equal area. The tangents at the vertices p, q, r form a second equilateral triangle inscribed in a second circle of radius $2a$ and consequently cut off three segments of equal area from the second circle.

Let us subject this configuration to an orthogonal projection given by the equations $x=x'$, $y=ay'/b$. The triangles project into triangles and the circles project into ellipses. The area of the triangle pqr is in a constant ratio, a/b, to the area of its projection PQR. The same is true of any two corresponding areas. This is easily seen from the transformations above. Hence the triangle PQR cuts off three equal area segments from the ellipse. The same can be said about the triangle formed by the tangents at the vertices P, Q, R.

The triangle PQR is a triangle of maximum area inscribed in the ellipse. Since there is a single infinity of equilateral triangles inscribed in the first circle, there is a single infinity of triangles inscribed in the ellipse that cut off three segments of equal area in the ellipse. See, C. Smith, *Conic Sections*, p. 173.

Also solved by R. Goormaghtigh, V. F. Murray, O. J. Ramler, and F. Underwood.

3533 [1932, 116]. *Proposed by R. E. Gaines, University of Richmond.*

The generating circle of a cycloid in one position is tangent to one arch of the cycloid and intersects the next following arch in P_1 and P_2; another such circle intersects the same arch in P_2 and P_3; another, in P_3 and P_4; and so on indefinitely. If α_1, α_2, α_3, $\cdots$ are the angles which the chords P_1P_2, P_2P_3, P_3P_4 $\cdots$ make with the base of the cycloid, then

$$6\sum_{n=1}^{\infty}\alpha_n = 4\pi - 3^{3/2}.$$

Solution by Wm. B. Campbell, Rangoon, Burma

When the unit generating circle has its center at $(\theta_1, 1)$, the point P_1 of this circle generating the cycloid has the coordinates $(\theta_1-\sin\theta_1,\ 1-\cos\theta_1)$; and the circle cuts the cycloid again in P_2, with the parameter θ_2 which satisfies the equation

$$(1)\qquad \begin{gathered}(\theta_2-\sin\theta_2-\theta_1)^2+(1-\cos\theta_2-1)^2=1,\ \text{or}\\ \sin\theta_2=\tfrac{1}{2}(\theta_2-\theta_1).\end{gathered}$$

If P_1 is on the first half arch, that is, if $0<\theta_1<\pi$, then $\theta_1<\theta_2<\pi$. If α_1 is the inclination of the line P_1P_2,

$$\tan\alpha_1=\frac{\cos\theta_1-\cos\theta_2}{\theta_2-\theta_1+\sin\theta_1-\sin\theta_2}=\tan\tfrac{1}{2}(\theta_2-\theta_1),$$

where the reduction follows from (1). Since α_1 and $\frac{1}{2}(\theta_2-\theta_1)$ are in the first quadrant, we have

$$\alpha_1 = \tfrac{1}{2}(\theta_2 - \theta_1). \tag{2}$$

Similarly, we have for P_2P_3, $\alpha_2=\frac{1}{2}(\theta_3-\theta_2)$, etc. The θ's form an increasing sequence which approaches a limit $\leqq \pi$, and (1) shows that this limit is precisely π. Hence

$$\sum_{n=1}^{\infty}\alpha_n = \tfrac{1}{2}(\pi - \theta_1). \tag{3}$$

If the first circle with center $(\theta_1, 1)$ is tangent at P_0 to the preceding arch of the cycloid, the common normal at P_0 has the slope $-(1-\cos\theta_0)/\sin\theta_0 = -\tan\frac{1}{2}\theta_0$. Here $-\pi<\theta_0<0$, and we find for $(\theta_1, 1)$ the values

$$\theta_0 - \sin\theta_0 + \cos\tfrac{1}{2}\theta_0,\ 1 - \cos\theta_0 - \sin\tfrac{1}{2}\theta_0.$$

The resulting two equations give $\theta_0=-\pi/3$, $\theta_1=3^{1/2}-\pi/3$; whence

$$6\sum_{n=1}^{\infty}\alpha_n = 3(\pi - \theta_1) = 4\pi - 3^{3/2}.$$

A Note by Otto Dunkel. The required relations may be easily obtained from a figure. Let the rolling circle with center C_1 be tangent to the base at M_1 and tangent externally to the preceding arch at P_0. Let the circle with center C_0 which passes through P_0 and determines that point be tangent to the base at M_0. Since M_0 is the instantaneous center of rotation for P_0 on the circle, M_0P_0 is normal at P_0 to the preceding arch, and M_0P_0 passes through C_1. In the rectangle $M_0M_1C_1C_0$, $C_0P_0=C_1P_0=\frac{1}{2}M_0C_1$; hence the triangles $M_0P_0C_0$ and $M_1P_0C_1$ are equilateral. We have then $M_0O=\pi/3=-\theta_0$, $M_0M_1=3^{1/2}$. The circle C_1 cuts the cycloid again in the nearer point P_1 and in the more distant point P_2, where $\theta_1=\text{arc } M_1P_1=OM_1=3^{1/2}-\pi/3$. Let C_2 be the point on C_0C_1 extended so that $P_2C_2=P_2C_1$; then C_2 is the center of the generating circle for P_2. Let this circle be tangent to the base at M_2. We see from the figure that $M_1M_2=\theta_2-\theta_1=2\sin\theta_2$, and that $\angle M_1C_1P_2=\angle M_2C_2P_2=\theta_2<\pi$. If P_1' on circle C_1 is symmetric to P_1 with respect to M_1C_1, then $\angle P_1'C_1P_2=\theta_2-\theta_1$, and hence $\angle P_1'P_1P_2=\alpha_1=\frac{1}{2}(\theta_2-\theta_1)$. The rest easily follows.

A solution of 3488 by Mannis Charosh was overlooked in the list of solvers [1932, 242]. A solution of 3483 by A. E. Fuller was received too late for credit at the proper time.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ooio.

Mr. R. C. Bullock, of the University of North Carolina, has been appointed to an acting professorship at Lambuth College, Jackson, Tenn.

Dr. Fay Farnum has been promoted to an assistant professorship, at Washington Square College, New York University.

Dr. Eberhard Hopf has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Professor Alfred Hume, of the University of Tennessee, has been reinstated as chancellor of the University of Mississippi.

Assistant Professor F. W. John has been promoted to an associate professorship at Washington Square College, New York University.

Assistant Professor A. L. Meder, of the department of mathematics at the New Jersey College for Women, has been appointed acting dean of that college.

Professor O. M. Norlie, of Hartwick College, has been appointed director of the new graduate school of Hartwick Seminary, Brooklyn.

Dr. Jerry H. Service, of Henderson State College, has been appointed assistant professor of mathematics at Michigan College of Mining and Technology.

Dr. W. R. Thompson has been appointed research assistant in pathology at Yale University.

Assistant Professor Marion M. Torrey, of Goucher College, has been promoted to an associate professorship.

Rev. Joseph Wilczewski, of St. Louis University, has been appointed to a professorship at Mount St. Michael.

Miss Marguerite L. Zeigel has been appointed to a position in the department of mathematics and physics at Lander College, Greenwood, S.C.

The following appointments to instructorships have been made:

Long Beach Junior College, Mr. A. L. Buckman.
Michigan College of Mining and Technology, Mr. Nels Johnson.
University of Minnesota, Dr. C. H. Fischer.
New York University, University Heights Branch, Mr. F. C. Hall.
Tulane University, Mr. J. F. Thomson.

Professor R. L. Green, professor emeritus of mathematics at Stanford University, died on Nov. 19, 1932, in his seventy-first year. He was a charter member of the Mathematical Association.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. B. Carver, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. Johnson, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

Illinois.
Indiana, Bloomington, May 5-6.
Iowa, Cedar Rapids, Apr. 21-22.
Kansas, Topeka, Feb. 11.
Kentucky, May.
Louisiana-Mississippi, Ruston, La., Mar. 3-4.
Maryland-District of Columbia-Virginia, Charlottesville, Va., May 13.
Michigan, Ann Arbor, Mar. 18.
Minnesota.
Missouri.
Nebraska, Lincoln, Apr. 28.
Ohio, Columbus, Apr. 6.
Philadelphia, Philadelphia, Dec. 2.
Rocky Mountain, April.
Southeastern, Athens, Ga., March.
Southern California, Claremont, Mar. 4.
Texas, Dallas, Feb. 11.
Wisconsin, Beloit, Apr. 8.

Affiliated Organizations: The New England Association of Teachers of Mathematics. The National Council of Teachers of Mathematics.

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THE SEVENTEENTH ANNUAL MEETING OF THE ASSOCIATION

The seventeenth annual meeting of the Mathematical Association of America was held at Atlantic City, New Jersey, on Tuesday and Wednesday, December 27 and 28, 1932, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Two hundred eighty-nine were in attendance at the meetings, including the following one hundred seventy-seven members of the Association:

C. R. ADAMS, Brown University
V. W. ADKISSON, University of Arkansas
R. P. AGNEW, Cornell University
BEATRICE AITCHISON, Johns Hopkins University
R. C. ARCHIBALD, Brown University
H. E. ARNOLD, Wesleyan University
C. S. ATCHISON, Washington and Jefferson College
FRANK AYRES, JR., Dickinson College

R. W. BABCOCK, Kansas State Agricultural College
CLARA L. BACON, Goucher College
N. H. BALL, Princeton University
ETHELWYNN R. BECKWITH, Milwaukee-Downer College
A. A. BENNETT, Brown University
G. D. BIRKHOFF, Harvard University
ARCHIE BLAKE, U. S. Coast and Geodetic Survey
HENRY BLUMBERG, Ohio State University
JOSEPH BOWDEN, Adelphi College
JULIA W. BOWER, University of Chicago
N. R. BRYAN, University of Maine
R. W. BURGESS, Western Electric Company
J. H. BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota
W. E. BYRNE, Virginia Military Institute

S. S. CAIRNS, Lehigh University
W. D. CAIRNS, Oberlin College
R. H. CAMERON, Cornell University
B. H. CAMP, Wesleyan University
P. A. CARIS, University of Pennsylvania
MILDRED E. CARLEN, Brown University
W. B. CARVER, Cornell University
G. G. CHAMBERS, University of Pennsylvania
W. F. CHENEY, Jr., Connecticut Agricultural College
A. B. COBLE, University of Illinois
ABRAHAM COHEN, Johns Hopkins University
J. B. COLEMAN, University of South Carolina
T. F. COPE, Marietta College
LENNIE P. COPELAND, Wellesley College
C. H. CURRIER, Brown University
E. H. CUTLER, Lehigh University

TOBIAS DANTZIG, University of Maryland
MARGUERITE D. DARKOW, Hunter College
H. T. DAVIS, Indiana University
J. E. DAVIS, Drexel Institute
F. F. DECKER, Syracuse University
C. E. DIMICK, U. S. Coast Guard Academy
H. A. DOBELL, New York State College for Teachers
J. L. DORROH, Princeton University
B. F. DOSTAL, University of Florida
ARNOLD DRESDEN, Swarthmore College
MELVIN DRESHER, Lehigh University
L. A. DYE, Cornell University

L. P. EISENHART, Princeton University
G. C. EVANS, The Rice Institute
G. W. EVANS, Lynn, Mass.
H. S. EVERETT, University of Chicago

F. J. FEINLER, St. Peter's Church, Loudonville, Ohio
IRVING FISHER, Yale University
M. M. FLOOD, Princeton University
TOMLINSON FORT, Lehigh University
T. C. FRY, Bell Telephone Laboratories

A. S. GALE, University of Rochester
F. J. GERST, Loyola University, Chicago, Ill.
D. C. GILLESPIE, Cornell University
J. S. GOLD, Bucknell University
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Dept., Washington, D.C.
H. V. GUMMERE, Haverford College
HARRY GWINNER, University of Maryland

C. H. Harry, Johns Hopkins University
W. L. Hart, University of Minnesota
E. R. Hedrick, University of California at Los Angeles
Robert Henderson, Equitable Life Assurance Society
T. R. Hollcroft, Wells College
I. O. Horsfall, Cornell University
E. V. Huntington, Harvard University

Louis Ingold, University of Missouri
M. H. Ingraham, University of Wisconsin

Dunham Jackson, University of Minnesota
R. P. Johnson, Carnegie Institute of Technology
B. W. Jones, Cornell University

Ralph Keffer, Aetna Life Insurance Company
A. J. Kempner, University of Colorado
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H. F. MacNeish, Brooklyn College
Dorothy McCoy, Belhaven College
N. H. McCoy, Smith College
D. L. McDonough, Overbrook High School
A. E. Meder, Jr., New Jersey College for Women
W. M. Miller, Tufts College
H. H. Mitchell, University of Pennsylvania
E. C. Molina. American Tel. and Tel. Co.
Ethel I. Moody, Sweet Briar College
C. N. Moore, University of Cincinnati
Eugenie M. Morenus, Sweet Briar College
W. K. Morrill, Johns Hopkins University
Richard Morris, Rutgers University
Marston Morse, Harvard University
G. W. Mullins, Barnard College
F. D. Murnaghan, Johns Hopkins University

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W. H. Roever, Washington University
C. F. Roos, American Association
M. F. Rosskopf, Brown University

F. H. Safford, University of Pennsylvania
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Pincus Schub, Gratz College
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C. H. YEATON, Oberlin College

OSCAR ZARISKI, Johns Hopkins University

The sessions of the American Association for the Advancement of Science began on Tuesday evening with an address on "The aims of anthropological research" by Doctor Franz Boas, retiring president, and the general reception following this in the Vernon Room of Haddon Hall. Various general addresses of interest to mathematicians included one by Professor Harlow Shapley of Harvard University on "Fact and fancy in cosmogony" Wednesday evening; and the first Maiben Lecture by Doctor H. N. Russell of Princeton University on "The constitution of the stars" Friday evening. In the series of afternoon general lectures Professor D. C. Miller of Case School of Applied Science spoke on "The Pipes of Pan, old and new, and how the musical scale grew," and Professor David Eugene Smith of Columbia University on "Oriental book collecting."

The Council of the American Association met Tuesday afternoon and each morning thereafter for the transaction of general business, the Mathematical Association being represented by Secretary W. D. Cairns and Professor Dunham Jackson. The Council elected Professor C. N. Moore vice-president and chairman of Section A for the year 1933, and Professor E. R. Hedrick secretary of Section A.

The mathematicians stayed for the most part at the Hotel Morton, where they had comfortable quarters and suitable rooms for the programs. The meeting places of the American Association and the related societies were conveniently near at hand and the Boardwalk with its attendant diversions furnished opportunity for such recreations as the individual members might choose.

The annual dinner was held Wednesday evening with an attendance of two hundred twenty-six, Professor Eisenhart, retiring president of the Society, presiding. Professor Archibald read extracts from a letter giving an account of a photo-electric congruence machine invented by Doctor D. H. Lehmer; by means of a series of discs and a photo-electric cell the machine determines the factorability of very large integers in an amazingly brief time. Professor Arnold Dresden, the newly elected president of the Association, humorously referred

to himself as the first socialist to be elected president of anything; he said that the Association has been doing a peculiar work and that it must continue to act to raise the standard of the work done by our undergraduates and to contribute materially to the high character which should be found also in high schools, rather than treat students entirely too much as little children with the consequence that they act mathematically as little children. Professor Virgil Snyder told of the origin of the International Mathematical Union with its purpose to realign the mathematical societies of the allies and associates, of the development of the Union in connection with the congress at Toronto in 1924, of the insistence by the majority that the membership of the Union should be reconstituted, and of the final decision at Zürich last summer to suspend the Union, to liquidate its affairs, and to ask a committee of the congress to work out *de novo* the consideration of the question as to whether there should be any such organization. In describing the new Institute for Advanced Study at Princeton, Professor Veblen said that a few years ago Mr. Bamberger decided to devote his wealth to some useful purpose and through the influence of Mr. Abraham Flexner decided to devote it to a project for the furtherance of pure scholarship. The plan contemplates a small group of mathematicians who will be free to do scientific work involving no bestowal of degrees, large liberty being allowed to the professors in conducting their activities in the form of seminars or formal lectures or none, as they may wish. It is expected that the students will be beyond the stage of the usual graduate student and that mathematicians will come to the Institute for limited periods of time for the purpose of doing some particular piece of work, for writing a book, etc. Professor Coble, the incoming president of the Society, called to mind his association with President Eisenhart in the past years and spoke of himself as deeply conscious of the high standard of the mathematical research and the well organized status of mathematical publication to the official direction of which he comes. He called attention to the meetings which are to be held at Chicago in connection with the Century of Progress and urged the mathematicians with their colleagues to cooperate with their attendance at the meetings next summer.

The American Mathematical Society held sessions on Tuesday evening, Thursday morning and afternoon for the reading of papers. On Tuesday morning the Society had a joint session with Sections A and K and the Econometric Society at which Doctor W. A. Shewhart of the Bell Telephone Laboratories gave a paper on "Probability as a basis for action." The tenth Josiah Willard Gibbs lecture on "Thermodynamics and relativity" was given by Professor R. C. Tolman of California Institute of Technology on Thursday afternoon under the joint auspices of the Society and the American Association. On Friday morning about one hundred fifty mathematicians went by train or automobile to Princeton where they inspected the sumptuous mathematics building, Fine Memorial Hall of Princeton University, and had lunch at Nassau Inn. At two o'clock the Society held a session in Fine Memorial Hall with an address by Professor J. von Neumann on "Application of the operational calculus to

mechanics" and a discussion by Professor G. D. Birkhoff of Harvard University and Doctor Eberhard Hopf of Massachusetts Institute of Technology. After the meeting a delightful and elaborate tea was served in the social rooms of Fine Hall by the ladies of the faculty and their assistants.

The program of the Mathematical Association consisted of two sessions on Tuesday afternoon and Wednesday morning, a joint session with Section K (Social and economic sciences) and the Econometric Society Tuesday afternoon, and a joint session with Section A and the Mathematical Society on Wednesday afternoon. Vice-Presidents Bussey and Evans presided respectively at the two sessions of the Association, and Professor Fort at the joint session Wednesday afternoon. The good services of the program committee, consisting of Professors Jewell C. Hughes, Oystein Ore, J. H. M. Wedderburn, and Arnold Dresden, Chairman, and those of the committee on arrangements under the chairmanship of Professor J. R. Kline, were recognized at the business meeting Wednesday noon. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

First Session of the Association

1. "Mathematical education and life insurance" by E. W. Marshall, vice president and actuary, Provident Mutual Life Insurance Co.

2. "Life insurance and its mathematical problems" by Ralph Keffer, assistant actuary, Aetna Life Insurance Co.

1. Mr. Marshall's paper discusses the educational aspects of the relation of mathematics and life insurance, particularly in the field of actuarial science, so that intelligent guidance can be given to students both educationally and vocationally. The mathematical requirements embraced by the examinations of the Actuarial Society of America and the American Institute of Actuaries were outlined, and deficiencies of college students as revealed by them pointed out. Only 30% of the college trained candidates for part I of the examinations passed it in 1931, and the situation was similar for other examinations involving pure mathematics. Apparently most of the college courses do not provide the knowledge and facility required to pass the examinations without much further preparation.

The qualifications of prospective actuaries were discussed. In addition to the mathematical background, they should possess analytical ability, a practical turn of mind, a creative imagination, good health to endure the long study for the examinations after office hours, and a personality which will help rather than hinder in human relations. The need for a cultural as well as a mathematical and vocational background was stressed.

The examination requirements were considered critically from both present and prospective viewpoints. They seem as stringent as they should be, except for a possible extension in the study of statistical method. A few suggestions were made regarding possible courses of study in college which will more closely meet the needs of the actuarial student. Actuarial work presents a satisfying com-

bination of theory, practise, and human contacts, which the well rounded mathematical education of the prospective actuary should take into account.

Following Mr. Marshall's address, Professor Camp asked whether parts of Hall and Knight's algebra as required in actuarial examinations are really needed in actuarial practice. The speaker replied that the requirements have been colored by the fact that the actuaries in the United States were originally English and Scotch. Others said that the development of actuarial formulas does depend on the material in Hall and Knight. Professor Rietz said that we have often heard that the examinations of the Actuarial Society were old-fashioned, and added that the actuaries have never claimed that they were setting what may be called fair examinations, but that they have been successful in selecting men. He advised an introduction into the theory of functions, advanced geometry, and the theory of surfaces by way of supplementing the direct preparation for actuarial work itself. Professor Hedrick added further criticisms of Hall and Knight, and noted that the social scientists had taken it into their own hands to pronounce on the kind of mathematics they need, as have also the S.P.E.E. and the American Chemical Society, the latter in a report in the January MONTHLY. In a further discussion of the manner of choosing young men, Mr. Molina said that his company gives very complete consideration to personal qualities, but assumes that a graduate of a good college has a certain minimum of attainment; we must allow for the fact that the qualifications differ extremely according to the department in which the young man is to be employed.

2. The paper by Mr. Keffer gave consideration to the fundamental assumptions regarding mortality and the application of the theory of probability to the development of certain problems of life insurance. Three points of view were discussed, (1) that of the individual purchaser of insurance, (2) that of the insurance company and (3) that of the state supervisory department.

Probabilities of death applicable to an individual are constantly changing and all problems from the individual viewpoint depend on the value of these probabilities at the moment. Mortality tables for an individual consist only of tables of the probabilities that the individual now living will die in specified future intervals of time. The averages used by insurance companies and state insurance departments for valuation are only correct when considered as a whole. The average legal reserve is quite different from the true reserve on an individual contract and the general misconception outside of actuarial circles that the average legal reserve will be applicable to an individual as a true reserve throughout life has led to certain inequities in the determination of guaranteed cash surrender values and in other respects.

The theoretical problem of reinsurance and limit of risk that a company might assume was discussed and it was shown that the risk to an insurance company of deviation of actual death claims from expected will be less if lives of different ages are insured than if all lives insured were of the same age and subject to mortality rates equal to the average. Attention was called to the dif-

ference in this problem for stock companies and for mutual companies. In a stock company the true expected mortality is estimated as accurately as possible and the risk of fluctuation in the actual death rate is borne by stockholders. In the mutual company the policyholders themselves assume this risk hence must pay a premium high enough to make a surplus very probable in every year. The company then has the problem of distributing this surplus to policyholders.

The paper closed with a discussion of mortality investigations and the limitations upon the interpretation of statistics.

Joint Session of the Association with Section K of the American Association and the Econometric Society

"The theory of money" by Professor G. C. Evans, The Rice Institute. This paper will appear in an early issue of *Econometrica*.

Second Session of the Association

1. "The life and work of T. H. Gronwall" by Professor J. A. Shohat, University of Pennsylvania.
2. "The Zürich Congress of Mathematicians" by Professor Virgil Snyder, Cornell University.
3. "Some interpolation series" by Professor J. L. Walsh, Harvard University.

1. On May 9, 1932, at the age of fifty-five, there died in New York City Thomas Hakon Gronwall, one of the leading mathematicians in this country, with an international reputation in the field of analysis.

Gronwall was born in 1877 in Sweden, the son of a gentleman farmer-engineer and a well educated mother from Värmland, the Swedish province which gave so many writers and poets during the last century, e.g., Selma Lagerlöf. Early attracted to mathematics, he fortunately came in contact with Mittag-Leffler, whose inspiring influence as teacher and scholar is so well known. Graduated from Upsala at twenty-one, Gronwall first went to Berlin to study, there graduated in engineering, and finally crossed the Atlantic, presumably in 1904. He passed several years in practicing engineering, until in 1912 he definitely returned to his first love, mathematics. In the course of the next few years, he published several papers dealing with difficult and important analytical problems, which established his reputation here and abroad as a man of broad training and erudition and a skillful master of the refined modern analytical tools.

Essentially a research man, Gronwall spent two years in teaching at Princeton University, and later a few years in Washington as mathematical expert on the technical staff of the chief of ordnance. For the last five years of his life, Gronwall was associate in the department of physics of Columbia University, generously giving of his vast analytical skill and knowledge to physics and physical chemistry in cooperation with Professor V. K. La Mer.

Gronwall published 85 papers in leading European and American mathematical journals, and 24 abstracts of papers presented to the American Mathematical Society. His publications show broad scientific interests in analysis, geometry, theoretical physics, physical chemistry, ballistics, etc. His best papers deal with Laplace and Legendre Series (convergence, summability). Here many of his papers constitute final and elegant answers to important problems, a stimulating source for research workers. In the words of Professor La Mer, "His achievement will remain as a worthy monument of the service which mathematics can render physics and chemistry."

2. Professor Snyder gave an informal talk on the recent congress of mathematicians held at Zürich, September 5–12, 1932, and traced the history of the development of congresses.

The last decade of the nineteenth century was an unusually active one mathematically. It witnessed the organization of the German and the American mathematical societies, started the encyclopedia of the mathematical sciences, and provided for the international catalogue of scientific literature.

The congress held at Chicago in connection with the Columbian exposition, though attended by only 44 persons, was significant. Europeans returning from that experience began to formulate plans for an international congress which was to be a regularly periodic affair. The first was held at Zürich in 1897, with a very informal organization. The purpose announced was "to further personal relations among the mathematicians." 204 persons attended, 12 general addresses were given and 34 sectional papers read. All these papers were published in the Proceedings of the Congress. At the close of the session those present voted to hold the next congress in Paris in 1900. This congress was attended by 280 persons, and 68 papers were read. The next, under the same informal auspices, was held at Heidelberg in 1904, attended by 396 persons and 76 papers were presented. Various undertakings were provided for, particularly the report on mathematical models and apparatus and the publication of the works of Euler, already begun by the Swiss, was supported. But the most important extension of the activities of the congress was the enlarged provision for entertainment and social intercourse.

At the Rome congress in 1908, the international commission on the teaching of mathematics was formally organized, and a detailed report on the progress of the encyclopedia was submitted. This congress was attended by 700, and the Proceedings filled three large volumes. The Cambridge congress in 1912 had a unique feature, as the 700 guests were housed in the colleges of the University; this permitted a very free social access among the participants.

An invitation to hold the next congress in Stockholm in 1916 was extended by Professor Mittag-Leffler, and enthusiastically accepted. But the war made it impossible and no congress was held.

In 1920 a congress of restricted internationality and under different auspices was held at Strasbourg with 200 active participants. At this meeting the international mathematical union was founded, a political organization under

governmental control, and severely limited as to membership. The Toronto congress was held under its auspices in 1924; over 500 persons were present and over two hundred and forty papers read. At the business meeting, the time and place of the next congress were not decided, but the matter left rather vaguely in the hands of the newly elected officers. It was held at Bologna in 1928 without the support of the union; over 1100 persons attended, 400 papers were read, and the Proceedings fill six splendid volumes. The congress at Zürich in 1932 was provided for under the early simple procedure. This was not so large as that at Bologna, but in consequence of the efficient arrangements for comfort and pleasure as well as for mathematics, the 900 present had a particularly pleasant and profitable time. The old purpose "to further personal relations among mathematicians" was admirably carried out.

It was decided by those present to hold the next congress at Oslo in 1936. The union was declared to be terminated.

3. Professor Walsh considered some results recently proved by him [Trans. Amer. Math. Soc., vol. 34 (1932), pp. 22–74; Proc. Nat. Acad., vol. 18 (1932), pp. 165–171] together with the following extension:

Let closed regions R_1, R_2, R_3 be given and let L denote the locus of all points t when α, β, z (varying independently) have R_1, R_2, R_3 as their respective loci and satisfy the relation

$$|(t, \alpha, z, \beta)| = \left|\frac{(t-\alpha)(z-\beta)}{(z-\alpha)(t-\beta)}\right| \geqq 1.$$

If the points $\alpha_1, \alpha_2, \cdots$ have no limit point exterior to R_1, if the points $\beta_1, \beta_2, \cdots$ have no limit point exterior to R_2, if $f(z)$ is meromorphic in the closed region L, and if all the poles of $f(z)$ in L belong to the sequence α_n, then the formal development

$$f(z) = a_0 + a_1\frac{z-\beta_1}{z-\alpha_1} + a_2\frac{(z-\beta_1)(z-\beta_2)}{(z-\alpha_1)(z-\alpha_2)} + \cdots, \quad \alpha_i \neq \beta_j,$$

found by interpolation to $f(z)$ in the points β_i, converges to $f(z)$ uniformly for z in R_3.

Meetings of the Board of Trustees

Seven members of the outgoing Board of Trustees and eight members of incoming Board were present at the Atlantic City meetings.

The following twenty-eight persons were elected to membership on applications duly certified:

N. C. Brown, M.S. (Maine) Instr., Wagner Coll., Staten Island, N.Y.

Melvin Dresher, Senior, Lehigh Univ., Bethlehem, Pa.

Myrtle Edwards, A.M. (Georgia) Head of Dept., State Coll., Bowdon, Ga.

F. J. Feinler, Pastor, St. Peter's Church, Loudonville, Ohio

J. R. Hadley, B.S. (Ohio State) Teacher, Mt. Sterling School, Mt. Sterling, Ohio; Grad. student, Ohio State Univ., Columbus, Ohio.

C. H. Harry, Ph.D. (Johns Hopkins) Instr., Johns Hopkins Univ., Baltimore, Md.

M. C. Hartley, Ph.D. (Illinois) Asst., Univ. of Illinois, Urbana, Ill.

E. H. C. Hildebrandt, Ph.D. (Michigan)

Asst. Prof., DePauw Univ., Greencastle, Ind.

R. D. JAMES, Ph.D. (Chicago) National Research Fellow, Calif. Inst. of Tech., Pasadena, Calif.

G. R. KAELIN, A.M. (California) Instr., Los Angeles Jr. Coll., Los Angeles, Calif.

RALPH KEFFER, A.M. (Wisconsin) Asst. Actuary, Aetna Life Ins. Co., Hartford, Conn.

EVELYN M. KENNEDY, A.M. (Cincinnati) Laws Fellow, Grad. School, Univ. of Cincinnati, Cincinnati, Ohio.

C. F. LUTHER, Ph.D. (Stanford) Instr., Stanford Univ., Stanford University, Calif.

CHRISTINE H. MACMARTIN, A.B. (Northwestern) Teacher, Deerfield Shields Twp. High School, Evanston, Ill.

H. W. MARCH, Ph.D. (Munich) Prof., Univ. of Wisconsin, Madison, Wis.

MORRIS MARDEN, Ph.D. (Harvard) Asst. Prof., Univ. of Wisconsin, Exten. Div., Milwaukee, Wis.

SISTER MARY CORDIA, Ph.D. (Johns Hopkins) Prof., Notre Dame Coll., Baltimore, Md.

A. B. MEWBORN, B.S. (Arizona) Instr., Univ. of Arizona, Tucson, Ariz.

HUGH MUNN, Student, Univ. of New Mexico, Albuquerque, N. M.

R. J. MUNRO, B.S. in M.E. (Iowa) Instr., Mech. Engg., Univ. of New Mexico, Albuquerque, N.M.

MARDELLE NEWSOM, A.B. (Wheaton) Teacher, High School, Poplar Grove, Ill.

IRENE PRICE, Ph.D. (Indiana) Instr., State Teachers Coll., Oshkosh, Wis.

ETHEL A. RICE, M.S. (Colorado) Pampa, Texas

H. N. SCHMELLNER, M.S. (New York Univ.) Grad Asst., West Virginia Univ., Morgantown, W. Va.

I. J. SCHOENBERG, Ph.D. (Jassy) Asst., Univ. of Jassy, Jassy, Rumania. *3 Shaler Lane, Cambridge, Mass.*

LAURA N. TURNER, A.M. (Michigan) Asso. Prof., Prairie View State Coll., Prairie View, Texas

R. N. WALTER, Student, Brooklyn Coll., Brooklyn, N. Y.

C. L. WILSON, M.E. (Kansas State) Prof. Mech. Eng., Prairie View State Coll., Prairie View, Texas

The financial report of the Secretary-Treasurer for the year 1932 was accepted. Professor Slaught for the finance committee had examined the report; Professors A. A. Bennett and W. L. Hart examined the report and the evidences of assets and declared them satisfactory. The Trustees voted to transfer from the general treasury to the General Endowment Fund, Liberty Bonds \$1000, Iowa Railway and Light Co. 5% bonds \$3000, and a Texas Power and Light Co. 5% bond \$885.

After a thoughtful consideration of papers published in the triennium 1929–31, the committee on the Chauvenet Prize, consisting of Professors C. C. MacDuffee, Virgil Snyder, and W. B. Ford, chairman, recommended that the prize of \$100 be awarded to Professor G. H. Hardy of the University of Cambridge, England, for his paper entitled "An introduction to the theory of numbers" which appeared in the Bulletin of the American Mathematical Society, Vol. 35 (1929) pages 778–818. The Trustees voted to adopt this report.

The Trustees approved the by-laws of the new Wisconsin Section and expressed their pleasure at the organization of this section.

It was voted to accept with thanks the invitation from the authorities of the University of Pittsburgh and Carnegie Institute of Technology and the teachers in the departments of mathematics to hold the annual meeting of the Association in Pittsburgh during the Christmas holidays of 1934, along with the meetings of the American Association and the Mathematical Society.

The Trustees voted to reappoint Mrs. Anna Pell Wheeler and Professor H. S. Vandiver as associate editors of the Annals of Mathematics, representing the Association, for the year 1933.

The following were appointed associate editors of the Monthly for the year 1933, as nominated by Professor Carver:

W. F. Cheney	R. E. Gilman	R. E. Sanger
N. A. Court	R. A. Johnson	D. E. Smith
Otto Dunkel	B. W. Jones	J. H. Weaver
B. F. Finkel	J. R. Musselman	F. M. Weida.
	H. L. Olson	

Some other items of business were discussed which had to do with (1) a possible modification of conditions of eligibility for the Chauvenet Prize; (2) the difficult situation involved in adapting the training of candidates for the Ph.D. in mathematics to teaching positions in secondary schools and junior colleges; (3) ways and means to finance the publication of one or more monographs under the auspices of the Carus Monograph Fund; (4) a slight modification of the arrangement between the Annals of Mathematics and the Mathematical Association; and, (5) tentative plans for the Chicago meetings in the week of June 19, 1933.

Annual Business Meeting of the Association

The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He reported also the deaths of the following members:

Anna H. Andrews, Teacher, High School, Hartford, Conn. (March 11, 1932)
L. A. Bauer, Director emeritus, Dept. of Terrestrial Magnetism, Washington, D.C. (April 12, 1932)
W. M. Brodie, Professor of mathematics, Virginia Polytechnic Institute, Blacksburg, Va. (April 1932)
A. B. Chace, Chancellor, Brown University. (February 28, 1932)
C. N. Dickinson, Professor of mathematics, Hollins College (May 25, 1932)
J. E. Donahue, Associate professor of mathematics, University of Vermont (August 13, 1932)
J. C. Fields, Professor of mathematics, University of Toronto (August 9, 1932)
J. D. Grant, Assistant in mathematics, University of Illinois (July 9, 1932)
R. L. Green, Professor of mathematics emeritus, Stanford University (November 19, 1932)
A. H. Jekel, President, Colorado Clay and Mining Co., Boulder, Colo. (March 8, 1932)
O. D. Kellogg, Professor of mathematics, Harvard University (August 27, 1932)
Lylah Kryder, Instructor in mathematics, Brooklyn College (March 29, 1932)
C. A. Pettersen, Assistant principal, Schurz High School, Chicago, Ill. (March 20, 1932)
C. G. Simpson, Professor of mathematics, College of Electrical Engineering, Milwaukee, Wis. (February 5, 1932)
L. C. Walker, Ceresco, Nebraska (December 15, 1930)
H. A. West, Professor of mathematics, Marion College (January 17, 1932)
Heinrich Wieleitner, Director, Neues Realgymnasium, Munich (December 27, 1931)
Rose B. Wood, Formerly instructor in mathematics, Greenville Woman's College, (July 28, 1931)
J. W. Young, Professor of mathematics, Dartmouth College (February 17, 1932)

The election of officers for the year 1933 resulted in the following, as reported by the tellers, Professors L. L. Smail and C. H. Yeaton:

For President: W. H. Bussey, 204 votes; Arnold Dresden, 245 votes.

For Vice-Presidents: A. A. Bennett, 240 votes; A. B. Coble, 220 votes; Tomlinson Fort, 194 votes; E. B. Stouffer, 225 votes.

For additional members of the Board of Trustees, to serve until January 1936: C. R. Adams, 181 votes; B. F. Finkel, 245 votes; W. L. Hart, 246 votes; E. V. Huntington, 280 votes; D. N. Lehmer, 226 votes; Mayme I. Logsdon, 203 votes; E. J. Moulton, 228 votes; W. M. Whyburn, 150 votes.

The following were accordingly declared elected:

President: ARNOLD DRESDEN, Swarthmore College.

Vice-Presidents; A. A. BENNETT, Brown University, E. B. STOUFFER, University of Kansas.

Additional members of the Board of Trustees: B. F. FINKEL, Drury College; W. L. HART, University of Minnesota; E. V. HUNTINGTON, Harvard University; E. J. MOULTON, Northwestern University.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 12, 1932

RECEIPTS

Balance Dec. 14, 1931		$10,551.10
1930 indiv. dues	97.10	
1931 indiv. dues	346.40	
1931 instit. dues	56.00	
1932 indiv. dues	6,529.52	
1932 instit. dues	779.50	
1932 subscriptions	803.21	
Initiation fees	154.00	
Life memberships	116.97	
Advertising	363.50	
Sale *Register*	11.00	
Sale copies of MONTHLY	264.10	
Sale Catalog	8.00	
Sale First Carus Mon.	22.50	
Sale Second Carus Mon.	21.25	
Sale Third Carus Mon.	22.50	
Sale Fourth Carus Mon.	26.25	
Sale Rhind Papyrus	262.00	
Carrying charges Papyrus	16.84	
Received *Annals* subscriptions	5.00	
Refund on section expense	12.47	
Int. Oberlin Savgs. Bk.	103.26	
Int. Peoples Bkg. Co.	71.60	

EXPENDITURES

Publisher's bills (Nov. '31–Oct. '32)		$ 5,385.04
President's office		10.74
Committee on Funds		5.25
Manager's office		42.72
Editor-in-Chief's office, 1931		105.00
Editor-in-Chief's office 1932		439.30
Committee on Membership		183.04
Register		550.78
Secretary-Treasurer's office		
Postage	$ 366.83	
Bond	5.00	
Safety deposit	4.00	
Office supplies	92.65	
Express, tel., etc.	80.85	
Clerical work	1,757.58	
Printing	256.09	
Library expense	99.02	
Ins. back copies of MONTHLY	30.20	
Paid copies MONTHLY	97.09	
New Orleans meeting	100.00	
Los Angeles meeting	12.75	
Refund subscriptions	14.40	2,916.46
Annals subvention		300.00

Int. Liberty Bonds....	85.00	
Int. Hardy Fund......	120.00	
Int. certifs. of deposit..	3.73	
Int. from Genl. Endowment Fund Bonds...	357.50	
Int. Carus Fund......	100.00	
Int. Chace Fund......	181.67	
Int. Chauvenet Fund..	22.08	
Int. from investment of current funds.......	171.12	
Profit from conversion of bonds (Chace, Chauvenet and current funds).........	20.70	11,154.41
Total 1932 receipts..............		$21,705.51
Total expenditures..............		$11,125.11
Balance to the end of 1932 business.		10,580.40
Received on 1933 business......		619.57
Book balance Dec. 12, 1932.......		$11,199.97

Paid to sections from initiation fees........................	205.51
Forwarded *Annals* subscriptions	5.00
Paid *Annals* subscriptions......	5.00
Paid B. F. Finkel int. Hardy Fund................	120.00
Sustaining memb. in American Math. Society...............	100.00
Transfer to Chace Fund........	558.75
Transfer to Carus Fund........	192.52
Total expenditures............	$11,125.11
Cash on hand.................	8.98
Checking account.............	1,787.17
Oberlin Savgs. Bk. acct.........	2,659.52
Peoples Bkg. Co. acct..........	1,776.84
Liberty Bonds................	1,000.00
Iowa Elec. Lt. & Pow. Co. 5% Bonds.....................	3,000.00
Texas Power & Light Co. 5% Bonds, market value.........	885.00
Part certif. of deposit..........	82.46
Bank balance Dec. 12, 1932....	$11,199.97

CARUS MONOGRAPH FUND

Balance Dec. 14, 1931....		$5,925.46
Receipts: Sales....	$ 92.50	
Interest....	227.51	320.01
		$6,245.47
Expenditures: Accrued interest on new bond....		23.19
		$6,222.28
Certificates of deposit....	$4,452.28	
Cleveland Trust Securities Co. Gold Bond....	1,000.00	
Pacific Power & Light Co. 5% Bond, market value....	770.00	
Balance December 12, 1932....		$6,222.28

ARNOLD BUFFUM CHACE FUND

Balance Dec. 14, 1931....		$4,094.98
Receipts: Sales....	262.00	
For carrying charges....	16.48	
Interest....	204.29	
Profit from conversion of matured bonds....	4.60	487.37
		$4,582.35
Iowa Elec. Lt. & Pow. Co. 5% Bond....	1,000.00	
Western United Gas and Elec. Co. Bonds....	2,370.00	
Certificates of deposit....	1,161.08	
Cash in bank....	51.27	
Balance Dec. 12, 1932....		$4,582.35

Chauvenet Prize Fund

Balance Dec. 14, 1931		$ 595.00
Interest	22.08	
Profit from conversion of matured bond	2.30	24.38
		$ 619.38
Iowa Lt. & Pow. Co. 5% Bond	500.00	
Cash in bank	119.38	
Balance December 12, 1932		$ 619.38

Life Membership Fund

Liability on life memberships Dec. 14, 1931	$ 556.99
Received on life membership payments	116.97
	$ 673.96
To be transferred to current funds, surplus	15.83
Liability on life memberships as of Jan. 1, 1933	$ 658.13

General Endowment Fund

Balance Dec. 14, 1931		$8,000.00
Depreciation Cleveland Terminals Land Trust Certificate		300.00
		$7,700.00
Liberty Bond	$1,000.00	
Land Trust Certificate	700.00	
Cleveland Trust Investment Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Northwestern Electric Co. Bonds	3,000.00	
Balance Dec. 12, 1932		$7,700.00

Of the funds on hand, indicated in the first division of this financial report, $51.27 belongs to the Arnold Buffum Chace Fund, $119.38 belongs to the Chauvenet Prize Fund, and $658.13 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1933. Aside from these amounts, the various funds of the Association are carried in the form shown in the inventories under the exhibit above.

When the accounts were closed Dec. 12, 1932, there remained on the total business for 1932 the following items:

Bills Receivable

1932 individual dues	$200.00
Advertising	50.00
	$250.00

Bills Payable

Publisher's bills (Nov.-Dec. '32)	$1,150.00
Manager's office	30.00
Editor-in-Chief's office	80.00
Other editors' postage	30.00
Secretary-Treasurer's office	350.00
Annals subvention	75.00
Initiation fees due to sections	850.00
Chauvenet Prize Fund	119.38
Chace Fund	51.27
Life Membership Fund	658.13
	$3,393.78

If to the balance on 1932 business shown in the report, $10,580.40, there be

added the bills receivable, \$250.00, and there be subtracted the estimated bills payable, \$3,393.78, there results an estimated final balance on 1932 business of approximately \$7,435, which represents the accumulated surplus in current funds. This is to be compared with the corresponding figure of \$6,040 for a year ago. The profit in this year's business is due chiefly to a gratifying amount of interest on general endowment and current funds, over seven hundred dollars, and to the continued carefulness and devotion of numerous members who serve the Association with a minimum of expense. In spite of the troublous times the Association and its activities are receiving the continued support of its members.

W. D. CAIRNS, *Secretary-Treasurer*

ARNOLD BUFFUM CHACE

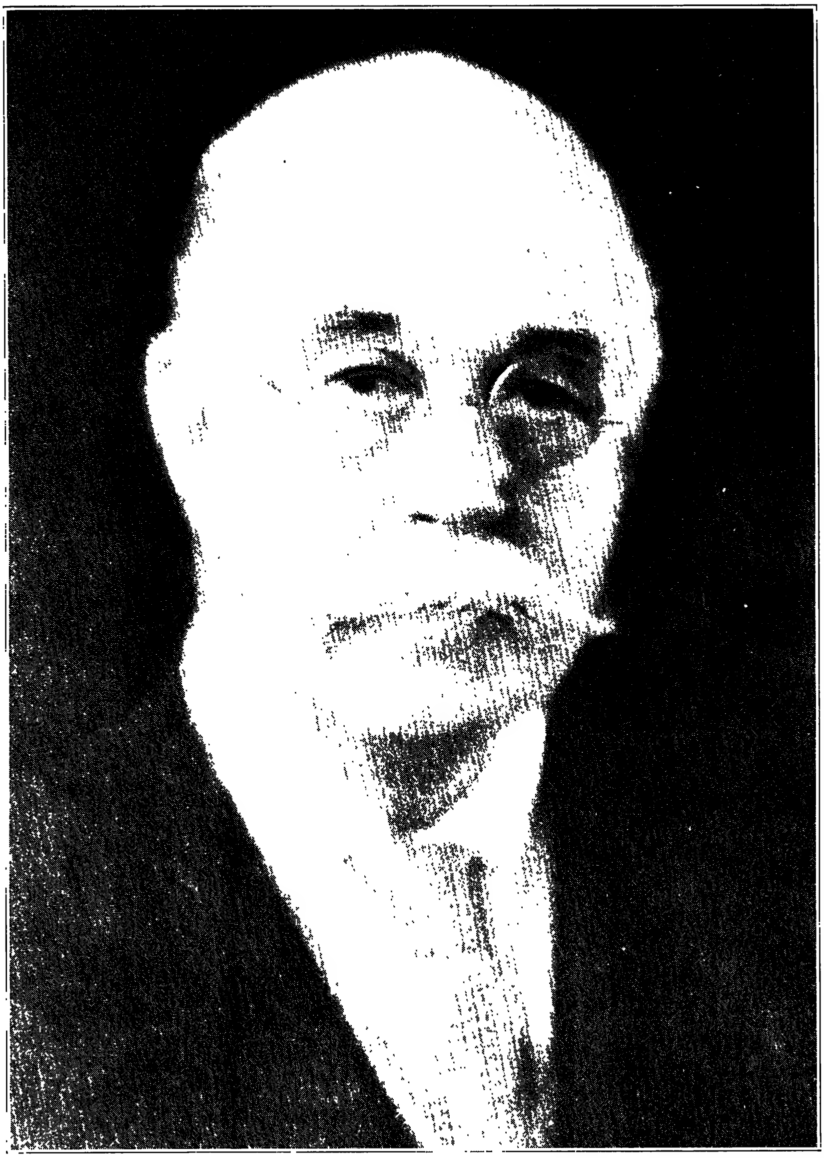
By R. C. ARCHIBALD, Brown University

Arnold Buffum Chace, friend and benefactor of the Mathematical Association, for fifty-six years a member of the Corporation, for eighteen years the Treasurer, and for twenty-five years the Chancellor of Brown University, died at his home in Providence, Rhode Island, on 28 February 1932, in the eighty-seventh year of his age.

He was born of Quaker parents at Valley Falls, Rhode Island, on November 10, 1845. After taking his Bachelor of Arts degree at Brown University in 1866, he spent two years in carrying on his scientific studies, the first at Harvard University, and the second, with special reference to chemistry, at the École de Médecine, Paris, in the laboratory of Professor Wurtz. During the following year he was an instructor of chemistry at Brown University, but his brilliant and inquiring mind was already reaching out in other directions, since the young instructor went regularly to Harvard to hear lectures in biology.

All hopes and plans for an academic career were, however, dashed after only this one year of apprenticeship. A death in his family suddenly left him the only surviving member to care for important business interests in connection with the Valley Falls Company; this care continued to be exercised for over sixty years. In writing to the Chancellor about a year ago, a life-long friend, the late great engineer John R. Freeman, expressed himself as follows: "I now venture to say to you, what I have often said to mutual friends, that it was a great misfortune to American science that you inherited a cotton mill, and did not continue as a mathematical physicist or chemist."

But such onerous business demands were wholly incapable of suppressing the cotton manufacturer's seething spirit of scientific inquiry. He had become interested in the subject of quaternions as developed by that extraordinary Irish genius, Sir William Rowan Hamilton, and he wished for further guidance. What was more natural than that he should try to get the help of Professor Benjamin Peirce of Harvard with whom mathematical research in American



Arnold Buffum Chace, 1845-1932

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universities may be said to have begun?[1] His request was met with cordial assent. In his reminiscences[1] of 1925 the Chancellor wrote:

"I went to his house one afternoon a week for nearly a year and, sitting in his pleasant study before an open fire, I would show him the work that I had done in the previous week, and he, an old man, and I, a young man, discussed quaternions and other matters in a most friendly way.

"He was one of the most stimulating men that I have ever known. I can picture him now with his large noble brow, his beautiful white hair, his flashing eyes, his animated but kindly face, and most inspiring personality."

The outcome of this contact was the publication in the *American Journal of Mathematics*, for 1879, of an excellent memoir on "A certain class of cubic surfaces treated by quaternions."

It was about this time that he became a director of the Manufacturers' Mutual Fire Insurance Company of which he was to remain a director for fifty years. Questions which arose here in connection with the mathematical theory of probabilities interested him constantly.

Instruction in modern and higher mathematics at Brown commenced with the advent of Henry Parker Manning. During the five years 1892–97, he gave courses on the Theory of Functions, on Higher Algebra, on Higher Analytic Geometry, and on Non-Euclidean Geometry; and in a class discussing such topics the Chancellor was a regular attendant for practically all of that time. Indeed for two years longer he used to meet Professor Manning once a week to talk about things mathematical, particularly the geometry of four dimensions.

The impulse to the crowning intellectual effort of his career was given in 1910 when he and Mrs. Chace made a trip to Egypt where they "became much interested in its monuments and literature." Two years later, when in England, they purchased British Museum copies of the Book of the Dead, and of the Rhind Mathematical Papyrus. Inspired by his interest both in the history of mathematics and in the ancient Egyptian civilization, he soon commenced an intensive study of the latter. He had purchased various works dealing with hieratic and hieroglyphic writing. After long study together, Mrs. Chace commenced the tremendous task of copying the hieratic from the British Museum copy, and of drawing under each sign the corresponding hieroglyphic transcription. (Mrs. Chace's skill in drawing was a great asset in such a task.) Under this transcription was put the transliteration; since this transliteration, in following the original text, was from right to left, a fourth row under the other three repeated the transliteration from left to right and in the fifth row came the literal English translation. The Chancellor's original idea was that some day he would complete an edition of the papyrus for depositing in the Brown University Library, the consultation of which would enable one unfamiliar with hieratic to be able to find out what each symbol meant.

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[1] Compare this *Monthly*, vol. 32, 1925, p. 1–30; reprinted by the Open Court Publishing Company (1925) as a booklet with additional material.

After ten years, great progress had been made in their work and American scholars were urging publication of the study. But the Chancellor was already seventy-seven years old, and his natural feeling was that an expert Egyptologist and a younger mathematician should be associated with him in revising the work and in putting it into a form worthy of scholarly ideals. This resulted in the happy collaboration of Doctor L. S. Bull, associate director of the Egyptian department of the Metropolitan Museum of Art, and of Professor Manning, who became such an expert in reading hieratic script that he was sometimes sharper-eyed than Doctor Bull himself. As the work evolved under the hands of these three, its scope increased. The Trustees of the British Museum granted permission to the Chancellor to publish a photographic facsimile of the papyrus. This alone would have made the work of great value to scholars. But a modern free translation and elaborate mathematical commentary were also prepared, and especially in the latter was the Chacellor able to make a large number of interesting contributions. A student of the Rhode Island School of Design was pressed into service to make 109 plates in two colors; and philological and critical commentaries on the contents of these plates dealt with matters primarily of interest to the Egyptologist.

The papyrus contains a fascinating variety of material. As the work on it was brought to a close, the Chancellor wrote:

"The pleasure derived from such study has been great and especially have I enjoyed the intimate knowledge which I thereby gained regarding the reasoning power of those ancient people. I venture to suggest that if one were to ask for that single attribute of the human intellect which would most clearly indicate the degree of civilization of a race, the answer would be, the power of close reasoning, and that this power could be best determined in a general way by the mathematical skill which members of the race displayed. Judged by this standard the Egyptian of the nineteenth century before Christ had a high degree of civilization."

The Chancellor decided that his ideals as to the distribution of his work would best be served by arranging for the Mathematical Association of America to be the publisher. He paid all bills for issuing the work in the finest form that an American press could turn out. The desire for this beauty of form was doubtless inspired by the fact that it was dedicated to the memory of Mrs. Chace. The proceeds from the sale of his work are to accumulate in an endowment fund of the Association, the Arnold Buffum Chace Fund, which now amounts to over four thousand dollars.

While the first volume was printed in 1927, it was not sent out until the second one was ready in November 1929, when the Chancellor was 84 years of age. In February 1928, shortly after the appearance of the first volume, a happy letter of felicitation signed by members of the faculty of Brown was sent to the Chancellor. His acknowledgment concluded with the following striking sentence: "How useful the work will be, I do not know, but it has interested me, and I think that that life is full in which one uses all of his facilities all the time."

Reviews by leading Egyptologists in England and Germany, and by mathematicians in Italy, Norway, and the United States, spoke of the work in terms of high praise. It was a contribution to the popularization of hieratic mathematical writing, a notable contribution to scholarship, and it contained much material dealt with to a degree of finality of enduring value.

From the time that the Chancellor began work on the papyrus, seventeen years had passed before final publication, but during those years his mind was ever alert for the acquisition of fresh ideas in other fields. During 1921–23, for example, he was a regular attendant at lectures on relativity given by members of the mathematics department at Brown. And every new number of *Science*, *Isis*, *Bulletin of the American Mathematical Society*, *American Mathematical Monthly*, and *Annals of Mathematics* was almost sure to suggest to his mind some question which he sought to discuss with an expert in that subject.

But the breadth of the Chancellor's interests, his zeal as a student, and the force of his personality were nowhere more strikingly shown than in his foundation and conduct of a non-scientific organization, namely, of the Greek Club, afterwards the Review Club, carried on without a break for forty-five years. The program he planned for the Greek Club, required eleven years, and called for the study of the writings and ideas of most of the better-known Greek and Roman authors. When the name was changed to the Review Club, the papers were of a more general nature. As president of the Club, the Chancellor continued to make careful preparation for every meeting, studying each announced topic so as to be in a position to make his own contribution to the evening.

Thus Mr. Chace was not only an extraordinarily successful manufacturer, banker, director of many companies, and college administrator, but also a man whose greater interest was in things of the spirit. He was a bookish man of wide research and information. He had a large frame, but nobility was the distinguishing mark of his bearing. He came near to dominating gatherings where he was present. The brilliancy of his perceptions, the singular clarity of his thought, his extraordinary ability to extract from a complicated mass of material what was essential, were indeed remarkable. But the detachment and serenity of his consideration of questions, even in the case of those concerning which he felt strongly, were outstanding and delightful characteristics. Members of his family[1] have told me that they never knew him to be ruffled. He had a wonderfully beautiful spirit.

Some one has said that God gave us memory so that we might have roses for December. One of my roses is memory of this spirit.

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[1] He had three sons and a daughter. His eldest son, Malcolm G. Chace, was at one time national tennis champion. The mother of Richard C. Tolman, professor of physics at the California Institute of Technology, was a sister of the Chancellor.

be associated with an affine transformation which merely scales down volumes (of the dimension of the space in which we are working) in a constant ratio, results such as equation (20) are readily extended to oblique systems by introducing the appropriate scale factor. The normal forms may be modified in the usual way, the "direction cosines" being replaced by ratios based on oblique projections, and perpendicularity by conjugacy in a quadric. However, the $(n-1)$ volumes in the denominator of equation (21) would not transform simply.

ON THE VALUATION OF LAND AWAITING CONVERSION TO A HIGHER USE

By H. A. BABCOCK, Chicago, Illinois

In and around every large American city there are, in the aggregate, very large areas of vacant land and land improved with obsolete and partially obsolete buildings. The owners of these properties are waiting for the growth of the city to absorb the vacant lands and to permit the conversion of lots supporting obsolete buildings to some higher[1] use. The *present* value of such land is of importance to these owners, to the tax assessor, and to those concerned with zoning regulations and the rehabilitation of blighted districts.

The problem of determining the present value of a large tract of ground, to be developed over a period of years, has been given attention recently.[2] The distinguishing characteristic of this type of valuation problem is that the present value is a reflection of a *future* utilization of the several parcels comprised in the tract, to take place progressively, immediate development of the entire area not being possible because of the absence of sufficient demand. It is the probability of such future progressive utilization which creates the present value. The incidental income from the undeveloped portions of such areas contributes but little to the value, it is the ultimate conversion to some higher use, parcel by parcel, which contributes the major portion of the present value.[3]

To solve such a problem it is necessary to ascertain the total area of the tract, estimate the rate of absorption or conversion, approximate the ultimate sales price or developed value per parcel and the incidental income per parcel, and to allow for the annual taxes to be levied against the unsold or undeveloped portions. The methods of determining or approximating these data will not be discussed here.

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[1] A "higher" use of a particular tract of land is one which develops a higher unit land value than the current use; it does not necessarily involve height of building or amount of building area and may not even require a building.

[2] *Report on the Economic and Engineering Feasibility of Regrading the Bunker Hill Area, Los Angeles*, Wm. H. Babcock & Sons, Real Estate Consultants & Valuators. Privately printed, June 1, 1931. Copies available in a number of university, municipal and other libraries.

[3] In the case of fully improved properties the reverse is true; it is the *immediate* earning expectancy (or utilization) which contributes the major portion of the present value and the ultimate conversion of the land to some other use which contributes but a minor portion.

Let it be assumed that no lands are added, during the conversion, to the area initially available for conversion, and that the rate of conversion, the net sales price per unit area, the incidental income per unit area, and the annual tax rate do not change during the process of conversion. Further, let it be assumed that the value of the unsold or undeveloped portion, at any time, as computed by applying these assumptions to the data, will be the value as determined by the tax assessor and that the tax rate will be applied to such values.[1]

The method of deriving the valuation formula consists of setting up the series of net receipts from the entire property, year by year, computing the present worth of each member of the series, and summing these present worths.

The calculation of the present worth of an amount of money to be received at a future date involves the use of an interest rate. In any actual valuation problem the uncertainties in the data and the predictions of future events are compensated by the selection of a suitable interest rate.[2]

Value of Entire Tract

Let the total area initially available for conversion be A, and the total number of years required to complete the conversion be N. Then A/N = area sold per year, and is constant by hypothesis. Let the net sales price per unit area be p, which is also constant by hypothesis. Then the constant net sales revenue per year will be Ap/N. Let n be a time variable taking on the positive integral values $n=0, \cdots, N$. Then the unsold area at the beginning of the $(n+1)$st year will be $A(1-n/N)$. If β is the annual incidental net income per unit area before taxes, constant by hypothesis, then the incidental income during the nth year will be $\beta A\,[1-(n-1)/N]$.

Let it be assumed that the annual tax will be levied on the actual value at the beginning of each year and paid one year later. Let the annual tax rate on full value be ρ and let the value of the unsold portion of area A at the beginning of the $(n+1)$st year be V_n. Then the taxes payable at the end of the nth year will be ρV_{n-1}.

Let the net receipts (sales receipts + annual incidental incomes − taxes) received at the end of the nth year be r_n, then

$$(1) \qquad r_n = \left[\frac{A}{N}p + \beta A\left(1 - \frac{n-1}{N}\right) - \rho V_{n-1}\right] = (P - nB - \rho V_{n-1}),$$

where $P = A\,[p + (N+1)\beta]/N$ and $B = A\beta/N$.

[1] It is not within the scope of this paper to discuss the validity of these assumptions or to show the effect, on the value, of variations from year to year in the rate of conversion, unit price, tax rate, and incidental income. This paper deals with the method of computing the value, at the beginning of and during the conversion period, assuming these propositions to be valid.

[2] In the limiting case of a riskless investment, that is, a case in which the predicated future events are certain to occur, the interest rate does not reduce to zero but to a rate called the "pure interest rate." In any actual problem, an interest rate is selected which is greater than this pure interest rate by an amount deemed sufficient to compensate for the uncertainties involved in the prediction of future events.

Now, the general formula for the initial (or present) value of any series of net receipts ($r_1, r_2, \cdots, r_n, \cdots, r_N$) is

$$V_0 = \sum_{n=1}^{N} r_n v^n, \tag{2}$$

where v = the present value of 1 due 1 year hence at the interest rate (or rate of return on value) i. By definition $v = 1/(1+i)$.

The value of the series after n of the payments have been received is

$$V_n = \sum_{m=1}^{N-n} r_{n+m} v^m, \tag{3}$$

where m is a time variable taking on the positive integral values $m = 1, \cdots, N-n$.

In the specific problem under consideration,

$$r_{n+m} = [P - (n+m)B - \rho V_{n+m-1}] \tag{4}$$

by (1), and substituting this value in (3),

$$V_n = \sum_{m=1}^{N-n} [P - (n+m)B - \rho V_{n+m-1}] v^m. \tag{5}$$

The tract will be sold out after N years, by hypothesis, and there will then be no further receipts, so that $V_N = 0$, and V_{N-1} will be the last value of V greater than zero. To determine the value of V_{N-1}, let $n = N-1$ in (5), hence

$$V_{n-1} = (P - NB - \rho V_{N-1})v$$

and

$$(1 + \rho v)V_{N-1} = (P - NB)v.$$

Now, let

$$\frac{1}{1+\rho v} = \mu, \text{ and } \mu v = u. \tag{6}$$

Then

$$V_{N-1} = \mu v(P - NB) = u(P - NB). \tag{7}$$

To determine the value of V_{N-2}, let $n = N-2$ in (5), hence

$$V_{N-2} = [P - (N-1)B - \rho V_{N-2}]v + [P - NB - \rho V_{N-1}]v^2.$$

By transposition of the term in V_{N-2} and substitution from (6), it can be shown that

$$V_{N-2} = u[P - (N-1)B] + u^2[P - NB]. \tag{8}$$

Similarly, by placing $n = N-3$ in (5), it can be shown that

$$V_{N-3} = u[P - (N-2)B] + u^2[P - (N-1)B] + u^3[P - NB]. \tag{9}$$

Eq. (9) can be written in the form

$$V_{N-3} = \sum_{m=1}^{3}[P - (N - 3 + m)B]u^m, \tag{10}$$

and can be generalized by letting s = the upper limit to give

$$V_{N-s} = \sum_{m=1}^{s}[P - (N - s + m)B]u^m. \tag{11}$$

Now, let $N - s = n$, then

$$V_n = \sum_{m=1}^{N-n}[P - (n + m)B]u^m = \sum_{m=1}^{N-n}(P - nB)u^m - \sum_{m=1}^{N-n}Bmu^m. \tag{12}$$

The equation

$$\sum_{m=1}^{N-n} [P - (n + m)B - \rho V_{n+m-1}]v^m = \sum_{m=1}^{N-n} [P - (n + m)B]u^m,$$

which has been inferred, may be proved by mathematical induction.

In (12), P and B are both constants, and n is constant so far as the summation is concerned, so that

$$V_n = (P - nB)\sum_{m=1}^{N-n}u^m - B\sum_{m=1}^{N-n}mu^m. \tag{13}$$

Before carrying out these summations it is advisable to examine the quantity u^m. By (6), u was defined as $\mu v = v/(1+\rho v)$, but since, by definition, $v = 1/(1+i)$ the subscript i may be used to indicate that v is at the interest rate i. Thus

$$u = \frac{v_i}{1 + \rho v_i} = \frac{\dfrac{1}{1+i}}{1 + \dfrac{\rho}{1+i}} = \frac{1}{1 + (i + \rho)} = v_{i+\rho}. \tag{14}$$

This is an important result inasmuch as it states that u is the present value of 1 due 1 year hence at the rate $(i+\rho)$.

The present value of 1 per annum for T years at the rate i, is denoted by the symbol $a_{\overline{T}|i}$ and is equal to $\sum_{t=1}^{T} v_i^t$, therefore, it follows that

$$\sum_{t=1}^{T}u^t = a_{\overline{T}|i+\rho} \tag{15}$$

is the present value of 1 per annum for T years at the rate $(i+\rho)$.

It may be shown that the increasing annuity

$$\sum_{t=1}^{T} tv_i^{\,t} = \frac{1}{i}\{[i(T+1)+1]a_{\overline{T}|i} - T\};$$

therefore, it follows that

$$\sum_{t=1}^{T} tu^t = \frac{1}{i+\rho}\{[(i+\rho)(T+1)+1]a_{\overline{T}|i+\rho} - T\}. \tag{16}$$

The results of (15) and (16) when substituted in (13) give

$$V_n = \left[P - (N+1)B - \frac{B}{i+\rho}\right]a_{\overline{N-n}|i+\rho} + (N-n)\frac{B}{i+\rho}\,.$$

By substitution of

$$P = \frac{A}{N}[p + (N+1)\beta] \text{ and } B = \frac{A}{N}\beta$$

and letting $\beta/(i+\rho) = k$, it can be shown that

$$V_n = \frac{A}{N}[(p - k)a_{\overline{N-n}|i+\rho} + (N-n)\,k]. \tag{17}$$

This is the general equation giving the value of the unsold portion of the tract, at any time n, in terms of the data, A, N, p, β, and ρ, and in terms of i, the rate of return on the value.

The initial value of the entire tract is obtained by placing $n = 0$ in (17).

$$V_0 = \frac{A}{N}[(p - k)a_{\overline{N}|i+\rho} + N\,k]. \tag{18}$$

If β, the incidental income, is zero, then $k = 0$, and

$$V_0 = \left(\frac{A}{N}p\right)a_{\overline{N}|i+\rho}, \tag{18.1}$$

where Ap/N = net sales revenue per year.[1] This simple equation states an important fact; namely, that the initial value of a tract of land producing no net income, to be sold out at a constant rate and at a constant price per unit, subject to an annual ad valorem tax at a constant percentage, is obtained by multiplying the net sales revenue per year by the annuity factor for the number of years required to dispose of the tract *at the interest rate which is the sum of the rate of return and the tax rate.*

[1] Formula (18.1) is the one used in computing the initial value of the proposed regrade area involved in the Bunker Hill Project, to which reference was previously made.

Value per Unit Area

The value per unit area of the unsold portion at any time n, denoted by w_n, is obtained by dividing V_n, in (17), by the unsold area, $A(N-n)/N$, at the beginning of the $(n+1)$st year, which gives

$$w_n = \frac{p-k}{N-n}\, a_{\overline{N-n}|i+\rho} + k. \tag{19}$$

The initial value per unit area, obtained by placing $n=0$, is

$$w_0 = \frac{p-k}{N}\, a_{\overline{N}|i+\rho} + k. \tag{20}$$

This may be written in the form

$$w_0 = p\frac{a_{\overline{N}|i+\rho}}{N} + \beta\left(\frac{1 - \dfrac{a_{\overline{N}|i+\rho}}{N}}{i+\rho}\right). \tag{20a}$$

By definition, $a_{\overline{N}|i+\rho} = \sum_{n=1}^{N} u^n$ so that

$$\frac{a_{\overline{N}|i+\rho}}{N} = \frac{1}{N}\sum_{n=1}^{N} u^n$$

which is the arithmetical average value of u^n over all integral values of n from 1 to N. Denoting this average value by u^λ,

$$w_0 = pu^\lambda + \beta\left(\frac{1-u^\lambda}{i+\rho}\right) = \beta a_{\overline{\lambda}|i+\rho} + pu^\lambda. \tag{20b}$$

Comparison with value of Unit Area for Which the Date of Sale is Certain

The initial value of unit area producing, before taxes, β per annum, and with a sale for the amount p certain to occur at the end of N years, may be derived by noting that, under this hypothesis

$$w_n = \sum_{m=1}^{N-n}(\beta - \rho w_{n+m-1},)v^m + pv^{N-n}$$

and, by the same reasoning used in deriving (12) from (5), that

$$w_n = \beta a_{\overline{N-n}|i+\rho} + pu^{N-n}, \tag{21}$$

which, with $n=0$, gives[1]

[1] Formula (22) is useful because it gives the initial value of a specific property in terms of the tax rate and net earnings *before taxes*, assuming a total earning life of N years, β constant, and a reversion value of p. This may be compared with the well known formula for the initial value of a constant *net* return, r, for N years at rate i, with a reversion value of p, which is

$$w_0 = ra_{\overline{N}|i} + pv^N.$$

The effect of the taxes is to raise the rate from i to $(i+\rho)$ but involves the substitution of β, (net earnings *before* taxes), for r, (net earnings *after* taxes). Note, however, that in the case of terminable earnings, if β is constant, r is not.

Formula (22) has a practical application in valuing income producing property to determine the value subject to tax.

$$(22) \qquad w_0 = \beta a_{\overline{N}|i+\rho} + pu^N.$$

The form of (22) is similar to that of (20b) but the values are quite different, as they should be. In (20b), the income may terminate and the sale take place at the end of any year from $n=1$ to $n=N$, each of these values of n being equally likely; whereas, in (22), the income is assumed to continue for the definite period of N years with the sale definitely occurring at the end of the Nth year.

THE CAYLEY-HAMILTON THEOREM

By A. K. MITCHELL, Trinity College

In a recent note[1] the author pointed out that the characteristic determinant of a square matrix is readily obtained, in powers of the latent roots, by using the generalized Kronecker delta in defining the determinant. For the n square matrix $\|E_s^r\|$ the characteristic polynomial so obtained was written thus

$$(1) \qquad \lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} + \cdots (-1)^s I_s \lambda^{n-s} + \cdots (-1)^n I_n$$

where

$$(2) \qquad I_p = 1/p!\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} E^{\beta_1}_{\alpha_1} \cdots E^{\beta_p}_{\alpha_p},$$

$\delta^{m_1\cdots m_s}_{n_1\cdots n_s}$ being the generalized Kronecker delta[2] and the repeated indices being summation labels, the summation running from 1 to n. The characteristic equation of the matrix $\|E_s^r\|$ is obtained by equating the expression (1) to zero.

According to the well known Cayley-Hamilton theorem,[3] every square matrix satisfies its own characteristic equation; and it is natural to expect that the notation and definitions used in the above mentioned note should yield an easy proof of this theorem. That this is so will be seen as follows:

Writing $E\equiv\|E_s^r\|$ for the n square matrix we have to prove that

$$(4) \qquad \delta^r_s I_n - I_{n-1}E + I_{n-2}E^2 + \cdots (-1)^{t+1} I_{n-t}E^t + \cdots (-1)^n E^n \equiv 0.$$

Proof: By differentiating the expression (2) and collecting terms, we find by an easy calculation[4] that

$$\partial I_p/\partial E^s_r = \delta^r_s I_{p-1} - I_{p-2}E + I_{p-3}E^2 + \cdots (-1)^{t+1} I_{p-t}E^{t-1} + \cdots (-1)^{p+1}E^{p-1}$$

and

$$(5) \quad \partial I_{n+1}/\partial E^s_r = \delta^r_s I_n - I_{n-1}E + I_{n-2}E^2 + \cdots (-1)^{t+1} I_{n-t}E^t + \cdots (-1)^{n+2}E^n$$

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[1] *A Note on the characteristic determinant of a matrix.* This MONTHLY, vol. 38 (1931), p. 386.

[2] See F. D. Murnaghan, in this MONTHLY, vol. 32 (1925), p. 233; and O. Veblen, *Invariants of quadratic differential forms*, p. 3.

[3] See H. W. Turnbull, *Theory of determinants, matrices and invariants*, p. 99.

[4] See *The derivation of tensors from tensor functions*, American Journal of Mathematics, vol. 53 (1931), p. 198; also H. W. Turnbull, *On differentiating a matrix*, Proceedings of the Edinburgh Math. Society, (2) vol. 1 (1928), pp. 111–128.

But

$$I_{n+1} = 1/(n+1)!\delta^{\alpha_1\cdots\alpha_{n+1}}_{\beta_1\cdots\beta_{n+1}} E^{\beta_1}_{\alpha_1} \cdots E^{\beta_{n+1}}_{\alpha_{n+1}} \,,$$

where the summation in the repeated labels is from 1 to n only. There being $n+1$ places to fill both in subscripts and superscripts and only n numbers to fill them, in each summation at least one number must be repeated, which makes the Kronecker deltas vanish by definition, hence

$$I_{n+1} \equiv 0 \text{ and } \partial I_{n+1}/\partial E^s_r \equiv 0$$

from which because of (5) we see that (4) is true.[1]

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION. PART IV

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

The evidence of inscriptions

We have seen that the works of the elder Āryabhaṭa and his successors contain indisputable evidence to show that the modern place-value notation has been in use in India ever since he wrote his *Āryabhaṭīya*. The evidence of inscriptions also supports this view.[2] Bühler writes:[3] "The earliest epigraphic instance of the use of the decimal notation occurs in the Gurjara inscription of the Cedi year 346 or A.D. 595, where the signs are identical with the numeral symbols of the country and of the period." In the Morbi copper-plate inscription of A.D. 663, which seems to have escaped Kaye's notice, the year 585 is expressed in decimal figures according to the modern notation.[4] Dates expressed in this notation by means of word-numerals instead of figures "are found in the Kamboja (Cambodia) and Campā inscriptions of the 7th century.[5] In Java they occur in the 8th century. And about the same time appears the first trace of such a notation in an Indian document,"[6] Devendravarman's Cicacole inscription of the year 183. The year "is given first in words and next expressed by the symbol for 100, the decimal 8, and the syllable *lo*, i.e. *loka* 3, while the day of the month, 20, is given only in decimal figures."[7] Similar mixtures of the modern

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[1] See also H. W. Turnbull, *The invariant theory of a general bilinear form*, Proceedings of the London Mathematical Society, (2), vol. 33 (1931) p. 10.

[2] Kaye holds a diametrically opposite view, which will be discussed in this paper.

[3] *Indian Antiquary*, Vol. XXXIII, 1904, Appendix, p. 83.

[4] *Indian Antiquary*, July and September, 1873.

[5] If Indians outside India (*i.e.*, in Kamboja and Campā) used this notation by words in inscriptions of the 7th century, it is very likely that they only copied the practice prevailing in India at the time.

[6] Bühler, *Indian Antiquary*, Vol. XXXIII (1904), Appendix, p. 86.

[7] *Ibid.*, p. 78.

notation with the old occur in other inscriptions also.[1] Such mixtures prove the existence of the modern notation just as the presence of a mixture of nitrogen and oxygen in the atmosphere proves the existence of either of these two gases. For an explanation of the use of a mixture of the old and new notations we may refer the reader to Kaye's *Hindu Astronomy*[2] where he writes:[3] "The persistence of old ideas and the neglect of new ones are among the commonest phenomena pertaining to the history of intellectual development. Indeed, these phenomena have prevailed (and still prevail) with the 'school-men' of almost every country."

What are the views of such high authorities on ancient Indian epigraphy as Bühler, Kielhorn, V. A. Smith, Bhandarkar and Thibaut? "Their work" write Smith and Karpinski "is accepted by Indian scholars the world over, and their united judgment as to the rise of the system with a place value—that it took place in India as early as the sixth century A.D.—must stand unless new evidence of great weight can be submitted to the contrary."[4] If Kaye's examination of Indian inscriptions tends to disprove their judgment, it is necessary that some reliable epigraphist should scrutinise his method of examination. I have elsewhere[5] shown that it is not safe to rely on his statements and quotations from authorities. Although the present writer is quite ignorant of epigraphy, Kaye's examination of some of the inscriptions appears objectionable to him. Kaye has taken no notice of the inscriptions in which dates are expressed either in a mixture of the old and new notations or with the help of word-numerals after the manner of Jīva Śarmā, Varāhamihira and subsequent astronomers. We have seen that such methods of expressing numbers presuppose the existence of the modern notation. Of the list of seventeen inscriptions before the tenth century, which he has considered,[6] I have already referred to the Gurjara inscription of the year 595 A.D. No epigraphist of any repute has declared it to be spurious. Yet Kaye writes: "There cannot be the remotest doubt as to the unsoundness of this particular piece of evidence of the early use of the modern system of notation in India."[7] In this case, as in some other cases,[8] Kaye, like an advocate who has a weak case to argue, tries to make up for the deficiency in the strength of his evidence by emphatic assertions or denials. No. 4 of his list is the Kanheri inscription of A.D. 674. Kaye does not state that it is forged. He begins examination of this document by quoting Thomas who is said to write: "The next date in order of priority, which I can refer to, occurs among the Kanheri inscriptions, but the date is expressed in numerals only and the Saṃvat is not

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[1] *Ibid.*, p. 78, foot-note 4.

[2] *Memoirs of the Archæological Survey of India*, No. 18, 1924.

[3] *Ibid.*, p. 129.

[4] *The Hindu-Arabic Numerals*, pp. 47 and 48.

[5] *Isis*, Vol. XII, pp. 134–145. *Am. Math. Monthly*, Vol. XXXVII, p. 19, foot-note 3.

[6] *Journal of the Asiatic Society of Bengal*, July, 1907, pp. 482–486.

[7] *Journal of the Asiatic Society of Bengal*, July, 1907, p. 484.

[8] *Isis*, vol. XII, pp. 137, 138, and 140.

specially defined . . . supposing the date to refer to the Khramaditya era, it will correspond with A.D. 674." Then Kaye writes: "Mr. West gives the figures of this date, . . . , which he interprets as 731 or 732."[1] In spite of doubt as to the actual figure in the units' place it cannot be denied that the modern notation has been used in this inscription. No. 7 of Kaye's list is the Samangad plates of A.D. 754. Although Thomas, Bühler, Bāla Gangādhar Śāstrī, and Fleet differ as to the actual figures in the units', tens' and hundreds' places, they hold that the date was expressed in the modern notation with decimal figures. The next document in the list is the Baijnath inscription of the Saka year 726 or A.D. 804, which contains, according to Bühler, "three numeral signs, the first of which is clearly 7. The following two may have been 26, as Sir A. Cunningham has read them and has represented them . . . , but in the impression they are by no means certain."[2] Although there may be difference of opinion as to the actual figures in the units' and tens' places, the use of the modern notation cannot be denied. It has not been suggested that the inscription is spurious. Similar remarks apply to Nos. 11 and 17. Nos. 5 and 10 are respectively the Kaira plates of A.D. 683 and the Kanheri inscription of A.D. 843. The figures of these documents are said to be marked doubtful. But we have just seen that even doubtful figures may imply the use of the place-value notation. It has not been suggested that these documents are spurious. Nos. 12 and 16 have been rejected for want of published plates. Want of opportunity to examine an inscription is no proof that the modern notation was not used in it. Thus Cajori is not justified in writing on the authority of Kaye that "of 17 citations of inscriptions before the 10th century displaying the use of place value in writing numbers, all but two are eliminated as forgeries; these two are for the years 813 and 867 A.D."[3]

It will thus be seen that, even if we reject the inscriptions which are regarded as forged or suspicious, there still remain others to show that the modern notation was used in Indian inscriptions at least as early as the seventh century. The forged or suspicious inscriptions also, which are not without a historical value of their own, point to the same conclusion. Colebrooke writes: "The necessity of rendering the forged grant credible would compel a fabricator to adhere to history and conform to established notions; and the traditions which prevailed in his time and by which he must be guided, would probably be so much nearer the truth, as it was less remote from the period which it concerned."[4] Smith and Karpinski maintain that even in those remote times at least a century must elapse between the invention of a system and its appearance in inscriptions and write that "it was more than two centuries after the introduction of the numerals[5] into Europe that they appeared there upon coins

[1] *Journal of the Asiatic Society of Bengal*, July, 1907, p. 484.

[2] *Ibid.*, p. 485. It is Kaye's quotation from Bühler.

[3] *A History of Mathematical Notations*, Vol. I, p. 48.

[4] Quoted by Smith and Karpinski in *The Hindu Arabic Numerals*, p. 46.

[5] Here the word 'numerals' is used to mean numerals with place value (*ibid.*, p. 46, foot-note).

and inscriptions."[1] Even Kaye admits that "it is *possible* that the new notation was in use long before it appeared in inscriptions."[2] Hence, it is not unreasonable to suppose, on the evidence of inscriptions, that the modern notation must have been known in India at least as early as the sixth century. But, if the Gurjara inscription of the Cedi year 346, *i.e.*, A.D. 595, which Bühler quotes as "the earliest epigraphic instance of the use of the decimal (*i.e.*, the modern place-value) notation" in India and which Kaye considers to be unreliable, be proved, on further examination, to be undoubtedly genuine, the earliest period in which the modern notation was known in India may be pushed back into the fifth century. Thus we see that the evidence of inscriptions is quite in agreement with literary evidence.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

ON CERTAIN TYPES OF POLYGONS

By J. R. MUSSELMAN, Western Reserve University

§1. Recently R. Goormaghtigh[3] and J. M. Feld[4] have made some interesting generalizations of several theorems which appeared in a note[5] of mine concerning sets of equilateral triangles. In a paper read before the Ohio section of The Mathematical Association on April 7, 1932, another form of generalization was mentioned. This leads to a type of polygon of n sides whose properties are of sufficient interest to invite further study.

Throughout this note we shall represent any point in the plane by a single complex number. Let $M_{1,k}$ $(k=0, 1, \cdots, n-1)$ represent the n vertices of a positively-ordered polygon; the coordinates of these vertices $a_{1,k}$ are subject to one and only one condition, namely that

$$\sum_{k=0}^{n-1} \epsilon^k a_{1,k} = 0, \tag{1}$$

where ϵ is a primitive nth root of unity. Hereafter we shall call any polygon satisfying condition (1) a *positive n-gon of the M type.*

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[1] *Ibid.*, p. 47.

[2] *Journal of The Asiatic Society of Bengal*, July, 1907, p. 487, foot-note 2.

[3] This Monthly, vol. 39 (1932), p. 535.

[4] This Monthly, vol. 40 (1933), p. 36.

[5] This Monthly, vol. 39 (1932), p. 290.

Let $M_{2,k}$ $(k=0, 1, \cdots, n-1)$ be a positive n-gon of the M type, the coordinates of whose vertices $a_{2,k}$ are subject to the condition

$$\sum_{k=0}^{n-1} \epsilon^k a_{2,k} = 0. \tag{2}$$

If we multiply (1) by λ and (2) by μ and then divide their sum by $\lambda+\mu$ we obtain the theorem that *given two positive n-gons of the M type* $M_{1,k}$ *and* $M_{2,k}$, *those points which divide the n segments* $M_{1,k}\,M_{2,k}$ $(k=0, 1, \cdots, n-1)$ *in the ratio* $\lambda:\mu$ *form a positive n-gon of the M type.* Since condition (2) is true if the coordinates $a_{2,k}$ are permuted cyclically, we can for every value of $\lambda:\mu$ select n such n-gons of the M type.

Let $M_{3,k}$ be a third positive n-gon of the M type, the coordinates of whose vertices $a_{3,k}$ are subject to the condition

$$\sum_{k=0}^{n-1} \epsilon^k a_{3,k} = 0. \tag{3}$$

If we take one-third the sum of (1), (2), and (3) we have the theorem that *given three positive n-gons of the M type* $M_{1,k}$, $M_{2,k}$, $M_{3,k}$; *the centroids of the n three-points* $M_{1,k}M_{2,k}M_{3,k}$ $(k=0, 1, \cdots, n-1)$ *likewise are a positive n-gon of the M type.* By permuting the coordinates $a_{2,k}$ and $a_{3,k}$ in conditions (2) and (3) we can obtain n^2 positive n-gons of the M type. Moreover we note that *the centroid of all* n^2 *n-gons thus formed is the same point.*

By a similar method of proof one is lead to the following theorem: *given n positive n-gons of the M type,* $M_{i,k}$, $(i, k=0, 1, \cdots, n-1)$; *if the points* $M_{i,0}$, $M_{i,1}, \cdots, M_{i,n-2}$ $(i=0, 1, \cdots, n-1)$ *are each positive n-gons of the M type, then the n points* $M_{i,n-1}$ $(i=0, 1, \cdots, n-1)$ *are also a positive n-gon of the M type.* Perhaps a more interesting theorem is that *given* $(n-1)$ *positive n-gons of the M type,* $M_{i,k}(i=0, 1, \cdots, n-2; k=0, 1, \cdots, n-1)$; *if we construct the point* N_0 *which to-gether with* $M_{i,0}$ *forms a positive n-gon of the M type, if we construct the point* N_1 *which to-gether with* $M_{i,1}$ *forms a positive n-gon of the M type, etc. then the n points* N_k $(k=0, 1, \cdots, n-1)$ *themselves are a positive n-gon of the M type.*

These n-gons of the M type are well-known for certain values of n; thus $n=3$ gives the equilateral triangle and $n=4$ is the pseudo-square.[1] If we define a square as an ordered four-point, whose diagonals are equal, perpendicular and bisect each other, when we eliminate one of these conditions we obtain respectively the rhombus, the rectangle, and the pseudo-square. From this point of view the pseudo-square has been ill-treated in our elementary geometry books.

To show the importance of these n-gons of the M type let me mention two further theorems[2] concerning them. *If we construct externally, on each side of a*

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[1] Probabiy named by Neuberg. See problem proposed for solution in Mathesis, vol. 13 (1893), p. 216. Is there a more appropriate name?

[2] Proofs of these two theorems for $n=3, 4$ can be found in the literature.

positive-ordered n-point, the positive-ordered regular polygon of n sides; the centers of these n regular polygons is a positive n-gon of the M type. For, if we assign the complex numbers $a_i(i=0,1, \cdots, n-1)$ to the vertices of the n-point, the coordinate b_i of the center of the regular polygon on the side A_iA_{i+1} is given by the relation

$$(4) \qquad (\epsilon^{n-1} - 1)b_i = \epsilon^{n-1}a_i - a_{i+1}.$$

It is a simple matter to show that these n centers satisfy the condition for a positive n-gon of the M type. Also, if we construct internally on each side of the positive ordered n-point, the positive-ordered regular polygon of n sides, it can be easily shown that the centers of these n regular polygons form a negative ordered n-gon of the M type. Both the associated n-gons have the same centroid as the original n-point.

On the sides of a positive n-gon of the M type, construct externally directly similar triangles, these n vertices form a positive n-gon of the M type. For, if we assign the complex numbers $a_i(i=0, 1, \cdots, n-1)$ to the vertices of the n-gon of the M type, the vertex v_i of the triangle on the side A_iA_{i+1} is given by the relation

$$(5) \qquad (t_1 - t_3)v_i = (t_2 - t_3)a_i + (t_1 - t_2)a_{i+1},$$

where t_1, t_2, t_3 are the complex numbers assigned to the triangle with which all the constructed triangles are directly similar. One can easily show that these n vertices v_i satisfy the condition for a positive n-gon of the M type, if the vertices of the original polygon do. A similar state exists if all the triangles be constructed internally; in both cases the centroids of the constructed n-gons coincide with the centroid of the given n-gon of the M type.

§2. The condition that two polygons of n sides $A_{1,i}$ and X_i $(i=1, 2, \cdots, n)$ be directly similar is that two be the rank of the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \cdots a_{1,n} \\ x_1 & x_2 & x_3 \quad \cdots x_n \\ 1 & 1 & 1 \quad \cdots 1 \end{pmatrix}.$$

By methods similar to those in the preceding section one can prove the two following theorems: *given two polygons $A_{1,i}$ and $A_{2,i}$ both directly similar to the polygon X_i $(i=1, 2, \cdots, n)$, then the points which divide the n segments $A_{1,i}A_{2,i}$ $(i=1, 2, \cdots, n)$ in the ratio $\lambda:\mu$ form a polygon directly similar to X_i*; and *given three polygons $A_{1,i}$, $A_{2,i}$, $A_{3,i}$ all directly similar to the polygon X_i, then the centroids of the n three-points $A_{1,i}A_{2,i}A_{3,i}(i=1, 2, \cdots, n)$ likewise are a polygon directly similar to X_i.*

Let us close with a proof of this theorem: *if we are given n polygons $A_{r,s}$ $(r, s=1, 2, \cdots, n)$ all directly similar to a polygon X_s, so arranged that the points $A_{r,1}$ and $A_{r,2}$ both form polygons directly similar to X_s, then the points $A_{r,3}$,*

$A_{r,4}, \cdots, A_{r,n}$ *likewise form polygons directly similar to* X_s. I shall prove this for the points $A_{r,3}$ only, in order to keep the notation and method concise; the argument is perfectly general. By virtue of the assumptions in the theorem, every determinant, for $k=3, 4, \cdots, n$,

$$\begin{vmatrix} a_{k,1} & a_{k,2} & a_{k,3} \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Hence

$$(x_2 - x_1)a_{k,3} = (x_2 - x_3)a_{k,1} + (x_3 - x_1)a_{k,2}. \tag{6}$$

But

$$\begin{aligned} (x_2 - x_1)a_{k,1} &= (x_2 - x_k)a_{1,1} + (x_k - x_1)a_{2,1} \\ (x_2 - x_1)a_{k,2} &= (x_2 - x_k)a_{1,2} + (x_k - x_1)a_{2,2}. \end{aligned} \tag{7}$$

Substituting the values for $a_{k,1}$ and $a_{k,2}$ from (7) in (6) and simplifying the resulting equation, leads to

$$(x_2 - x_1)a_{k,3} = (x_2 - x_k)a_{1,3} + (x_k - x_1)a_{2,3} \tag{8}$$

or

$$\begin{vmatrix} a_{1,3} & a_{2,3} & a_{k,3} \\ x_1 & x_2 & x_k \\ 1 & 1 & 1 \end{vmatrix} = 0$$

for $k=3, 4, \cdots, n$; whence the polygons $A_{r,3}$ and X_s are directly similar.

In the special case of similar triangles the algebra can also be interpreted to give us this theorem: *if $A_1B_1C_1$ and $A_2B_2C_2$ are two directly similar triangles in the plane, if we construct $A_1A_2A_3$, $B_1B_2B_3$ and $C_1C_2C_3$ directly similar to the two given triangles, then $A_3B_3C_3$ is itself directly similar to the other triangles.*

Further, if we consider the special case of regular polygons, for $n=3$ there results my former theorem on equilateral triangles; for $n=4$ we have *given four positive ordered squares of any size or position in the plane, if we move them about so the four vertices marked* 1, *and the four vertices marked* 2 *form squares, then likewise will the four vertices marked* 3 *and the ones marked* 4 *form squares.*

On the Evaluation of Certain Trigonometric Integrals

By J. A. Bullard, University of Vermont

Instead of using a reduction formula to evaluate the integral

$$\int_0^\alpha \sin^m x \cos^n x \, dx,$$

it is often easier to express the integrand as a series of sines or cosines, as the case may be, of multiples of x and then to evaluate the integral.

First,

$$\begin{aligned}\cos^n x &= \frac{1}{2^n}(e^{ix} + e^{-ix})^n = \frac{1}{2^n} \sum_{h=0}^{n} {}_nC_h e^{i(n-2h)x} \\ &= \frac{1}{2^n} \sum_{h=0}^{n} {}_nC_h[\cos (n - 2h)x + i \sin (n - 2h)x] \\ &= \frac{1}{2^n} \sum_{h=0}^{n} {}_nC_h \cos (n - 2h)x.\end{aligned}$$

Then

$$\cos^{2p} x = \frac{1}{2^{2p}}\left[2 \sum_{h=0}^{p-1} {}_{2p}C_h \cos 2(p - h)x + {}_{2p}C_p\right],$$

$$\cos^{2p+1} x = \frac{1}{2^{2p}} \sum_{h=0}^{p} {}_{2p+1}C_h \cos (2p - 2h + 1)x.$$

Similarly,

$$\sin^{2p} x = \frac{1}{2^{2p}}\left[2(-1)^p \sum_{h=0}^{p-1} {}_{2p}C_h(-1)^h \cos 2(p - h)x + {}_{2p}C_p\right],$$

$$\sin^{2p+1} x = \frac{(-1)^p}{2^{2p}} \sum_{h=0}^{p} {}_{2p+1}C_h(-1)^h \sin (2p - 2h + 1)x.$$

The general case is of most interest.

$$\begin{aligned}\sin^m x \cos^n x &= \frac{1}{2^n(2i)^m}(e^{ix} - e^{-ix})^m(e^{ix} + e^{-ix})^n \\ &= \frac{1}{2^n(2i)^m} \sum_{h=0}^{m} \sum_{k=0}^{n} {}_mC_h \, {}_nC_k(-1)^h e^{i(m+n-2h-2k)x} \\ &= \frac{1}{2^n(2i)^m} \sum_{s=0}^{m+n} \sum_{h=0}^{s} {}_mC_h \, {}_nC_{s-h}(-1)^h e^{i(m+n-2s)x} \\ &= \frac{1}{2^n(2i)^m} \sum_{s=0}^{m+n} \Delta_s^m [\cos (m+n-2s)x + i \sin (m+n-2s)x].\end{aligned}$$

where Δ_s^m denotes the s-th difference of the m-th order formed from ${}_nC_k(k=0,$

$1, 2, \cdots, n)$; that is, $\Delta_s^1 = {}_nC_s - {}_nC_{s-1}$, $\Delta_s^2 = \Delta_s^1 - \Delta_{s-1} = {}_nC_s - 2{}_nC_{s-1} + {}_nC_{s-2}$, etc. Then

$$\sin^{2p} x \cos^{2q} x = \frac{(-1)^p}{2^{2p+2q}}\left[2\sum_{s=0}^{p+q-1} \Delta_s^{2p} \cos 2(p+q-s)x + \Delta_{p+q}^{2p}\right],$$

$$\sin^{2p} x \cos^{2q+1} x = \frac{(-1)^p}{2^{2p+2q}} \sum_{s=0}^{p+q} \Delta_s^{2p} \cos (2p+2q-2s+1)x,$$

$$\sin^{2p+1} x \cos^{2q} x = \frac{(-1)^p}{2^{2p+2q}} \sum_{s=0}^{p+q} \Delta_s^{2p+1} \sin (2p+2q-2s+1)x,$$

$$\sin^{2p+1} x \cos^{2q+1} x = \frac{(-1)^p}{2^{2p+2q+1}} \sum_{s=0}^{p+q+1} \Delta_s^{2p+1} \sin 2(p+q-s+1)x.$$

For example let us express $\sin^4 x \cos^6 x$ as a series of functions of multiples of x.

${}_6C_s$:	1	6	15	20	15	6	1				
Δ_s^1:	1	5	9	5	-5	-9	-5	-1			
Δ_s^2:	1	4	4	-4	-10	-4	4	4	1		
Δ_s^3:	1	3	0	-8	-6	6	8	0	-3	-1	
Δ_s^4:	1	2	-3	-8	2	12	2	-8	-3	2	1

Thus

$$\Delta_0^4 = 1,\ \Delta_1^4 = 2,\ \Delta_2^4 = -3,\ \Delta_3^4 = -8,\ \Delta_4^4 = 2,\ \Delta_5^4 = 12,$$

and

$$\sin^4 x \cos^6 x = \frac{1}{2^{10}}[2(\cos 10x + 2\cos 8x - 3\cos 6x - 8\cos 4x + 2\cos 2x) + 12].$$

The above scheme for computing the coefficients I have recently found in De Morgan's *Trigonometry and Double Algebra* (Chapter V), but formulas for the general case are not given there and I have not found them elsewhere.

The following integrals are now readily written:

$$\int_0^\alpha \sin^{2p} x \cos^{2q} x\,dx = \frac{(-1)^p}{2^{2p+2q}}\left[\sum_{s=0}^{p+q-1} \Delta_s^{2p} \frac{\sin 2(p+q-s)\alpha}{p+q-s} + \Delta_{p+q}^{2p}\,\alpha\right],$$

$$\int_0^\alpha \sin^{2p} x \cos^{2q+1} x\,dx = \frac{(-1)^p}{2^{2p+2q}} \sum_{s=0}^{p+q} \Delta_s^{2p} \frac{\sin (2p+2q-2s+1)\alpha}{2p+2q-2s+1},$$

$$\int_0^\alpha \sin^{2p+1} x \cos^{2q} x\,dx = \frac{(-1)^p}{2^{2p+2q-1}} \sum_{s=0}^{p+q} \Delta_s^{2p+1} \frac{\operatorname{hav} (2p+2q-2s+1)\alpha}{2p+2q-2s+1},$$

$$\int_0^\alpha \sin^{2p+1} x \cos^{2q+1} x\,dx = \frac{(-1)^p}{2^{2p+2q+1}} \sum_{s=0}^{p+q} \Delta_s^{2p+1} \frac{\operatorname{hav} 2(p+q-s+1)\alpha}{p+q-s+1}.$$

It should be noted that these integrals are forms of the Beta-function,

$$\tfrac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)},$$

when $\alpha = \pi/2$.

A METHOD OF SOLVING NUMERICAL EQUATIONS[1]

BY S. A. COREY, Des Moines, Iowa

Many iterative methods of solving algebraic equations have already been given in the literature on the subject, but the following method seems to contain some points of novelty and merit worthy of attention.

Let $f(y) = 0$ be a numerical equation having a root y_1; and let $\phi(y, x)$ be $f(y) + (1-x)(Cy+D)$ where C and D are constants. Now let a be an approximation to the root y_1, and let $f(a) = A$, $f'(a) = A_1$, $f^{(i)}(a) = A_i$. Choose C so that $2C^2 + 2A_1C + AA_2 = 0$, and D so that $A + Ca + D = 0$; and we then have $\phi(a, 0) = 0$. Let us further suppose that at the point $(a, 0)$ $\partial\phi/\partial y \neq 0$, that is, $A_1 + C \neq 0$, and that the equation $\phi(y, x) = 0$ defines y as an analytic function of x in a circle about $x = 0$ and including $x = 1$. For $x = 1$ we will then have $y = y_1$. By the usual procedure for implicit functions we find the values of the successive derivatives of y with respect to x at $x = 0$, $y = a$ to be

$$(1)\quad \frac{dy}{dx} = \frac{-A}{A_1 + C}, \quad \frac{d^2y}{dx^2} = \frac{-A(2C^2 + 2A_1C + AA_2)}{(A_1 + C)^3} = 0, \quad \frac{d^3y}{dx^3} = \frac{A^3A_3}{(A_1 + C)^4}\cdot$$

Hence the Maclaurin expansion for y in the neighborhood of $x = 0$, $y = a$ is

$$(2)\qquad y = a - \frac{A}{A_1 + C}x + \frac{A^3A_3}{6(A_1 + C)^4}x^3 + \cdots .$$

Putting $x = 1$, we have, approximately,

$$(3)\qquad y_1 = a - \frac{A}{A_1 + C} + \frac{A^3A_3}{6(A_1 + C)^4} + \cdots .$$

But for the purpose of computation (3) may be better written in the form,

$$(4)\qquad y_1 = a - E + (E^4A_3)/(6A) + \cdots$$

where

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[1] This paper was presented to the Iowa Academy of Science, Mathematics Section, at its meeting held at Davenport, Iowa, May 1st and 2d, 1931.

$$E = \frac{A_1 \mp (A_1^2 - 2AA_2)^{1/2}}{A_2},$$

the ambiguous sign to be so interpreted as to make the modulus of E as small as possible.

To illustrate the rapid rate of convergence obtainable in certain cases let us take the old familiar equation,

$$y^3 - 2y - 5 = 0.$$

Letting $a=2.1$ we get $A=.061$, $A_1=11.23$, $A_2=12.6$, $A_3=6$ and $E=.005,$-448,532,9, so that by substituting in (4) we get

$$y_1 = 2.1 - .005,448,532,9 + .000,000,014,4 = 2.094,551,481,5,$$

correct to the last figure.

For greater accuracy the same process may be repeated employing a more accurate value of a.

The method may be used to calculate real as well as imaginary roots to almost any degree of accuracy, and applies also to transcendental equations. Logarithms may usually be employed to advantage.

A little care must be used when $A_2=0$ as E then becomes 0/0; but in that case (3) may be employed, taking $C=0$.

As transcendental equations are not solvable by Horner's method, Newton's Rule and the Regula Falsi are largely relied on to furnish solutions of such equations. The following example is added to further illustrate the use of the foregoing method by solving the transcendental equation, $\sin y - \frac{1}{2} = 0$, the correct value of y_1 in this case being $\pi/6$.

Let $a=\frac{1}{2}$, we then get

$$\begin{aligned}
A &= \sin a - .5 = -.020,574,461,396 \\
A_1 &= \cos a = .877,582,561,891, = -A_3, \\
A_2 &= -\sin a = -.479,425,538,604, \\
-E &= \frac{\cos a - (1 + \sin^2 a - \sin a)^{1/2}}{\sin a} = .023,596,562,6 \\
(E^4A_3)/(6A) &= .000,002,206, \\
a &= .5 \\
\text{sum} &= .523,598,768,6 \\
\pi/6 &= .523,598,775,6 \\
\text{error,} &= .000,000,007.
\end{aligned}$$

To attain still greater accuracy a more accurate value of a may be employed, e.g., .5236.

A Note on Trigonometric Algebraic Numbers

By D. H. Lehmer, Altadena, California

In connection with some of his researches in the theory of partitions, Professor E. T. Bell has met with the following problem:

To find all angles commensurable with 2π whose sine (or cosine) is an algebraic number of given degree.

Although no explicit general solution of this problem can be expected, since it depends as we shall see on the inverse of the celebrated totient function of Euler, nevertheless the inverse of the problem (in which the angle is given and the degree of its sine is required) can be completely solved in terms of known functions. By merely tabulating these results a solution of the original problem is obtained. From this point of view the solution may be derived from certain treatises by Sylvester, Kronecker, and others. However it is possible to give a quite elementary treatment of the problem.

We recall that if an irreducible polynomial $x^m + \cdots$ has rational coefficients its roots are algebraic numbers of degree m. In particular if the coefficients are integers the roots are algebraic integers. If θ is commensurable with 2π, one of these polynomials will be found to have the root $2\cos\theta$. In fact we shall prove

Theorem 1. Let $r = k/n$, where $n > 2$, be a rational number with the integers k and n relatively prime. Let[1] $\phi(n)$ be the number of integers less than n and relatively prime to n. Then $2\cos 2\pi r$ is an algebraic integer of degree $\phi(n)/2$.

Proof: From the identity

$$x^\nu + x^{-\nu} = (x + x^{-1})(x^{\nu-1} + x^{-(\nu-1)}) - (x^{\nu-2} + x^{-(\nu-2)})$$

follows the familiar fact that $x^\nu + x^{-\nu}$ can be expressed as a polynomial in $x + x^{-1}$ with integer coefficients, the first being unity. The same type of expression can be found for $x^{-m}f(x)$ where f is a symmetric polynomial $x^m + ax^{m-1} + \cdots + ax + 1$. Now the polynomial $Q_n(x)$ whose roots are the primitive n-th roots of unity (and whose leading coefficient is unity) is just such a polynomial as $f(x)$, since its roots occur in reciprocal pairs. Hence there is a polynomial $\psi_n(t)$ such that

$$x^{-d}Q_n(x) = \psi_n(x + x^{-1}) \tag{1}$$

where d is the degree of ψ_n and half that of Q_n. If possible, let $h(x)$ be a polynomial with rational coefficients which divides $\psi_n(x)$. If in (1) we replace ψ_n by h there is determined uniquely a polynomial $q(x)$ with rational coefficients to take the place of Q_n. But $q(x)$ having a root in common with $Q_n(x)$, will be a factor of $Q_n(x)$ contrary to the fact, proved by Kronecker[2] and others, that

[1] For the very simple properties of Euler's ϕ-function needed in what follows, the reader may consult any book on the theory of numbers. For facts concerning its inverse see Dickson's *History of the Theory of Numbers*. A small table in Lucas's *Théorie des Nombres*, p. 395, and a large one by Carmichael in the American Journal of Mathematics, vol. 30, pp. 394–400 may be of interest.

[2] See, for example, Kronecker: Journal de Mathématique (1), vol. 19, (1854), pp. 177–192.

$Q_n(x)$ is irreducible. In short ψ_n is irreducible. Now since the fraction $r=k/n$ is in its lowest terms, $Q_n(x)$ will have the root $e^{2\pi ik/n}$ and from (1) ψ_n has the root $e^{2\pi ik/n}+e^{-2\pi ik/n}=2\cos 2\pi r$. Hence this quantity is an algebraic integer of degree d. To find d we observe that $Q_n(x)$ has as many roots as there are proper fractions j/n in their lowest terms. This is the number of integers $j<n$ and prime to n, represented by $\phi(n)$. Hence the degree of $2\cos 2\pi r$ is precisely $d=\phi(n)/2$.

For sines we have the following result.

Theorem 2. The fraction $r=k/n$ being in its lowest terms, the quantity $2\sin 2\pi r$ is an algebraic integer of degree $\phi(n)/2$ or $\phi(n)$ according as n is or is not a multiple of 4.

Proof. In fact we have

$$2\sin 2\pi r = 2\cos 2\pi(r-\tfrac{1}{4}) = 2\cos 2\pi(4k-n)/4n. \tag{2}$$

Hence, by Theorem 1, $2\sin 2\pi r$ is an algebraic integer. Let its degree be δ. Three cases now arise in considering the fraction $(4k-n)/4n$ occurring in (2). Since k is prime to n, either this fraction is in its lowest terms or else it may be reduced to an equal fraction in its lowest terms with the denominator $2n$ or n. These cases occur when n is odd, twice an odd number, or a multiple of 4 respectively. Applying Theorem 1 in each case we find that $\delta=\phi(4n)/2=\phi(4)\phi(n)/2=\phi(n)$ in the first case; $\delta=\phi(2n)/2=\phi(n)$ in the second case; and $\delta=\phi(n)/2$ in the last case. Hence the theorem.

It follows from these theorems that if the ordinary table of natural trigonometric functions could be given exactly, each entry would be an algebraic number. Thus sin 5° 46′ 31″ is an algebraic number of degree 345,600. If we increase this angle by 83/187 of a second we obtain an algebraic number of degree 160. The numerator k of the fraction r is not involved in the degree of $\sin 2\pi r$ or $\cos 2\pi r$ except that it must be prime to n. Hence our problem may be reduced to finding all integers n for which the sine or cosine of $2\pi/n$ is an algebraic number of given degree. The following table gives these values of n for the first 7 values of d.

Sines

d	n
1	1, 2, 4
2	3, 6, 8, 12
3	none
4	5, 10, 16, 20, 24
5	none
6	7, 9, 14, 18, 28, 36
7	none

Cosines

d	n
1	1, 2, 3, 4, 6
2	5, 8, 10, 12
3	7, 9, 14, 18
4	15, 16, 20, 24, 36
5	11, 22
6	13, 21, 26, 28, 36, 42
7	none

Remarks on the Mathematics of Finance

By Meyer Salkover, University of Cincinnati

During the past school year, I had occasion to give a course in the mathematics of finance, a subject with which I had been unfamiliar. This experience, combining the standpoints of teacher and student, led me to form certain opinions regarding presentation which possibly might interest others. My notation is that of Crenshaw, Pirenian and Simpson, *The Mathematics of Finance* (Prentice-Hall, 1930).

I. *Amortization.* As presented in all texts accessible to me, the derivation of the formula for the periodic payment to amortize a debt is followed up by a bookkeeping device called amortization schedule. This is a scheme for exhibiting, in columns, the outstanding principal at the beginning of each conversion period, the interest incurred during the period, the amount of each payment, the part of each payment applied to the reduction of principal and the total reduction of principal to date. Since, once the periodic payment has been figured, the work in drawing up an amortization schedule involves only simple arithmetic of a repetitious sort, students take kindly to this procedure.

But if the problem is put: Without making a schedule, find how much of the debt remains unpaid just after the pth payment, there is apt to be trouble. Of course the texts explain that the problem simply calls for the present value of the annuity constituted by the remaining $n-p$ payments. This point, however, appears a bit subtle to the student because he has not fathomed the connection between amortization schedule and annuity theory.

I would suggest that an amortization schedule be drawn up in algebraic form for the case of a debt A, the periodic payment being $R=A/a_{\overline{n}|i}$. By induction from the tabulated entries one finds—and the result is capable of immediate interpretation in terms of compound interest formulas—that the debt just after the pth payment is

$$A(1+i)^p - R[(1+i)^p - 1]/i$$

which reduces to

$$R[1-(1+i)^{-(n-p)}]/i = Ra_{\overline{n-p}|i}$$

and this is exactly the present value of the remaining payments. I believe that by tactics of this nature there is a possibility of unifying the bookkeeping and algebraic points of view, and of clarifying the latter.

II. *Amortization of Premium.* The idea of *amortization of premium* is rather obscure because the student cannot see how it fits in with the process called amortization of a debt. The fact is that the word *amortization* as applied to bond premiums has quite another meaning though the texts, addicted as they are to accounting terminology and practice, fail to bring out the distinction. Even aside from the matter of nomenclature, however, it seems to me that a different treatment, also applicable, of course, to accumulation of discount, is indicated.

If F is the face value of a bond redeemable at par, i the yield rate, and R the amount of each dividend, and the bond is bought at a premium P, this is given by the formula

$$P = (R - Fi)\, a_{\overline{n|}i}$$

while the purchase price is

$$V = F + P = F(1 + i)^{-n} + Ra_{\overline{n|}i}.$$

By substituting for F from the second of these formulas into the first and making use of the relation

$$a_{\overline{n|}i} = (1 + i)^{-n} s_{\overline{n|}i}$$

and the definition of $s_{\overline{n|}i}$ one finds

$$P = (R - iV) s_{\overline{n|}i},$$

an illuminating formula which the texts are not justified in ignoring.

The interpretation of this result is plain. $R - iV$ is the excess of each dividend over the expected return on the investment V at the yield rate i. If these excesses are accumulated for the term of the bond at the yield rate, they will, as the result signifies, in the end amount to the premium P, and when the bond is redeemed the investor will be in possession of $P+F$ which is V; his capital will thus be intact. The excesses $R-iV$, then, are the regular contributions that must be made to a sinking-fund created by the investor who does not want his capital impaired by the declining book value of the bond. It is easily shown that the size of the sinking-fund just after a dividend plus the book value of the bond at that time (under unchanged market conditions, of course) equals the original purchase price. If the premium were really amortized in the technical sense of the word, instead of $P = (R - iV)s_{\overline{n|}i}$ we should have $P = (R - iV)$ $a_{\overline{n|}i}$, which latter relation is not true; then why not face the issue squarely and call the accumulating fund a sinking-fund and not an amortization fund? For the so-called amortization of premium is in every way analogous to the sinking-fund method of providing for the depreciation of an asset.

III. *Annuities: General Case.* When the interval between periodic payments does not agree with the conversion period, all texts I have seen except one[1] set up an algebraic machinery much too elaborate for the usual types of problems. The use of heavily indexed symbols and much harping on "installments" and nominal and effective rates of interest serve only to bewilder the student. With reference to ordinary annuities, the situation so far as the solution of actual problems goes may be summed up as follows:

Case I. The interval between payments R consists of p conversion periods, the interest rate being i per conversion period. Here the given annuity can be replaced by an equivalent one of payment $R' = R/s_{\overline{p|}i}$ at intervals coinciding with the

[1] C. H. Forsyth: *Introduction to the Mathematical Theory of Finance*, Wiley, 1928.

conversion periods, because the condition to be satisfied is that p payments of R' accumulate to R.

Case II. There are p payments R in one conversion period. That is, the interval between payments is $1/p$ conversion periods. If the validity of compound interest for a fraction of a conversion period is granted, then automatically all compound interest functions are defined for fractional as well as integral exponents. In particular, the equivalent annuity would here consist of payments $R' = R/s_{\overline{1/p}|i}$ at intervals coinciding with the conversion periods. Replacing the symbol by its exponential expression, one obtains

$$R' = Ri/[(1+i)^{1/p} - 1] = Rpi/p[(1+i)^{1/p} - 1] = Rps_{\overline{1}|i}^{(p)}$$

where the last factor for various values of p and i can be looked up in tables.

When the given annuity is replaced by the equivalent one, the problem in which it figures is correspondingly reformulated and readily solved.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Integralgleichungen unter besonderer Berücksichtigung der Anwendungen. By G. Wiarda. Leipzig, B. G. Teubner, 1930. 183 pages.

This book deals with the linear integral equation of the second kind and the corresponding homogeneous integral equation. The author treats the classical theory of this equation from the point of view of Schmidt's theory. No new results are obtained in this work although many of the proofs given for the classical theorems are ingenious.

In the brief introduction and the first chapter the author states the problem and the assumptions made on the given functions of the integral equation. These assumptions are not clearly stated by the author. As a consequence a beginner in the subject may have a little difficulty in this respect. Most of the first chapter is devoted to the discussion of the vibrating string and how integral equations arise from its consideration.

In the second chapter the author develops the Schmidt theory for the symmetrical integral equation. It is in this chapter particularly that the beginner may have difficulty in understanding just what the assumptions are which the author imposes on the given functions. Some of the assumptions not explicitly stated or sufficiently emphasized are that Riemann or Lebesgue integration may be used; the given functions may be bounded or unbounded, measurable, integrable and of integrable square. The proofs of the theorems are given in detail. There are three exceptions. In each of these exceptions part of the proof of the theorem depends on the following type of argument; if $h(s)$ is a real

function and $\int_a^b h^2(s)ds = 0$ then $h(s) \equiv 0$. In each of these cases the author omits the details of the proof that allows him to conclude that $h(s) \equiv 0$. This may confuse the student regarding the assumptions used.

Chapter three deals with applications of the preceding theory to a problem on heat and a problem on the bending of a beam. A brief treatment of the Sturm-Liouville differential equation is also given.

In chapter four the author considers the linear integral equation of the second kind with the unsymmetrical kernel. He bases the treatment of this integral equation on the previous theory developed for the symmetrical kernel. A brief discussion is given at the end of this chapter to the application of the theory of the unsymmetrical kernel to a problem in potential theory.

Finally in the last ten pages of the book the author gives a brief discussion of the Fredholm method for solving the integral equation of the second kind.

This book in the opinion of the reviewer is not suitable as a textbook for classroom use. Aside from the physical applications incorporated in the text the author uses very few examples to illustrate the theory. No problems are given for the student to do. The assumptions the author imposes on the given functions of the integral equation are not fully and properly emphasized. Finally, the book is restricted to just one type of integral equation.

There are few misprints in the text and these are quite obvious.

L. J. PARADISO

Elementary Mathematics from an Advanced Standpoint. Arithmetic, Algebra Analysis. By Felix Klein. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, The Macmillan Company, 1932. ix+274 pages, 125 figures. \$3.00.

The popularity of the German original, delivered as a course of university lectures in 1908, is attested by the fact that it ran through three editions and still has an extensive sale.

A work of this kind can be successful only when the author has a comprehensive knowledge of the whole field of mathematics, and the skill to explain the essential features without a mass of detail. In both of these capacities, Professor Klein was probably better fitted than any other person then living.

In each chapter, two questions are considered; first, what is the status of our present knowledge of the subject, and second, what effect should this knowledge have on school instruction. In the first chapter, that on arithmetic, the growth of the concept of number is told in a fascinating way, supplemented by a detailed description of calculating machines (as of 1908), and of hypercomplex number systems, in particular of quaternions. The problem in algebra is the solution of the algebraic equation. By means of the regular body forms equations of degree n not greater than four are reduced to pure equations $x^n = c$, and it is shown that the general quintic can not be so reduced, but can be solved in terms of elliptic modular functions.

The chapter on analysis discusses logarithmic and trigonometric functions. The discussion includes an extensive history and explanation of tables, and shows the intimate relation between the two kinds of functions. A section on spherical trigonometry gives a rapid survey of the Study group defined by three diameters of a sphere. Trigonometric series are used to approximate various functions in a strikingly direct and convincing manner.

A chapter on the elements of the calculus includes the derivative, integral, Taylor's series and a careful criticism of pedagogical principles. An appendix furnishes a detailed proof of the transcendency of e and of π, and an introduction to the theory of assemblages.

The translation, the printing and the proof reading have been excellently well done. The only error (perhaps not typographical?) noticed that might cause confusion is the use of the word *surface* on pages 178, 179.

The book is supplied with a generous number of citations to original sources, and to other books containing more details of proofs. As these are largely German, one might be tempted to ask why the translation of the book under review?

During the last quarter of a century there has been a universal effort to improve the quality of teaching in the elementary and secondary schools. Whenever a change is made in this country in the curricula for the training of teachers, it has been in the direction of more "education," pedagogy and psychology, always at the expense of further courses in subject matter. The results are already apparent; for the grade schools the new method is an improvement, but for high schools, especially the last two years, it is lamentably deficient. However desirable the other things may be in themselves, for a teacher of mathematics nothing has yet been discovered to replace a knowledge of mathematics. May the present volume take its place in American and English schools, to extend the service it has so admirably rendered in Germany.

VIRGIL SNYDER

Determinanten. (Sammlung Göschen) By Paul B. Fischer. Berlin and Leipzig, Walter de Gruyter and Co., 1932. 136 pages. RM 1.62.

This small volume contains an excellent development of the usual theorems on determinants; the application of determinants to the solution of linear equations; illustrations of their use in algebraic geometry and some of their special forms. Among the special forms briefly discussed are Vandermonde's Determinant, symmetric, skew-symmetric, pseudo-symmetric and reciprocal determinants, in addition to the Hessian, Jacobian, resultant and discriminant. Owing to the scant mention of matrices, the treatment of linear equations lacks the usual theorems on linear independence and the case of m equations in n unknowns.

L. T. MOORE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities and topics for club programs and, material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Washington University

Greetings and best wishes to all chapters! The Missouri Beta chapter reports a very successful year under the following officers elected at the meeting held on May 16, 1931: Elizabeth Harris, Southwestern Bell Telephone Company, Director; Bayard R. Brick, Instructor in Mathematics, Vice Director; Jessica M. Young, Assistant Professor of Mathematics and Astronomy, Secretary; Charles O. Quade, Graduate in Civil Engineering, Assistant Secretary; Ross R. Middlemiss, Assistant Professor of Mathematics, Treasurer; William E. Stephens, Senior in College of Liberal Arts, Librarian.

The student members of the Executive Committee were: Jessie Best, Senior in College of Liberal Arts; Cecilia Lehmann, Senior in College of Liberal Arts; Richard Singer, Senior in College of Liberal Arts; Richard Torrance, Senior in Civil Engineering.

There were fifty six active members during the year 1931–1932. Thirty-three new members were initiated on April 16, 1932, distributed as follows: From the College of Liberal Arts—twelve; Schools of Engineering and Architecture—twenty; School of Graduate Studies—one.

Ten meetings were held during the year as follows:

October 7, 1931: 8:15 P.M. Regular program meeting, 319 Rebstock Hall. "What is a Sumner line?" by Professor H. R. Grummann; "Making the Sun's shadow move backwards" by Professor E. Stephens; "What is sampling?" by Professor P. R. Rider; Refreshments.

November 10, 1931: 8:15 P.M. Regular program meeting, 319 Rebstock Hall. "A few pages from the History of Mathematics" by L. G. Starrett; "The remainder in Taylor's series" by W. C. Guse; Refreshments.

December 7, 1931: 8:15 P.M. Regular program meeting, 319 Rebstock Hall. "A tensor notation as developed from determinants" by Cecilia Lehmann; "Some simple nomographs" by P. M. Arnold; Refreshments.

January 6, 1932: 8:15 P.M. Regular program meeting, 319 Rebstock Hall. "Report on the New Orleans Convention" by Professor E. Stephens; "Some illustrative examples in variational methods in the solution of problems in Mechanics" by Sol Gleser; Refreshments.

February 9, 1932: 8:15 P.M. Regular program meeting, 319 Rebstock Hall. "New Ways of measuring time" by W. M. Yates; "Extracts from the History of Mathematics" by A. S. Langsdorf, Jr.; Refreshments.

March 7, 1932: 8:15 P.M. Regular program meeting, 319 Rebstock Hall. "The Auto-Giro" by W. A. Langtry; "Electrical analogues in solving mechanical problems" by H. E. Zeffren; Refreshments.

March 16, 1932: 4:00 P.M. Business meeting, 221 Brookings Hall. Election of new members; Unanimous election of W. O. Pennell to Honorary Life Membership.

March 23, 1932: 8:15 P.M. Special meeting. 112 Wilson Hall. An open meeting held in conjunction with the Academy of Science of St. Louis. Mr. W. O. Pennell, Chief Engineer of the Southwestern Bell Telephone Company, gave a popular lecture on "A thousand mile telephone cable from mathematics to reality." The lecture was illustrated with lantern slides, moving pictures and exhibits. About two hundred attended the lecture and the reception following.

April 16, 1932: 6:30 P.M., Initiation, Women's Building, Alumnae Room. 7:00 P.M., Banquet, Cafeteria. 8:00 P.M., Program, Lounge: "A rotating magnetic field" by A. S. Langsdorf, Dean of the Schools of Engineering and Architecture and Director of Industrial Engineering and Research. 9:15 P.M., Social gathering.

May 14, 1932: 8:15 P.M. Business meeting and social gathering, Women's Building, Lounge. Presentation of Honorary Life membership to Walter Otis Pennell "in appreciation of his untiring efforts and faithful service to the fraternity." "The Chapter recognizes him as one of its most distinguished members and hopes that he will continue to inspire its younger members with enthusiasm as he has during the past years." Treasurer's report. Election of officers for 1932–1933.

JESSICA M. YOUNG, *Secretary*

Pi Mu Epsilon of The Ohio State University

The officers for the year 1931–1932 were: F. M. Brooks, Director; J. W. Suckau, Vice Director; Brandon Rightmire, Treasurer; L. E. Bush, Secretary.

A new election of officers will take place May 27, 1932. The Chapter has at present 44 active members. A class of 16 candidates was initiated May 16, 1931 and a new class of 13 candidates is to be initiated May 27, 1932.

The Chapter has held its meetings throughout the year in conjunction with the Graduate Mathematics Club of The Ohio State University. Meetings have been held about twice a month at which papers were read by members of the two organizations and by members of the mathematics faculty. Besides these regular meetings we have had two visiting lecturers.

Professor Louis Brand of the University of Cincinnati spoke on "Boundary value problems of the Sturm-Liouville type." Dr. S. Saks of the University of Warsaw, Rockefeller International Research Fellow, gave a series of three lectures on the subjects: "The existence of singularities on the basis of the Baire theory;" "The integral of Denjoy;" and "The classical method of integration of the nineteenth century."

L. E. BUSH, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the University of Virginia

The *Echols Mathematics Club* of the University of Virginia was organized October 13, 1931 to "promote better fellowship among its members and to foster a wider interest in the subject of Mathematics at the University of Virginia." Its membership is composed of graduate students and officers of instruction in mathematics, and those students of the College who show a marked proficiency in this field. The club meets on the second and fourth Thursdays during the school year.

The officers for the session of 1931–1932 were: J. W. Givens, Jr., President; W. T. Puckett, Jr., Vice President; T. L. Wade, Jr., Secretary-Treasurer.

The meetings and programs were as follows:

October 13, 1931: Organization.

October 29, 1931: "The square root of a number by means of sequences" by Professor W. H. Echols.

November 12, 1931: "Continued fractions" by J. W. Givens, Jr.

December 3, 1931: "A system of co-ordinates for three dimensional space consisting of one real and one complex number" by Professor J. J. Luck.

January 14, 1932: "Non-Euclidean geometry" by T. L. Wade, Jr.

January 28, 1932: "Sequences" by Professor B. Z. Linfield.

February 11, 1931: "Some theorems on limits" by W. T. Puckett, Jr.

February 25, 1932: "Attempted trisections of the general angle" by Irving Lindsey.
April 14, 1932: "The integral solutions of equations of the first and second degree in two variables" by W. M. Aylor.
April 28, 1932: "Finite differences" by Professor E. J. Oglesby. Election of officers for 1932–1933.
May 12, 1932: "Problem of the one and many in Mathematics and Physics" by Professor F. S. C. Northrop of Yale University, visiting Professor of Philosophy.

A short social meeting is held in connection with each regular meeting of the club.

T. L. WADE, *Secretary*

The Mathematics Club of the Oshkosh State Teachers College

The club was organized by Dr. May Beenken in February, 1931. Membership is restricted to those who have completed at least one year of college mathematics.

The purpose of the club is to promote interest in the study of mathematics, and to afford an opportunity to study certain interesting matters connected with mathematics that do not find a place in the usual class discussion.

The officers for the year 1931–1932 were: Marie Conrad, President; Florence Zelinske, Vice President; Emma Huffman, Secretary-Reporter; Hugh Williams, Treasurer. Dorothy Mortson was elected President the second semester to take the place of Marie Conrad who graduated.

Many interesting programs were given during the year. The programs presented were:

October 5, 1931: "Number lore" by Fred Moes; "Solution of the cubic equation" by Loretta Golz.
November 2, 1931: "Magic Squares" by Rose Schlegel, "Sir Isaac Newton" by John Wrage. Several questions which had been distributed from our "Question Box" were answered. Among the questions answered were: "What are Napier's Bones"? "Why are logarithmic tables made with base *e*?" "What is Pascal's triangle?"
December 7, 1931: "The contribution of the Greeks to geometry" by Ramona Paddock, "The foundations of Euclidean geometry" by Alton Davis.
January 11, 1932: "Fundamental propositions in algebra" by Dorothy Mortson; "The life and work of Kepler" by Hugh Williams. One of the questions answered from the "Question Box" was: "What proposition in plane geometry is known as 'Pons Asinorum'"?
February 16, 1932: "René Descartes as a mathematician and philosopher" by Irene Timm, "Cryptographs and Ciphers" by Gordon Kester.
March 7, 1932: "Nine-point circle" by Vailor Dumdie, "Our coordinate systems" by Rita Schuttler.
April 25, 1932: "Number concept, its origin and development" by Nina Kachur, "The duodecimal system and other bases" by Louis Gardipee; "Tests for the divisibility of numbers from 2 to 13 inclusive" by Fred Moes.
May 9, 1932: An enjoyable evening was spent at a party given by the faculty of the mathematics department at the home of Professor Irene Price.
May 27, 1932: The annual picnic was held at the home of Bertram Lyngass on the shores of Lake Winneconne.

EMMA HUFFMAN, *Secretary-Reporter*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSEN AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems to W. F. Cheney, Jr., Dept. Box 35, Storrs, Conn.

The Department of Elementary Problems and Solutions in the Monthly welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions. Solutions for different problems should be submitted on separate sheets of paper.

PROBLEMS FOR SOLUTION

E 26. *Proposed by H. T. R. Aude, Colgate University.*

All proper rational fractions in lowest terms can be separated into two classes:

1. Those in which the numerator and denominator are both odd.
2. Those in which the numerator and denominator are not both odd.

Show that for any fraction F in one class there is just one corresponding fraction F' in the other such that $\arctan F + \arctan F' = \pi/4$.

E 27. *Proposed by E. P. Starke, Rutgers University.*

Derive the algebraic formula for the sum of n fractions whose numerators are in arithmetic progression and whose denominators are in geometric progression.

E 28. *Proposed by H. D. Ruderman, Brooklyn, New York.*

Given m marbles of which m_1 are of a first color, m_2 of a second color, m_3 of a third color, and so on to m_n of an n-th color, all arranged in a circle. Determine the number of different arrangements possible if we can start from any point and move in either direction around the circle.

E 29. *Proposed by J. Rosenbaum, Milford, Connecticut.*

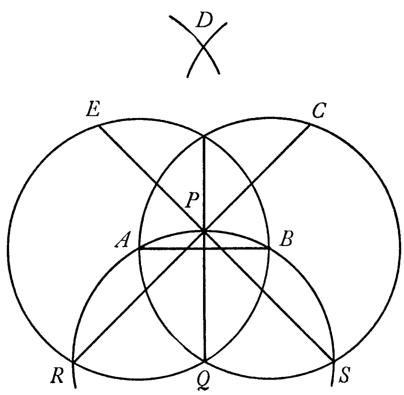
The faces of a tetrahedron are congruent triangles whose sides are a, b and c. If $2S = a^2+b^2+c^2$, show that the volume is $[(S-a^2)(S-b^2)(S-c^2)]^{1/2}/3$.

E 30. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The arithmetic mean of $N\ E\ D$ and $S\ A\ S\ H$ is $S\ H\ U\ N$. Their geometric mean is $S\ E\ N\ D$ and their harmonic mean is $S\ E\ E\ D$. Assuming that the seven letters here involved represent different digits, identify them.

SOLUTIONS

E 6. [1932, 547]. *Proposed by W. R. Ransom, Tufts College.*



This construction was given in 1525 by Albrecht Dürer, the great engraver, for a regular pentagon, $ABCDE$, and it is still given in books on mechanical

drawing. The circles are all drawn with the same radius, equal to the given length of the side AB, with centers at these points (in order) A, B, Q, C and E. Calculate the angle ABC to determine whether this is an exact or an approximate construction.

Solution by E. P. Starke, Rutgers University.

The construction of the pentagon is only a fair approximation, the error in angle ABC being an excess of about twenty-two minutes.

1. In rhombus $ABQR$, angle ARQ equals $60°$, and RB equals $AB\sqrt{3}$.
2. Since angle PRQ in the right isosceles triangle PQR equals $45°$, it follows that angle BRC is $15°$.
3. In the triangle RCB we know RB, BC and angle BRC. By the law of sines, $\sin RCB = \sqrt{3} \sin 15°$, whence angle $RCB = 26°38'2''$ and angle CBR equals $138°21'58''$.
4. Since angle $ABR = 30°$, angle $CBA = 108°21'58''$.

Also solved by H. E. H. Greenleaf, C. C. Richtmeyer and the proposer.

E. 7. [1932, 548]. *Proposed by C. O. Oakley, Brown University.*

The following letters represent the digits of a problem in simple addition.

$$\begin{array}{r} S\ E\ N\ D \\ M\ O\ R\ E \\ \hline M\ O\ N\ E\ Y \end{array}$$

Solve and show that the solution is unique.

Solution by W. E. Buker, Leetsdale, Pennsylvania.

1) Since neither S nor M exceeds 9, $MO < 19$ and $M = 1$.
2) $SE < 99$ and $MO < 19$, so $MON < 118$, and since no two letters may be equal, the letter O must be zero, and $S = 9$.
3) $N = E + 1$ and so 1 is carried from $N + R$.
4) Since $E = N - 1$, $N + 9 = E + 10$, and $R = 8$, and 1 is carried from $D + E$.
5) Since $S = 9$ and $R = 8$, D is 7 or less, N is 7 or less and E is 6 or less. Then $D + E = Y + 10 < 14$. But if $E = 6$, $N = 7$, $D < 6$ and $D + E = 10$ or 11, both of which are impossible. If E were < 5, $D + E$ would be less than 12, which is impossible. Therefore $E = 5$.
6) Then $N = 6$ and $D = 7$ and $Y = 3$, so that the addition problem reads

$$\begin{array}{r} 9567 \\ 1085 \\ \hline 10652 \end{array}$$

Since publishing this problem, information has been received that it appeared in the International Chess Review L'Echiquier (June 1928) Brussels, and also in Le Sphinx. Also solved by Mildred Beaty, T. A. Bickerstaff, Mrs. Anabel S. Boyce, John Breiland, M. L. Constable, H. E. H. Greenleaf, Arthur Haas, L. S. Johnston, Elmer Latshaw, F. L. Manning, W. R. Ransom, Simon Vatriquant and Henry Weick.

E 8. [1932, 548.] *Proposed by O. A. Spies, St. Paul, Minnesota.*

It is required to construct an inscriptible quadrilateral with ruler and compass, given the lengths of the four sides in order.

Solution by R. L. Korgen, Bowdoin College.

In the inscribed quadrilateral of successive sides a, b, c and d, let the diagonal from the beginning of a to the end of b be denoted by k, and the angle between a and b by q. The angle between c and d is $180° - q$.

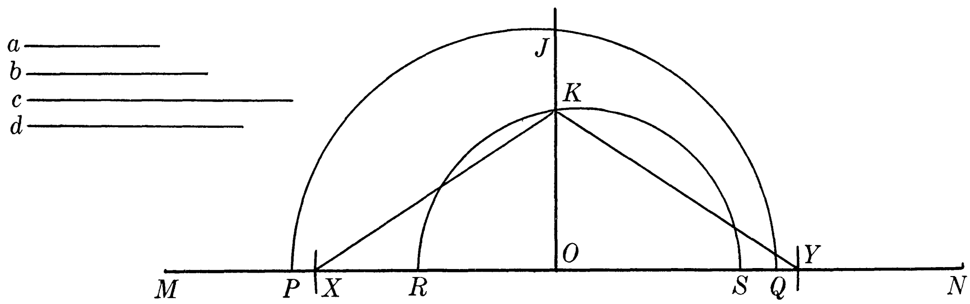
1) In the triangle abk, $k^2 = a^2 + b^2 - 2ab \cos q$.
2) In the triangle cdk, $k^2 = c^2 + d^2 + 2cd \cos q$.
3) Eliminating k^2, we have

$$\cos q = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

4) Let $a+b+c+d = 2s$, and since $\tan^2 \frac{1}{2}q = (1 - \cos q)/(1 + \cos q)$, it is found that

$$\tan \frac{q}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}}\,.$$

Construction.



1) Within the line-segment $MN = a+b+c+d$, locate the points P, Q, R and S such that $MP = a$, $QN = b$, $MR = c$ and $SN = d$.
2) Construct circles on the diameters PQ and RS.
3) At O, the midpoint of MN, erect a perpendicular cutting the two circles at J and K respectively.
4) With O as center and OJ as radius, swing arcs cutting MN at X and Y, and draw the angle XKY. It equals the desired angle q. The remainder of the construction is obvious.

Proof of Construction.

1) $MO = ON = s$. $PO = s - a$, $OQ = s - b$, $RO = s - c$, $OS = s - d$.
2) Therefore $XO = OJ = \sqrt{[(s-a)(s-b)]}$, and $OK = \sqrt{[(s-c)(s-d)]}$.
3) Therefore $XO/OK = \tan \frac{1}{2}q$, and angle $XKY = q$.

It should be remarked that interchanging the order of the sides does not change the radius of the circumscribing circle.

Since publishing this problem, information has been received that it is solved in Altshiller-Court's "College Geometry" page 111, and in R. A. Johnson's "Modern Geometry" page 82, as well as in this MONTHLY, where it was solved [1929, 289] by L. S. Johnson and others.

Also solved by W. E. Buker, Arthur Haas, E. P. Starke and the proposer.

E 10. [1932, 548]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The x's below show the relative positions of the digits in a certain exact division. Find the divisor.

```
x x x ) x x x x x x x x ( x x x x x x
        x x x
        -----
        x x x x
          x x x
        -----------
              x x x x
                x x x
              ---------
                  x x x x
                  x x x x
                  -------
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Solution by Simon Vatriquant, L'Athênée Royal d'Ixelles, Brussels, Belgium.

It appears from the third subtraction that the second remainder is 1, so that the second subtraction, which deducts a three place number from a four place number with a remainder of 1, must be 1000–999. Then the divisor, being a three place factor of 999, must be 111 or 333 or 999. But the first subtraction shows that 999 is impossible because of the three place remainder, and since the last product is a four place number, 111 is ruled out. Therefore the divisor must be 333, and the quotient is $A3003B$, where $A<3<B$.

Also solved by T. A. Bickerstaff, Mrs. Anabel S. Boyce, W. E. Burke, Arthur Haas, L. S. Johnston, H. R. Leifer, Ethel Mintzer, W. R. Ransom, C. A. Rupp, Bernard Zimmerman and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3601. *Proposed by M. Markowitz, Brooklyn, New York.*

Prove or disprove that

$$H = \left[(n-1)^2 F_x F_y - n(n-1) F F_{xy}\right]/xy,$$

where F is a homogeneous function of x and y of order n, and H is its Hessian, i.e., $H = F_{xx}F_{yy} - (F_{xy})^2$

3602. *Proposed by Sigmond Moroh, Brooklyn, New York.*

In a circle with the center O let CD be the diameter perpendicular at M to the chord AB. Through M two chords EF and ST are drawn arbitrarily; and let the lines ET and SF cut AB in Q and P, respectively. Prove that QM is equal to MP.

3603. *Proposed by J. B. Reynolds, Lehigh University.*

For what limits of density will a homogeneous cube float in stable equilibrium with a diagonal vertical?

3604. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Two cones of revolution with apex angles α_1 and α_2 are to intersect in an ellipse with the given length a as the long diameter; the angle between the axes of the two cones is ϕ. Find the short diameter b of the ellipse and the angles γ_1 and γ_2 which the axes of the cones make with the plane of the ellipse.

3605. *Proposed by H. S. Thurston, University of Alabama.*

Prove that every square matrix whose elements are given by the relation,

$$a_{ij} = \begin{cases} (-1)^{j-1}\dbinom{j-1}{i-1}, & i < j \\ (-1)^{i-1}, & i = j \\ 0, & i > j, \end{cases}$$

is a square root of the unit matrix I.

3606. *Proposed by A. Blake, Washington, D. C.*

Find a rational construction for the Hermitian matrices whose existence is provided for in this theorem. Given any field γ containing a conjugate function, such that for every two numbers, a and b, of γ we have $\bar{\bar{a}} = a$, $\overline{ab} = \bar{b}\bar{a}$, and $\overline{a+b} = \bar{a} + \bar{b}$. Given any square matrix κ on a finite range with elements in γ, such that the coefficients in the invariant factors of κ are all scalars (i.e., numbers equal to their conjugates). Then there exists a pair of Hermitian matrices θ_1 and θ_2 with elements in γ, such that θ_2 is non-singular and $\kappa = \theta_1\theta_2$.

Specialize the construction required above in such a way as to obtain useful canonical factors.

SOLUTIONS

3528. [1932, 115]. *Proposed by A. A. Bennett, Brown University.*

If a, b, c, be complex numbers such that

$$|a| = |b| = |c| = r \neq 0,$$

then $|(ab+bc+ca)/(a+b+c)| = r$. Generalize.

Solution by H. L. Schug, North Canton, Ohio.

The theorem may be generalized as follows:

If $a_1, a_2, \cdots, a_n$ be n complex numbers such that $|a_1| = |a_2| = \cdots = |a_n| = r \neq 0$, and ${}_nT_s$, $s < n$, is the sum of the products of these n numbers taken s at a time; then

$$\left| \frac{{}_nT_s}{{}_nT_{n-s}} \right| = r^{2s-n}. \tag{1}$$

To prove this, let

$$a_1 = re^{i\theta_1}, \ a_2 = re^{i\theta_2}, \cdots, a_n = re^{i\theta_n}.$$

Then r^s can be factored from ${}_nT_s$, and r^{n-s} from ${}_nT_{n-s}$. In the numerator there is left $e^{i\psi}\sum e^{-i\Phi}$, and in the denominator $\sum e^{i\Phi}$, where ψ is the sum of all the angles, and Φ is the sum of $n-s$ of the angles. Since $\sum e^{-i\Phi}$ and $\sum e^{i\Phi}$ are conjugates, their absolute values are the same; and the absolute value of $e^{i\psi}$ is unity. Hence the equality in (1) results. If n is an odd integer and $s = (n+1)/2$, the right side of (1) is r as in the given problem.

Also solved by R. P. Agnew, M. G. Boyce, W. B. Campbell, A. G. Clark, J. H. Edmonston, L. R. Ford, H. D. Grossman, Emma T. Lehmer, R. E. Moritz, O. J. Ramler, F. Underwood, and Paul Wernicke.

3536. [1932, 175]. *Proposed by Martin Rosenman, Brooklyn, New York.*

Consider fractions of the form $1/2$, $1/3$, $1/4$, $1/5$, $\cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7 = 41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807 \cdots$ give the desired result; in which series each denominator exceeds by 1 the product of all preceding.

Partial Solution by F. Underwood, University College, Nottingham, England.

Denote the series formed by the denominators 2, 3, 7, 43, 1807, $\cdots$ by $v_1, v_2, v_3, v_4, v_5, \cdots$. Then $v_n = v_1v_2v_3 \cdots v_{n-1}+1$. We note that

$$1/2 + 1/3 = 1 - 1/6 = 1 - 1/(v_3 - 1).$$

$$1/2 + 1/3 + 1/7 = 1 - 1/42 = 1 - 1/(v_4 - 1).$$

$$1/2 + 1/3 + 1/7 + 1/43 = 1 - 1/1806 = 1 - 1/(v_5 - 1).$$

Assume that, for a particular value of n,

$$S_n = 1/v_1 + 1/v_2 + \cdots + 1/v_n = 1 - 1/(v_{n+1} - 1).$$

Then, for this value of n,

$$\begin{aligned} S_{n+1} = S_n + 1/v_{n+1} &= 1 - 1/\{v_{n+1}(v_{n+1} - 1)\} \\ &= 1 - 1/(v_{n+2} - 1). \end{aligned}$$

Now $S_n + 1/w$, where $w > v_{n+1}$, is obviously less than S_{n+1} and $S_n + 1/u$, where $u \leqq v_{n+1} - 1$, is either equal to or greater than unity.

Hence if S_n is the nearest approximation to unity by adding n fractions of the given form, so also is S_{n+1} the best approximation that can be made by adding another fraction to S_n. But the theorem for this mode of formation is obviously true when $n=2$ or 3, and so is true universally if the restriction is made that $n+1$ fractions must always include the preceding n fractions. It remains to be proved that if this restriction is removed, the mode of formation of S_n used above is still the best possible.

Also solved by Mannis Charosh.

Note by the Editors. The proof of the final part seems difficult and may interest other readers; the editors are now examining a proof by the present solver.

3544. [1932, 239]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The orthogonal sphere of four variable spheres, with fixed centers, whose radii remain proportional, describes a coaxial pencil.

Solution by W. V. Parker, Mississippi Woman's College, Hattiesburg, Miss.

The equations of the four variable spheres may be written

$$(x - a_i)^2 + (y - b_i)^2 + (z - c_i)^2 = (\lambda r_i)^2; \qquad (i = 1, 2, 3, 4),$$

where λ is a parameter. The sphere

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

will be orthogonal to these if and only if

$$2aa_i + 2bb_i + 2cc_i = a_i^2 + b_i^2 + c_i^2 - \lambda^2 r_i^2 + a^2 + b^2 + c^2 - r^2; \quad (i = 1, 2, 3, 4).$$

Subtracting each of the last three of these equations from the first we get three equations which are linear in a, b and c and whose solution is of the form

$$a = x_1 + \lambda^2 x_2,\ b = y_1 + \lambda^2 y_2,\ c = z_1 + \lambda^2 z_2,$$

where $x_1, y_1, z_1, x_2, y_2, z_2$ are constants determined by a_i, b_i, c_i, r_i. Hence as λ varies, the center of the orthogonal sphere moves along the line through the point (x_1, y_1, z_1) with direction components (x_2, y_2, z_2).

A Note by Otto Dunkel. The three equations for a, b, c admit a unique solution since the determinant formed from the coefficients of a, b, c is not zero. This determinant is not zero, for the wording of the problem implies that there is a unique, actual, sphere orthogonal to the four spheres, and in this case the centers of the latter spheres do not lie in a plane. The equation of the orthogonal sphere may be written

$$x^2 + y^2 + z^2 - 2a(x - a_1) - 2b(y - b_1) - 2c(z - c_1) + \lambda^2 r_1^2 - (a_1^2 + b_1^2 + c_1^2) = 0,$$

where a, b, c are given by the above linear expressions in λ^2. This shows that the orthogonal spheres belong to a coaxial pencil.

Also solved by J. W. Blincoe.

3545. [1932, 239]. *Proposed by George Y. Sosnow, Newark, N. J.*

The equation

$$\frac{a_1}{x + b_1} + \frac{a_2}{x + b_2} + \frac{a_3}{x + b_3} + \cdots + \frac{a_n}{x + b_n} = 0,$$

will be an identical equation if

$$\sum_{i=1}^{n}(a_i) = 0, \quad \sum_{i=1}^{n}(a_ib_i) = 0, \quad \sum_{i=1}^{n}(a_ib_i^2) = 0, \cdots, \sum_{i=1}^{n}(a_ib_i^{n-1}) = 0.$$

Solution by Margaret M. Young, Brooklyn College of the City of New York.

Combining the fractions on the left hand side of the given equation we get in the numerator a sum of terms of degree $n-1$ of the type

$$a_i(x + b_1)(x + b_2) \cdots (x + b_{i-1})(x + b_{i+1}) \cdots (x + b_n).$$

Expanding this product we obtain a sum of terms of the type

$$a_ix^{n-1-t} \sum b_1b_2 \cdots b_t, \quad 0 \leqq t \leqq n - 1,$$

where no subscript i occurs in the sum. Denote by B_t the elementary symmetric function of the same kind as $\sum$ but which contains b_i. Then

$$\sum b_1b_2 \cdots b_t = B_t - b_i B_{t-1} + b_i^2 B_{t-2} - \cdots + (-1)^t b_i^t.$$

Hence the coefficient of x^{n-1-t} in the numerator is

$$B_t \sum a_i - B_{t-1} \sum a_ib_i + B_{t-2} \sum a_ib_i^2 - \cdots + (-1)^t \sum a_ib_i^t.$$

Since each sum which appears as a factor of a B is zero by the conditions of the problem, the numerator vanishes identically; and this concludes the proof.

Also solved by J. L. Botsford, J. M. Feld, L. S. Johnston, and W. V. Parker.

A Note by Otto Dunkel. The vanishing of the n sums given in the problem is both a necessary and a sufficient condition for the identical vanishing of the

sum of fractions. But this condition deserves further examination. Since the a_i's satisfy n homogeneous linear equations, either they are all zero or the determinant of these equations is zero. In the latter case at least two b_i's are equal, since the determinant is simply the product of all the differences of the b_i's. Instead of continuing the discussion in this manner, let us consider the identical vanishing of the given sum in a more direct manner. Suppose that b_1 occurs m_1 times, b_2 occurs m_2 times, $\cdots$, b_t occurs m_t times, where $b_1, b_2, \cdots, b_t$ are distinct in value, and where $m_1+m_2+\cdots+m_t=n$. We may then replace the given sum by

$$\frac{A_1}{x+b_1}+\frac{A_2}{x+b_2}+\cdots+\frac{A_t}{x+b_t},$$

where A_1 is the sum of m_1 of the a's, A_2 is the sum of m_2, etc., and where

$$A_1+A_2+\cdots+A_t=\sum_{i=1}^{n}a_i.$$

If the sum is identically zero for all values of x, its limit must be zero as x approaches $-b_1$; and since the other fractions are finite for this value, we must have $A_1=0$. Pursuing this reasoning, we find a necessary condition

$$A_1=A_2=\cdots=A_t=0.$$

But this is also sufficient. Since we have two necessary and sufficient conditions, one implies the other. If the b's have distinct values, and one would be inclined to assume that that is the meaning of the problem, we see again that each a_i is zero. We have in the statement of this problem an example of a necessary and sufficient condition which is not sufficiently reduced in its expression.

3546 [1932, 239]. *Proposed by W. H. Rasche, Virginia Polytechnic Inst.*

Given that r is the radius of a circle, S its plane, O its center, P any point in the plane of the circle, distant $d(<r)$ from the center; l is the line through P perpendicular to line OP which makes with plane S the angle $\alpha=\cos^{-1}(d/r)$; show that if the circle be rotated about l as an axis it will generate a torus whose meridian section is two equal non-intersecting circles of radius d, whose centers are equi-distant from axis l.

Note: A solution suitable for college freshmen not familiar with solid analytical geometry is desired.

Solution by Rufus Crane, Ohio Wesleyan University.

Let A be any point on the generating circle; B its projection on PO; D its projection on the plane through PO perpendicular to l, that is, the equatorial plane; θ the angle AOB.

By considering right angled triangles, we have

$$(PA)^2=(PB)^2+(BA)^2=(d+r\cos\theta)^2+(r\sin\theta)^2$$
$$\therefore (PA)^2=d^2+r^2+2dr\cos\theta$$

$$(PD)^2 = (PB)^2 + (BD)^2 = (d + r\cos\theta)^2 + (r\sin\theta\sin\alpha)^2$$
$$\therefore PD = d\cos\theta + r$$
$$AD = r\sin\theta\cos\alpha = d\sin\theta.$$

From the expression for $(PA)^2$ or from the expressions for PD and AD, it is obvious that the point A, in any of its positions, is at a distance d measured in the meridian plane PDA from a point on PD at distance r from P.

Also solved by F. L. Wilmer and the proposer.

3547. [1932, 239]. *Proposed by Martin Rosenman, Brooklyn, New York.*

Consider n points in a plane. Join these in any order to form a closed polygon. Repeat the operation on the n midpoints of the sides of the polygon thus formed, etc. Prove that the successive polygons converge to a point.

Solution by R. E. Huston, University of Chicago.

From the statement of the problem, it is not clear whether or not the order in which the vertices are to be joined is to be maintained for successive polygons. We shall prove the theorem allowing this order to change arbitrarily at any stage.

Let $P_1, P_2, \cdots, P_n$ $(n>2)$ be the given points. Take a system of coordinates with the origin at their center of gravity and let $z_k = x_k + iy_k$ $(k=1, 2, \cdots, n)$ be the corresponding complex numbers. Then

$$z_1 + z_2 + \cdots + z_n = 0. \tag{1}$$

If $\rho = \text{Max } |z_k|$, then

$$|z_k| \leqq \rho \qquad (k = 1, 2, \cdots, n). \tag{2}$$

Let π denote the original polygon and $\pi^1, \pi^2, \cdots, \pi^{n-1}$ the first $n-1$ successive polygons derived as indicated in the problem, and let $z_j, z_j^1, \cdots, z_j^{n-1}$ $(j=1, 2, \cdots, n)$ denote the complex numbers corresponding to their vertices. For any vertex of π^{n-1} we have $z^{n-1} = (1/2)(z_{j_1}^{n-2} + z_{j_2}^{n-2})$ $(j_1 \neq j_2)$. For a similar reason $z^{n-1} = 2^{-2}\ (z_{j_1}^{n-3} + z_{j_2}^{n-3} + z_{j_3}^{n-3} + z_{j_4}^{n-3})$ where at least three of the numbers j_1, j_2, j_3, j_4 are different. Similarly $z^{n-1} = 2^{-3}\ (z_{j_1}^{n-4} + z_{j_2}^{n-4} + \cdots + z_{j_8}^{n-4})$ where at least four of the numbers $j_1, \cdots, j_8$ are different. Finally we get

$$z^{n-1} = \frac{a_1 z_1 + a_2 z_2 + \cdots + a_n z_n}{2^{n-1}}, \tag{3}$$

where the a_k are *positive integers* of total sum $\sum a_k = 2^{n-1}$. From (1), (2), and (3) we get

$$|z^{n-1}| = \left| \frac{(a_1 - 1)z_1 + \cdots + (a_n - 1)z_n}{2^{n-1}} \right| \leqq \frac{2^{n-1} - n}{2^{n-1}} \rho. \tag{4}$$

Hence π^{n-1} is completely contained in the circle of radius $K\rho = 2^{-n+1}\ (2^{n-1} - n)\rho$,

$(0<K<1)$, around the origin. Repeating the same process m times we see that $\pi^{m(n-1)}$ is contained within the circle

(5) $$|z| \leqq K^m \rho,$$

which tends to the origin for $m \to \infty$. Our theorem follows immediately, since π^p $(p>m(n-1))$ lies in the same circle (5) which is arbitrarily small if m is sufficiently large.

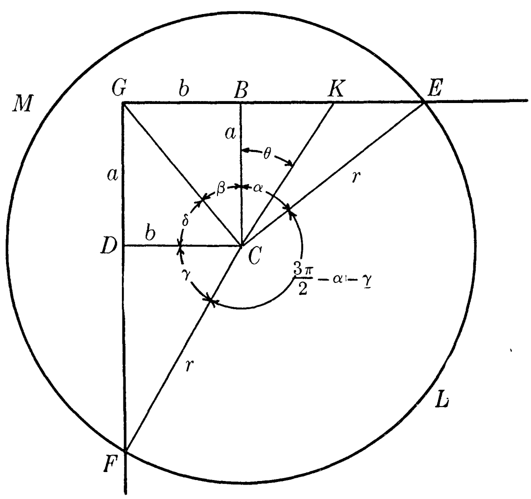
Exactly the same proof applies for the case of an r-dimensional space: $|z_k| = \overline{OP}_k$ means the distance from the origin O to the point P_k and (1) and (3) represent in condensed form the same r relations for the corresponding coördinates. The further arguments need no change.

Also solved by E. Alliot, W. T. Reid, I. J. Schoenberg, and the proposer.

3548 (1932, 239). *Proposed by J. B. Reynolds, Lehigh University.*

A coal chute delivers coal at a point at distances a and b from two vertical walls at right angles to each other. Find the volume of a conical pile of coal under the chute if its radius is r, $r>(a^2+b^2)^{1/2}$, and the angle of repose of coal is A.

Solution by V. F. Murray, Hoboken, N. J.



In the figure, in the plane of the base, GE, GF represent the vertical walls, $GC=(a^2+b^2)^{1/2}$, C is the centre of the base, radius r. If the vertex of the cone be at H, then $CH=r\tan A$.

In the figure the angles $\alpha=\cos^{-1} a/r$, $\beta=\tan^{-1} b/a$, $\gamma=\cos^{-1} b/r$, $\delta=\tan^{-1} a/b$.

The volume required is the sum of five parts; $FLEC-H$, $BEC-H$, $BGC-H$, $GDC-H$, $FDC-H$.

The volume $FLEC-H$ is part of the right cone, base radius r and height $r\tan A$, and, by proportion,

$$= \frac{1}{2\pi}\left(\frac{3\pi}{2} - \alpha - \gamma\right)\cdot\frac{1}{3}\pi r^3 \tan A,$$

$$(1) \qquad = \frac{1}{6}\left(\frac{3\pi}{2} - \alpha - \gamma\right)\cdot r^3 \tan A.$$

To obtain the volume $BEC-H$: Since the plane of GBE is parallel to the axis of the cone, the section GBE of the cone gives part of a hyperbolic segment, the transverse axis of the hyperbola passing through B. For integration of the volume, consider an elementary wedge at CK, the angle BCK, θ, being in the plane of the base. The edge of the wedge, CH, equals $r \tan A$, the height CK is $a \sec \theta$, and the height at K from the base of the cone to the hyperbola, Kk, depends on the angle θ. It can be easily shown that, for the given cone and the distance a, $Kk = \tan A(r - a \sec \theta)$. The volume of the wedge element thus is

$$\tfrac{1}{6}a^2(r \tan A + 2 \tan A\{r - a \sec\theta\})\cdot \sec^2\theta\cdot d\theta$$

and the volume $BEC-H$ equals

$$\tfrac{1}{2}a^2 r \tan A \int_0^\alpha \sec^2\theta\cdot d\theta - \tfrac{1}{3}a^3 \tan A \int_0^\alpha \sec^3\theta\cdot d\theta$$

$$(2) \qquad = \tfrac{1}{2}a^2 r \tan A\left[\tan\theta\right]_0^\alpha - \tfrac{1}{6}a^3 \tan A\left[\sec\theta\tan\theta + \log(\sec\theta + \tan\theta)\right]_0^\alpha$$

$$= \tfrac{1}{2}a^2 r \tan A\cdot\tan\alpha - \tfrac{1}{6}a^3 \tan A\{\sec\alpha\cdot\tan\alpha + \log(\sec\alpha + \tan\alpha)\}.$$

So also the volume $GBC-H$ is given by an expression (3) of the type (2), α being replaced by β.

Similarly, the volume $DFC-H$ is given by an expression (4) of the type (2), a being replaced by b, and α being replaced by γ.

Also the volume $GDC-H$ is given by an expression (5) of the type (4), γ being replaced by δ.

The total volume required is obtained by adding (1), (2), (3), (4), and (5).

It is interesting to note the change in the result as the point G moves outward from C, r being constant.

When G falls on the circumference $MFLE$, $\alpha = \beta$ and $\gamma = \delta$.

When G crosses the circumference, $\alpha = \beta$ and $\gamma = \delta$, and a new angle ϵ appears between β and δ. This angle will be added to the angle in expression (1).

When G takes a position so that GE and GF are tangents to FLE, $\alpha = \beta = \gamma = \delta = 0$, and $\epsilon = \frac{1}{2}\pi$, the volume becoming a cone.

The original problem may be further extended by adding two walls parallel to GE and GF respectively, forming a rectangular base. The volume when r varies can be found from appropriate expressions of the types (1) and (2). When the rectangular base becomes just covered, and the angles α, β, γ, etc. are unequal, eight expressions of the type (2) give the volume, the value of r being the maximum distance from C to a corner of the rectangle.

Also solved by W. B. Campbell.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

C. B. LePage, assistant secretary of the American Society of Mechanical Engineers, has distributed to the Sectional Committee on Scientific and Engineering Symbols and Abbreviations and to the affiliated committees copies of an important report entitled "American Tentative Standard Abbreviations for Scientific and Engineering Terms," with the request that all interested help in placing the standard into practice and in educating companies, both users and manufacturers, to the advantages to be secured from the use of this standard. It was approved by the American Standards Association in November 1932.

The abbreviations included in the report are those which are preferred for use in typewritten or printed text. The general omission of periods after abbreviations has been recommended not only as a measure of economy but also to avoid the confusion which has been found to result from attempts to employ abbreviations for the names of metric units without periods and abbreviations of the names of English units with periods in the same publication. The Committee endorses the omission of the period in abbreviations of metric units and the growing tendency toward the omission in abbreviations of other origin.

Copies of this standard may be secured from the American Standards Association and the A.S.M.E., 29 West 39th St., New York, N. Y.

The Institute of International Education has just issued the fourth edition of a publication listing fellowships and scholarships that are available to American students for foreign study under various auspices. Copies may be obtained from the Institute, 2 West 45th Street, New York City.

Markus Renier, visiting research professor of Lafayette College, an engineer, Department of Public Works, British government in Palestine, is giving a series of lectures on mathematical Rheology at the John C. Green School of Engineering, Princeton University.

Professor Vannevar Bush, at the meeting of the National Academy of Sciences held at Ann Arbor, November 14–16, reported on the applications of the differential analyzer, a machine for solving ordinary differential equations, to the solution of important problems. During the past year, about thirty problems have been treated, involving nearly one thousand solutions of equations. The Thomas-Fermi equation and the same equation after introducing the relativity correction illustrate the solution of non-linear equations. The Schrödinger wave equation, by means of the machine, has been successfully treated for several nuclear numbers in the case of the helium-like atoms. The solution of this problem involved a fourth order non-linear equation with boundary conditions at zero and infinity, together with a normalization condition.

At their meeting at Atlantic City, December 27, 1932, the Trustees of the Mathematical Association of America awarded the Chauvenet Prize of $100 to Professor G. H. Hardy of the University of Cambridge, England, for his paper entitled "An introduction to the theory of numbers" which appeared in the Bulletin of the American Mathematical Society, Vol. 35 (1929), pages 778–818. This prize is awarded every three years for the best expository paper on a mathematical subject published in English by a member of the Mathematical Association. This award covered the triennium 1929–31.

Professor I. A. Barnett, of the University of Cincinnati, lectured on "Infinitesimal transformations in certain types of function spaces" before the Mathematical and Physical Society of Budapest on November 24, 1932, at the invitation of that Society. He also delivered a series of lectures before the Mathematical Institute of the University of Cracow in November.

Professor Edward Kasner, of Columbia University, will deliver a series of public lectures in January and February, 1933, at the People's Institute, Cooper Union. He will also speak on "Squaring the circle" over the radio on March 3 (Columbia network, WABC).

At the Atlantic City meeting of the Association, announcement was made that Dr. D. H. Lehmer, has invented a machine for the solution of congruences. The machine is designed to solve Diophantine equations, and sorts out solutions at the rate of about 5,000 per second.

Professor O. K. Defoe, of the College of the Ozarks, has been appointed professor of mathematics and physics at the St. Louis College of Pharmacy.

Professor E. H. McAlister, of the University of Oregon, has been transferred to a professorship of mathematics at the Oregon State College.

Dr. J. von Neumann has been appointed professor of mathematical physics at Princeton.

Dr. Ruth W. Stokes has been appointed associate professor of mathematics at the North Texas State Teachers College, Denton.

A CORRECTION

In the News and Notices department of the February issue, page 124, it was stated that Professor Alfred Hume, of the University of Tennessee, has been reinstated as chancellor of the University of Mississippi. Chancellor Hume writes us that this notice is in error with regard to his connection with the University of Tennessee as he has never been a member of the faculty of that University.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The December meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University on Saturday, December 3, 1932.

Sixty-six persons attended the meeting including the following forty members of the Association: O. S. Adams, Beatrice Aitchison, Clara L. Bacon, G. A. Bingley, Archie Blake, C. C. Bramble, Paul Capron, C. N. Claire, Abraham Cohen, Orpha A. Culmer, Tobias Dantzig, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, J. H. Edmonston, Michael Goldberg, Harry Gwinner, C. H. Harry, F. E. Johnston, L. M. Kells, W. D. Lambert, Florence P. Lewis, B. Z. Linfield, J. J. Luck, Florence M. Mears, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, W. F. Reynolds, R. E. Root, J. H. Taylor, Mildred E. Taylor, Marian M. Torrey, F. M. Weida, John Williamson, C. H. Wheeler, G. T. Whyburn, E. W. Woolard, R. C. Yates, Oscar Zariski.

The annual Spring meeting will be held at the University of Virginia on Saturday, May 13, 1933.

Professor Oscar Zariski of the Johns Hopkins University was the invited speaker and he addressed the afternoon session on "Recent contributions to the problem of existence of curves with preassigned singularities."

The following papers were presented:

At the morning session:

1. "On mappings with functions of finite sections" by Beatrice Aitchison, Johns Hopkins University.

2. "Various formulas for the numerical integration of differential equations" by Dr. L. S. Dederick, Aberdeen Proving Ground.

3. "Perspective solid angles" by Professor B. Z. Linfield, University of Virginia.

4. "An irreducible system, complete through the fifth degree, of Euclidean invariants of the ternary cubic" by T. L. Wade (Introduced by Professor Linfield), University of Virginia.

5. "A geometry of acyclic spaces" by Dr. C. H. Harry (Introduced by Dr. G. T. Whyburn), Johns Hopkins University.

At the afternoon session:

6. "Recent contributions to the problem of existence of curves with preassigned singularities" by Professor Oscar Zariski, Johns Hopkins University.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Some necessary conditions for sets M to be transformed by a real continuous function $f(x)$ into a simple arc so that $f(x)=c$ has only a finite number of solutions, or for M to have "property P" have been reported by Čech. Mazurkiewicz has given a necessary and sufficient condition that an acyclic curve have this property. This condition suggested an extension to much more general

spaces, resulting in the following theorem: If every cyclic element C of a locally connected continuum M has property P and, moreover the set of cut points of M on C is transformed into a reducible set on the arc corresponding to C, then, if the end points of M are countable and the branch cyclic elements of M are reducible by the operation Φ' of bilateral coherence among the branch elements, M can be transformed into a simple arc by a real continuous function of finite sections.

2. The number of possible formulae for the numerical integration of ordinary differential equations is theoretically infinite. Even if we restrict our attention to those formulae having some element of practical convenience, a fairly extensive list may still be written. In this paper a method was given for deriving such formulae, and a few were discussed and compared.

3. Any solid (trihedral) angle d determines at once three more solid angles, namely (1) the solid angle d' whose edges are perpendicular to the faces of d at its vertex, (2) the solid angle D whose edges are the projections of the edges of d on the corresponding faces of d, and (3) the solid angle D' whose edges are perpendicular to the faces of D. By means of simple vector and matrix identities determined by the face angles and dihedral angles of d, Professor Linfield proved that any two of these four solid angles are *perspective*, i.e., planes through corresponding edges meet in one line, and therefore, corresponding faces intersect in three lines which are coplanar.

4. By means of the first and second fundamental theorems of the symbolic method for Euclidean invariants (R. Weitzenböck, "Invariantentheorie," Chapter XII) it is shown that any polynomial invariant of degree not greater than five of the ternary cubic $f(x) = (ax)^3$ can be expressed as a polynomial with numerical coefficients in the invariants

$$\underset{1}{\underline{\overline{\mathrm{II}}}} = (a\mid a)(a\mid b)(b\mid b) \qquad \underset{2}{\underline{\overline{\mathrm{II}}}}(a\mid b)^3$$

$$\underline{\overline{\mathrm{III}}} = (abc)(abl)(bcl)(cal) \qquad \underset{1}{\underline{\overline{\mathrm{IV}}}} = (abc)(abd)(acd)(bcd)$$

$$\underset{2}{\underline{\overline{\mathrm{IV}}}} = (abc)(bcd)(a\mid a)(b\mid c)(d\mid d) \qquad \underset{3}{\underline{\overline{\mathrm{IV}}}} = (abc)(bcd)(a\mid b)(c\mid d)(a\mid d)$$

$$\underset{4}{\underline{\overline{\mathrm{IV}}}} = (abc)^2(b\mid c)(d\mid d)(adl) \qquad \underset{5}{\underline{\overline{\mathrm{IV}}}} = (a\mid a)(abl)(b\mid c)^2(c\mid d)(d\mid d)$$

$$\underset{6}{\underline{\overline{\mathrm{IV}}}} = (a\mid a)(a\mid b)(b\mid c)^2(c\mid d)(d\mid d) \qquad \underset{1}{\underline{\overline{\mathrm{V}}}} = (abc)^2(ade)del)((b\mid c)(d\mid e)$$

$$\underset{2}{\underline{\overline{\mathrm{V}}}} = (abc)^2(ade)(bel)(d\mid d)(c\mid e).$$

Furthermore, no one of these invariants is expressible rationally and integrally in terms of the others.

F. M. WEIDA, *Secretary*

ELIAKIM HASTINGS MOORE

AN APPRECIATION BY H. E. SLAUGHT

In the death of Professor Eliakim Hastings Moore on December 30, 1932, the University of Chicago lost one of its most inspiring leaders and the world-wide fraternity of mathematicians lost one of its most gifted members. He was one of that remarkable company of scholars who from the very outset in 1892 determined the high character of the University of Chicago and gave to it great distinction. Professor Moore was one of the youngest of this group of men but his dauntless spirit, his boundless enthusiasm, his scientific integrity, and his unsurpassed logical insight soon established an influence which has extended far and wide and will continue to be felt wherever students of mathematics are seeking to discover the mainspring of their inspiration.

Looking back over a period of forty-one years of intimate association with Professor Moore, this writer, whose acquaintance with him began in the capacity of first fellow in the department of mathematics at the University of Chicago, now wishes to enumerate certain outstanding characteristics of this truly great man and, if possible, to set them forth with sufficient emphasis to command the attention and attract the emulation of the on-coming generation of students of mathematics.

First, and perhaps most important, was Professor Moore's exemplification of *intellectual honesty*. It was impossible for him to accept as established any proposition whose proof was in doubt or not fully shown, much less to promulgate any such proposition. There was no place for camouflage of any sort in his mental make-up. A striking and critical test of this characteristic occurred at the opening meeting of the Mathematical Club in one of the early summer quarters. Moore was presenting a paper on a highly technical topic to a large gathering of faculty and graduate students from all parts of the country. When half way through he discovered what seemed to be an error (though probably no one else in the room observed it). He stopped and re-examined the doubtful step for several minutes and then, convinced of the error, he abruptly dismissed the meeting—to the astonishment of most of the audience. It was an evidence of intellectual *courage* as well as *honesty* and doubtless won for him the supreme admiration of every person in the group—an admiration which was in no wise diminished, but rather increased, when at a later meeting he announced that after all he had been able to prove the step to be correct.

A second outstanding characteristic was the *intellectual freedom* which he claimed for himself and which he cheerfully granted to all others. He expected every man on the staff to do his full duty but he did not follow him up or visit his classroom. He trusted, rather, to the freedom based upon individual responsibility and, on the whole, the results were far more satisfactory than any system of espionage could possibly have been. There were many departmental open discussions where every instructor's views could be fully weighed and evaluated in the light of all other considerations; but in the end individual free-

dom was accorded to each one for working out the general plan. If in any case the results were unsatisfactory, it was up to the instructor to discover the trouble and to find the remedy. To this acid test of *freedom*, Professor Moore subjected himself on equal terms with all his instructors, even to the extreme test, on occasion, of admitting his own failure and seeking the remedy from the counsel of others.

Among many other characteristics which might be mentioned are: *fairness of judgment*, *unselfish devotion* to his students, and *whole-hearted confidence* in his chosen science. Professor Moore never passed judgment on any question until all the evidence had been examined and evaluated. If in any case it seemed that he had erred he was always ready to grant reconsideration in the light of new evidence. With his students, especially those who showed ability in research, his interest never flagged. He was constantly in touch with their progress and was a never failing source of inspiration. His favorite lecture period was eleven o'clock but he seldom heard or heeded the twelve o'clock bell, so engrossed was he in the development of the topic in hand, whether he himself or one of the class was making the report. His interest, however, was not limited to his own special students nor even solely to Chicago students or faculty. A common occurrence at the time of the mathematical meetings in Chicago was a scientific tête-à-tête in a secluded corner between Moore and some younger man who had presented a paper (perhaps his first attempt) showing worthy effort in independent thinking. His confidence in the worthwhileness of his chosen science was his supreme delight not only in an abstract, logical sense but also in its concrete significance for the comfort and progress of the human race, as he believed that it influences the thought and touches the daily life of mankind (though in most cases quite unconsciously) in a multitude of ways.

It is not the purpose of this writer to make any attempt at weighing or evaluating the scientific work of Professor Moore, nor even to give a bibliography of his published papers. These phases of his long life work will be treated by others in various publications. Suffice it to say here that he has left a great legacy of unpublished research material concerning General Analysis and that his published papers and abstracts include 29 in Group Theory, 10 in Geometry and Foundations, 17 in Function Theory, 11 in Integral Equations and General Analysis, and 7 on miscellaneous topics, making a total of 74. In this connection there should be emphasized Professor Moore's important and widely extended influence through his former students, and especially through the thirty-one doctors of philosophy whose theses were written under his supervision. Of these Ph.D.'s, there are three on the faculty at the University of Chicago, two each at the University of Iowa, and the University of the Philippines, one each at Harvard, Princeton, Cornell, Bryn Mawr, Vassar, Swarthmore, Toronto, Annapolis, Fisk University, the state colleges of South Dakota and Pennsylvania, and the state universities of Michigan, Wisconsin, Nebraska, Washington, Louisiana, Texas, Montana, and California. Three are deceased and two are not teaching, though they are engaged in research occupations. One of these

doctors, himself an internationally known mathematician, is quoted as saying: "For forty years Moore has been a leading figure in the mathematical world, not merely in fundamental research, but also in the nationalization of a local mathematical society, the institution and editing of its research journal, and finally inspiring and starting on their research careers so many of the younger generation. Moore's work easily places him among the world's greatest mathematicians. He was a leader who was universally loved because he was at the same time a prince of a man." Another of these doctors has said: "His genius for mathematical work, and the personal enthusiasm which spurred him on to great success, were inimitable and constituted a tremendous inspiration to all who came in contact with him, students and colleagues alike."

But there is another phase of Professor Moore's remarkable influence in the general domain of mathematics which might easily be overlooked, but which should not be underestimated, amidst the greater effulgence of his brilliant research career; namely, his deep interest in the teaching of mathematics in the secondary schools and colleges and his ardent support of the Mathematical Association of America and of this MONTHLY as its official journal dedicated to the interests of mathematics in the collegiate field. It will be worthwhile to trace the development of this pedagogical interest in Moore's mind. It began in 1894 when he gave personal encouragement to Benjamin F. Finkel who was then just starting the AMERICAN MATHEMATICAL MONTHLY. They both hoped that this new journal would attract and inspire high school teachers. In this they were greatly disappointed but the new periodical soon found a fitting place in the collegiate world. Moore and Dickson contributed articles in the early numbers which gave encouragement to Finkel and prestige to the MONTHLY and later, in 1902, with Moore's advice and consent, Dickson accepted the co-editorship and the University of Chicago began an annual subsidy contribution of \$50.00 which was continued till the MONTHLY became self-supporting in 1915.

Meanwhile in 1894, the American Mathematical Society was organized in New York and in 1897 the first Section of this Society was formed at Chicago largely through Moore's initiative. In 1901–1902 Moore was president of the Society and in December, 1902, he delivered his presidential retiring address in which he revealed his profound interest in the pedagogical situation with respect to mathematics in America and set forth with his keen logic and amazing skill and insight the relationship that should subsist between so-called "pure" and "applied" mathematics, or in the words of Klein, between "mathematics of precision" and "mathematics of approximation." Here we find a mathematician whose keenest delight was in the realm of the most abstract logical analysis pleading with his scientific confrères in astonishing earnestness for the recognition and promulgation of the "laboratory method" in the teaching of mathematics in the elementary, secondary and collegiate-schools; that is, for the *concrete* approach to all *abstract* mathematical ideas. He firmly believed that the methods of presenting mathematics in the schools could thus be modified, greatly to their pedagogical advantage, and so as to win for this queen of

the sciences the "very high position in general esteem and appreciative interest which it assuredly deserves."

A re-reading of this presidential address[1] will convince the most casual observer that Professor Moore had given very earnest and deep thought to the question of improvement (yes, of reform) in the teaching of mathematics in the schools and that he had weighed carefully similar movements in Europe, especially in England and Germany. There is now good evidence that he was a seer and a prophet many years in advance of his time. While he recognized the chief responsibility of the American Mathematical Society to be the promotion of research, he nevertheless felt that even research interests in mathematics were bound up with those of mathematical education. He hoped that the Society might give more attention to the pedagogy of mathematics and he felt sure that such a movement would further the highest interests of mathematics in this country. But this movement was delayed for fourteen years and then was brought about by another procedure which will now be fitted into this picture.

It was shown above that Moore's interest in the founding of the MONTHLY was based on his desire for a general extension of mathematical education in this country. This interest again appeared when, in 1908, he urged this writer to undertake the co-editorship of the MONTHLY which Professor Dickson was then vacating to take up other editorial duties. Again in 1912 Moore's cooperation and council were of the greatest importance when the periodical was rescued from financial disaster and taken over by representatives of twelve universities and colleges in the middle west. His joy was like that of a child with a new toy when the first number, in January 1913, of the enlarged publication in its new dress was presented to him. And, finally, his interest and satisfaction were approaching an upper bound when, in January 1916, the MONTHLY became the official journal of the newly organized Mathematical Association of America. Ample evidence cropped up now and then to show that he was an habitual reader of the MONTHLY—problems and all. The first complete set of bound volumes of the periodical to be deposited in the Association library was the gift of Professor Moore and served as a token of his abiding interest in the movement for which the MONTHLY was to stand.

One of the first official acts of the Mathematical Association of America was the appointment of a committee under its auspices to consider the whole question of the teaching of mathematics in the secondary schools. It was composed of six representatives of universities or colleges and six from secondary schools. Professor Moore was an active member of this committee and his brother-in-law, J. W. Young, was its chairman. At last, after fourteen years of waiting, Moore found himself in the company of a nationally representative body wholeheartedly devoted to reform in the teaching of mathematics. This committee's activities were financed by the General Education Board and its findings were published in a voluminous report on the "Reorganization of Mathematics in

[1] *Bulletin of the American Mathematical Society*, Vol. 9, p. 402–424.

Secondary Education" whose circulation was nation-wide and whose influence soon became and still remains phenomenal though it is a decade since it was issued.

One outcome of this committee's influence was the organization of the National Council of Teachers of Mathematics whose membership now numbers nearly five thousand and whose activities have justified the responsibility which it inherited from the committee. Another outcome has been the firm establishment of the Mathematical Association of America as the effective intermediator between university research on the one hand and general mathematical education on the other hand. Professor Moore would have been the last to claim any special credit for these developments in the mathematical domain during the last thirty years, but it is nevertheless true that his personal influence has been potent and profound. The very fact that his life work was bound up in the highly technical and abstract reaches of his beloved science gave the world undeniable evidence that the interest which he manifested in the teaching of mathematics was genuine and whole-hearted.

Professor Moore was born at Marietta, Ohio, on January 26, 1862. He graduated at Yale University in 1883 and took his Ph.D. there in 1885. He spent a year in study at Berlin and Göttingen and taught for six years at Yale and Northwestern before coming to the University of Chicago as professor and acting head in 1892. He had been head of the department from 1896 to his retirement in 1931 as professor emeritus. His early associates at Chicago were Oskar Bolza and Heinrich Maschke. These three men supplemented each other perfectly. In the words of Professor Bliss: "Moore was brilliant and aggressive in his scholarship, Bolza rapid and thorough, and Maschke more deliberate but sagacious and without doubt one of the most delightful lecturers on geometry of all time."

Moore was the recipient of many honors, including an honorary Ph.D. from Göttingen in 1899, an LL.D. from Wisconsin in 1904, and honorary doctorates of science or mathematics from Yale, Clark, Toronto, Kansas and Northwestern. Besides his memberships in American, English, and German mathematical societies, he was a member of the American Academy of Arts and Sciences, the American Philosophical Society and the National Academy of Sciences. Two contributed funds have been established in his honor. The first is held by the American Mathematical Society for the purpose of assisting in the publication of his researches and for the establishment of a permanent memorial to him in the activities of the Society. The second fund has been expended for an unusually fine portrait of Professor Moore which hangs in Eckhart Hall at the University of Chicago. The interest in these funds displayed by his friends and admirers, in Chicago and throughout the country, was a remarkable tribute to him.

CURVES ARISING FROM A SINGLE INFINITY OF TRIANGLES

By MABEL M. YOUNG, Wellesley College

As basal figure, unchanged throughout the discussion, let us assume a parabola with those two of its tangents which pass through a given point in its plane. These two fixed tangents and a variable tangent of the parabola form a single infinity of triangles with which are associated many interesting curves intimately connected with the parabola. In the geometry of a triangle we are familiar with a vast number of points, lines and circles uniquely defined with respect to the three fundamental lines. Here we have a triangle in which two sides are held fast while the third varies continuously according to a simple law. We may ask what law of motion is thereby imposed on some of the significant points and lines associated with the triangle. To attempt a comprehensive study of this problem would indeed be to essay a formidable task. Almost any notable point or line, however, selected for investigation will prove rewarding and may be trusted to present most stimulating questions of theory for consideration. The writer has been interested chiefly in the projective properties of the figure. Free use has been made of theorems of elementary geometry which describe the special points and lines discussed but the curves arising from the motion of these points and lines have been derived by projective methods. They are novel only in that they are simple, familiar forms seen in unfamiliar relationships. The entire configuration has moreover a practical importance in that it offers an easily constructed group of projectively related forms which may to advantage be used for the discussion and illustration of a wide variety of geometric theorems.

Let the given parabola be denoted by π, the fixed tangents by p and q, and the variable tangent by t_i. The lines p and q meet in the fixed point T and t_i meets p and q in the two projective ranges of points, P_i and Q_i. The triangles pqt_i will be denoted by Δ_i, the orthocenters by H_i, the circumcenters by O_i, the centroids by M_i and the centers of the nine-point circles by N_i. The subscripts $i=1, 2, 3$—will be used to denote arbitrarily selected triangles with their associated points; $i=a, b, c$ will be used to designate triangles having special properties.

Known theorems relating to a parabola give useful preliminary information and need only to be stated. Since the altitudes of every triangle of tangents to a parabola meet on the directrix, the locus of the orthocenters of the triangles Δ_i is h, the directrix of π. Since the circle through the intersections of any three tangents of a parabola passes also through the focus, the circumcircles of Δ_i, which pass through T, pass also through a second fixed point, F, and form an elliptic system. Let this system of circumcircles be denoted by S_i. Conversely, any circle of S_i cuts p and q in points which determine a tangent of π. The locus of the circumcenters of Δ_i is o, the line of centers of S_i.

Associated with every triangle is a second, called its orthic triangle, which has as vertices the feet of the altitudes of the given triangle. We now consider the envelope of that side of the orthic triangles of Δ_i which lies opposite t_i. If in the triangles Δ_i we drop perpendiculars from the pairs of variable vertices, P_i and Q_i, upon the opposite sides we form two perspective pencils of parallel lines, corresponding rays of which meet on the directrix of π and determine on q and p two new projective ranges of points, Q_i' and P_i'. Since the ideal points of these ranges correspond the lines $P_i'Q_i'$ envelope a parabola. Thus *the envelope of that side of the orthic triangles of Δ_i which lies opposite t_i is a parabola tangent to p and q.* We shall denote this parabola by π', the tangents $P_i'Q_i'$ by t_i' and the triangles pqt_i' by Δ_i'.

The figure consisting of π and π' and the two series of tangent triangles, Δ_i and Δ_i', repays detailed examination. Associated with the triangles Δ_i we have the locus of orthocenters, which is the directrix of π, and the system of circumcircles through T and the focus. Similarly the locus of the orthocenters of Δ_i' is h', the directrix of π', and the system of circumcircles, S_i', passes through T and the focus, F'. From the geometry of a triangle it is evident that this system may also be described as consisting of the circles which have as diameters the segments from T to the orthocenters of the triangles Δ_i. Since the locus of these points is h, the directrix of π, we see that the circles S_i' pass through T and the foot of the perpendicular from T to that line. Thus F', the focus of π', lies on the directrix of π and the locus of the circumcenters of Δ_i' is a line, o', parallel to h and midway from T to h. The complete figure before us accordingly includes π, π' and their foci and directrices; Δ_i, Δ_i' and the two systems of circumcircles with the line of centers of each. We remark particularly the incidence of the foci and certain of these forms.

The interrelationships of this somewhat complicated configuration are clearly brought out by the use of a certain involution of rays on T. Since the circumcenter and orthocenter of any triangle are a pair of isogonal points as to the triangle the lines joining these points to a vertex make equal angles with the bisectors of the corresponding angle. Thus the lines joining T to the pairs of points O_i and H_i of the triangles Δ_i form an involution of which the bisectors of the fixed angle (pq) are the double rays. We shall denote this involution by I. We show first that the lines from T to the foci of π and π' are a pair in the involution. To that ray TO_i which passes through F must correspond a ray TH_i which is parallel to the axis of π. Since TF' is perpendicular to h, TF' is the correspondent of TF. As an incidental result of this fact we see that TF must be perpendicular to the directrix of π' and that the four lines o, h, o', h' are parallel in pairs, o to h', o' to h. Moreover since the triangles Δ_i and Δ_i' have the sides p and q in common and TO_i' falls on TH_i it follows that each line-pair in the involution TH_i, TO_i arising from the triangles Δ_i, coincides with the line-pair TO_i', TH_i' arising from the triangles Δ_i'. Hence the lines from T to the orthocenter and circumcenter of any triangle of Δ_i meet the lines o' and h' in the circumcenter and orthocenter of the corresponding triangle of Δ_i'.

The relationships just established make possible a very simple construction for π and π' by means of the two systems of circles, S_i and S_i'. We may assume as S_i a system of circles on any two points, T and F. Through T are drawn two lines, p and q. The circles S_i cut p and q in points which determine the tangents of a parabola, π. Since the vertex tangent of a parabola is the pedal of the focus as to the curve the circle with diameter TF determines this tangent. The directrix may then be drawn. A perpendicular from T upon h gives the second base point of the system S_i' by which the tangents of π' may be constructed. Any point, O_1, on the line of centers of S_i determines a circle S_1 which cuts p and q in the points where these lines meet t_1. The perpendicular from T upon t_1 cuts the line of centers of S_i' in O_1' and hence determines the circle S_1' which passes through the intersections of p and q with t_1'. To the circle with diameter TF, which determines the vertex tangent of π, corresponds the circle of S_i which has diameter TF' and determines the vertex tangent of π'. The circle through TFF' determines the third common tangent[1] of π and π'. Those circles of S_i which are tangent at T to p and to q meet q and p again in the points at which these lines touch π.

Projective relationships among the forms just described give rise to many curves projective with π. First of all it is clear that π is projectively related to π'. This follows at once from the fact that the two series of tangents t_i and t_i' determine on each of the common tangents p and q two superposed projective ranges. That is, $P_i \barwedge P_i'$, $Q_i \barwedge Q_i'$. To p and q considered as tangents of π correspond q and p considered as tangents of π'. To the third common tangent correspond two distinct lines. The ideal tangent is self-corresponding. Closely connected with the projectivity of π and π' is the projectivity of the systems of circles, S_i and S_i'. It is evident that two circles S_1 and S_1' having radii TO_1 and TO_1' on two corresponding rays of the involution I have tangents at T which are equally inclined to the bisectors of the angle (pq). The tangent of S_1 at T accordingly makes with the chord TF an angle equal to that between the tangent of S_1' at T and TF'. Thus the common chord of the system S_i subtends an angle in every circle S_1 equal to the angle subtended by the common chord of the system S_i' in the circle S_1'. Hence the systems are projectively related. We may note that the degenerate circles of the systems correspond. The finite and ideal chords of the one correspond respectively to the finite and ideal chords of the other. The projectivity of the conics π and π' and of the two systems of circles S_i and S_i' leads at once to the consideration of the curves arising from the intersection of corresponding elements. These will be discussed in a later section.

Projective ranges on the most important auxiliary lines of our figure give rise to numerous conics projective with π. Of great assistance in the determination of such conics are the many superposed projective flat pencils having T as base point. Such pencils are those formed by (a) the lines through T parallel to

[1] This easily constructed figure furnishes a convenient means of verifying the known properties of the pencil of parabolas touching three finite lines.

the bases t_i of the triangles Δ_i; (b) the perpendiculars from T upon t_i; (c) the medians from T to t_i; (d) the symmedians from T to t_i; (e) the radii of the circumcircles of Δ_i through T; (f) the tangents of the circumcircles at T. If in these pencils corresponding rays are defined as those arising from the same triangle of Δ_i, it is clear that any two of these six pencils are projectively related and that each is projective to the tangents of π and π'. The corresponding rays of certain pencils are also so related as to form involutions. Thus (a) and (b); (a) and (c); (c) and (d); (e) and (f) are pencils in involution. We notice among these the involution TO_i, TH_i of which frequent use has already been made.

We may at once employ this involution I to determine a particular group of conics projective with π. We have seen that every line-pair of I meets o, h, o', h' in four especially defined points O_i, H_i, O_i', H_i'. The four ranges traced by these points are clearly projective with one another and hence give rise to the four conics which are the envelopes of the lines O_iO_i', H_iH_i', O_iH_i, $O_i'H_i'$. The conics are parabolas and are tangent to the double lines of the involution I. Of these curves the most interesting is the envelope of O_iH_i. Since the line joining the circumcenter and orthocenter of a triangle is its Euler line, we see that this parabola is the envelope of the Euler lines of the triangles Δ_i. The Euler line of a triangle passes also through the centroid and nine-point center. The segment OH is bisected by N and divided by M in the ratio $-\frac{1}{2}$. The two point-pairs MH and ON divide each other harmonically. To find the significance of these relationships for the parabola enveloped by O_iH_i we consider those tangents of the curve which with o, h and the ideal tangent have an anharmonic ratio of -1 and $-\frac{1}{2}$ respectively. Specifically we determine n and m where $(oh, nl_\infty) = -1$ and $(oh, ml_\infty) = -\frac{1}{2}$. Then the segment of every tangent of the parabola intercepted between o and h is bisected by n and divided by m in the ratio $-\frac{1}{2}$. Thus n is the locus of the nine-point centers of Δ_i and m the locus of the centroids. Since $(MH, ON) = -1$, we see that m, h, o and n are four harmonic tangents of the parabola. Hence, *the Euler lines of the triangles Δ_i envelope a parabola of which the loci of the centroids, orthocenters, circumcenters and nine-point centers are four harmonic tangents.*

Another conic projective with π is enveloped by the lines joining the mid-points of the segments determined by p and q on a pair of corresponding tangents of π and π'. For each parabola the locus of such mid-points is a straight line tangent to the curve. We denote these loci by t_m and t_m'. If we joint to T the points in which t_i and t_i' meet t_m and t_m' respectively we have the pencils of medians of Δ_i and Δ_i'. Since the latter coincides with the pencil of symmedians of Δ_i, corresponding pairs of lines are equally inclined to the bisectors of angle (pq) and form an involution. Thus the mid-points of the segments on t_i and t_i' trace projective ranges on t_m and t_m'. It may be shown that in this projectivity the ideal points of the ranges correspond. The curve enveloped by the lines joining pairs of corresponding points is a parabola, tangent to t_m and t_m' and to the double lines of the involution. This result may be stated otherwise. From the geometry of a triangle it is known that the diameter of the nine-point circle

which passes through the mid-point of a base bisects the opposite base of the orthic triangle and the segment from the opposite vertex to the orthocenter. In our figure the locus of the latter point is o' and the locus of N is n. Thus the lines of our envelope, which pass through the mid-points of the segments on t_i and t_i', pass also through N_i' and O_i'. The ranges determined on t_m and o' are projective since corresponding points are joined to T by corresponding rays of two projective pencils. The line o' is accordingly a tangent of the curve. Since n bisects the segment intercepted between t_m and o' on every tangent, n also touches the parabola. Hence, *that diameter of the nine-point circle of every triangle Δ_i which passes through the mid-point of the variable base envelopes a parabola tangent to the bisectors of the fixed angle (pq), to t_m, t_m', o' and n.*

Many curves covariant with the variable triangles Δ_i and of order higher than the second arise from the union of corresponding elements in pairs of projectively related forms of the figure. Some of these curves however break into curves of lower order because of self-corresponding elements in the projectivities. The two systems of circumcircles, S_i and S_i', are a special case of two projective systems of circles. Here the systems have a base point in common and the circles of infinite radius correspond in such a manner that the finite chord of one corresponds to that of the other and the ideal chords correspond. Thus instead of the bicircular quartic, which is in general the locus of the intersections of corresponding circles in such systems, we have a circular cubic with double point at T and the ideal line. Corresponding tangents of π and π' also meet in a degenerate curve of order four. Since through T and the ideal point of each bisector of angle (pq) pass two pairs of corresponding tangents of the parabolas, these points are double points on the new locus. Since the ideal tangents are self-corresponding, the curve breaks into this line and a rational cubic passing twice through T and once through each of these ideal points. Of the many rational cubics which arise from the intersection of a pencil of rays on T and the tangents of a projectively related conic, we may mention the pedal of T as to π and the strophoids which are the pedal curves of T as to the group of parabolas tangent to the bisectors of angle (pq). Such examples might be multiplied indefinitely.

We make use of the method just described to determine the locus of the symmedian points of the triangles Δ_i. The symmedian point of a triangle is defined as the isogonal conjugate of the centroid. It may also be constructed as the intersection of the lines joining the mid-point of a base to the mid-point of the corresponding altitude. One of several convenient ways of determining the desired locus is to consider the symmedian point of a triangle of Δ_i as the intersection of the symmedian line through T and the line joining the mid-point of the base TP_i to the mid-point of the altitude Q_iP_i'. Since the locus of the mid-points of the altitudes Q_iP_i' is a line through T, the successive pairs of mid-points trace on this line and on p two ranges projective to each other, to P_i and to Q_i. Thus the lines joining the mid-points of the bases on p to the mid-points of the corresponding altitudes on p envelope a conic projective to π and

to the pencil of symmedians on T. Hence, *the locus of the symmedian points of the triangles Δ_i is a rational cubic with double point at T.* Since in the triangles Δ_i the medians and corresponding symmedians on T form a pencil in involution, we see that *the line which is the locus of the centroids of the triangles Δ_i and the rational cubic which is the locus of the symmedian points are projectively related forms of which the centroid and symmedian point of each triangle are a pair of corresponding elements.*

The special relationships which may exist among the forms we have derived are illustrated by the following theorems. The first is based on the conclusions of the preceding paragraph. Since a conic is determined by three tangents and a focus and the second focus is the isogonal conjugate of the first as to the three lines we see that *tangent to the three sides of each triangle of Δ_i is a unique ellipse having the centroid and symmedian point of the triangle as foci. The principal axes of these ellipses envelope a curve of fourth class.* The second theorem is an application of the familiar fact that the parabola determined by four lines has as focus the point of intersection of the circles circumscribing the four triangles formed by the lines and as directrix the line on which lie the orthocenters of these triangles. We consider any triangle Δ_i with the three altitudes P_iQ_i', Q_iP_i', TT_i'. A parabola touching p, q, P_iQ_i', Q_iP_i' has as focus T_i' and as directrix $P_i'Q_i'$. But T_i' is a point on the pedal of T as to π; $P_i'Q_i'$ is a tangent of π', and there is projective correspondence between the pairs T_i' and t_i'. Hence, *the single infinity of parabolas which touch the fixed sides of Δ_i and the successive pairs of altitudes upon the fixed sides have as corresponding foci and directrices pairs of projectively correlated elements on the pedal of T as to π and on π'.* Similarly we may show that *the single infinity of parabolas which touch the common tangents of π and π', p and q, and the successive pairs of tangents t_i and t_i' have as corresponding foci and directrices projectively correlated elements on the cubic derived from the systems S_i and S_i' and on the parabola enveloped by the line H_iH_i'.*

In conclusion one example may be given of the infinite variety of curves to be derived from π and the triangles Δ_i by the use of different methods. Let us take the polar reciprocal of π as to the polar circle of a triangle Δ_1. This circle has its center at the orthocenter of the triangle and with respect to it the triangle is self-conjugate. The parabola is reciprocated into a rectangular hyperbola having asymptotes parallel to the rectangular tangents drawn from H_1 to π. It passes through the vertices and orthocenter of Δ_1 and is tangent at H_1 to the directrix of π. Since the inverse of any point of the circumcircle of a triangle as to the polar circle is the corresponding point on the nine-point circle, then the polar of F, a point on the circumcircle of Δ_1, is the line perpendicular to FH_1 at its mid-point, I_1, through which pass the nine-point circle of Δ_1 and also the vertex tangent of π. But since I_1 falls on v, a perpendicular to FI_1 at I_1 is itself a tangent of π. Hence F is that point of the rectangular hyperbola which corresponds to this tangent of π. When four points of a rectangular hyperbola lie on a circle the point diametrically opposite any one is the orthocenter of the

other three. The points F and H_1 are accordingly ends of a diameter of the hyperbola, I_1 is its center and the tangents at F and H_1 are parallel. Thus the polar reciprocals of π as to the polar circles of the successive triangles Δ_i form a single infinity of rectangular hyperbolas passing through the fixed points T and F, tangent at F to the latus rectum of π and having the directrix of π as a second common tangent. The locus of the centers is the vertex tangent of π. The asymptotes of any curve of the system are parallel to the two perpendicular tangents which may be drawn to π from the point in which the hyperbola touches the directrix of π. We wish to show that these asymptotes have an envelope. Let us consider the pair of perpendicular tangents t_1 and t_2 drawn to π from a point, H_1, on h. These lines meet v in two points, V_1 and V_2, such that FV_1, FV_2 are at right angles to t_1 and t_2. The asymptotes of the corresponding hyperbola, center I_1, are also perpendicular to FV_1, FV_2 and pass through the mid-points of these segments. Thus as the orthocenters of Δ_i and the centers of the corresponding hyperbolas move on the directrix and vertex tangent of π, the feet of the perpendiculars from F on the asymptotes of the successive hyperbolas fall on a line, g, parallel to v and halfway from F to v. If then the various points of v are joined to F the perpendicular to each segment at its mid-point is an asymptote of some hyperbola of the system considered. These asymptotes accordingly envelope a parabola having F as focus, g as vertex tangent and the vertex tangent of π as directrix. The parabola is similar to π and similarly placed, ratio $\frac{1}{2}$.

THERMOSTATIC CONTROL

By J. L. SYNGE, University of Toronto

1. *Introduction.*

The function of thermostatic control of a heating system is to maintain a constant temperature, and, when the temperature is not at the desired level, to bring the temperature to the desired level and to keep it there. We treat here the case of a room, heated by radiators which are heated by an oil-burner. With an ideal thermostatic control, the burner would operate whenever the temperature of the room fell below the desired level, and would cease to operate when that level was exceeded. Actually, two controlling temperatures (or thermostatic temperatures) are involved, namely, a temperature α_1 at which the burner commences to operate when the temperature is falling and a temperature α_2 at which it ceases to operate when the temperature is rising; and α_2 is greater than α_1, although the difference is small. It is obvious from a practical point of view that if the oscillations of the temperature of the room are large, we are justified in neglecting $\alpha_2-\alpha_1$, and treating the problem as one in which an ideal control with a single temperature α is involved. But when the oscillations are small this is no longer the case. However the problem furnished by

the ideal control is of interest for its own sake, and we shall consider it as well as the case of more practical interest where $\alpha_2 - \alpha_1$ is not zero.

For the ideal control ($\alpha_1 = \alpha_2$) it is found that if the temperature of the external atmosphere is constant, the temperature of the room oscillates about the desired level, the amplitude of the oscillations and their periodic time both tending to zero as the time increases indefinitely.

For the practical control ($\alpha_1 \neq \alpha_2$) it is found that if the temperature of the external atmosphere is constant, the oscillations of the temperature of the room tend asymptotically to a state of steady oscillation, whose amplitude is of the order of $h^{2/3}$ and whose periodic time is of the same order, where $h = \alpha_2 - \alpha_1$, assumed small. The limiting amplitude and periodic time are evaluated in terms of the constants of the system. There is an interesting and simple relation between the intervals during which the burner is, and is not, in operation in the steady oscillation: we find in fact that the ratio of these intervals is equal to the ratio of the differences between the temperature for which the thermostat is set and two fundamental temperatures. These fundamental temperatures are (i) the temperature of the external atmosphere, (ii) the temperature to which the room would tend were the burner to remain in continuous operation.

The problem of these thermal oscillations has an exact dynamical analogue in the oscillations of a damped harmonic oscillator of the dead-beat type, subject to a discontinuous disturbing force. Considering for simplicity the case where the two thermostatic temperatures coincide (ideal control), this disturbing force is a repulsive force of constant intensity when the particle approaches nearer to the centre of oscillation than a certain position (corresponding to the thermostatic temperature) and is zero when the particle is on the other side of this position. The ultimate fate of the particle is to perform oscillations of infinitesimal amplitude and periodic time about what we may call its "thermostatic position." This position can hardly be called a position of equilibrium—we do not define the force acting on it in that precise position.

The dynamical analogue to the case where the thermostatic temperatures are different is easily described.

2. *The system.*

The thermal system consists of four parts:

(1) the burner;

(2) the water in the radiators (temperature $= \theta_2$);

(3) the atmosphere in the room (temperature $= \theta_3$);

(4) the external atmosphere (temperature $= \theta_4$).

We assume, for simplicity, a uniform temperature throughout the water and a uniform temperature throughout the room. If we assume that, when in operation, the burner supplies heat at a uniform rate to the water, and that the linear law of cooling is applicable throughout, the variations of temperature in the system are governed by the differential equations,

$$
(1) \qquad \begin{cases} \dfrac{d\theta_2}{dt} = \epsilon k_{21} - k_{23}(\theta_2 - \theta_3), \\ \dfrac{d\theta_3}{dt} = k_{32}(\theta_2 - \theta_3) - k_{34}(\theta_3 - \theta_4), \end{cases}
$$

where $\epsilon = 1$ when the burner is in operation and $\epsilon = 0$ when the burner is not in operation; k_{21}, k_{23}, k_{32}, k_{34} are *positive* constants of the system; θ_4 is supposed to be a known function of t.

If we denote by α_1, α_2, $(\alpha_2 > \alpha_1)$ the operating temperatures of the thermostat, then

$$
(2) \qquad \epsilon = 1 \text{ if } \theta_3 < \alpha_1; \qquad \epsilon = 0 \text{ if } \theta_3 > \alpha_2.
$$

In the range $\alpha_1 < \theta_3 < \alpha_2$, ϵ is determined through the previous history of the system by the condition that ϵ *always retains its value except when compelled to change it in order to satisfy* (2).

The solutions of the differential equations (1) are uniquely determined if the values of θ_2, θ_3 are assigned for $t = 0$, provided that θ_3 lies outside the range (α_1, α_2); if θ_3 lies inside this range, we require also to be told the initial value of ϵ.

3. *Reduction to dynamical form.*

The equations (1) may be replaced by the equivalent equations

$$
(3) \qquad \frac{d^2\theta_3}{dt^2} + K_1\frac{d\theta_3}{dt} + K_2\theta_3 = \epsilon K_3 + f(t),
$$

$$
(4) \qquad \theta_2 = \frac{1}{k_{32}}\frac{d\theta_3}{dt} + \left(1 + \frac{k_{34}}{k_{32}}\right)\theta_3 - \frac{k_{34}}{k_{32}}\theta_4,
$$

where

$$
(5) \qquad \begin{cases} K_1 = k_{23} + k_{32} + k_{34} > 0, \\ K_2 = k_{23}k_{34} > 0, \\ K_3 = k_{21}(k_{32} + k_{34}) > 0, \\ f(t) = k_{34}\left(\dfrac{d\theta_4}{dt} + k_{23}\theta_4\right). \end{cases}
$$

Equation (3) is the equation of motion of a damped harmonic oscillator, subject to the discontinuous disturbing force ϵK_3 and the varying disturbing force $f(t)$. Since temperatures must be continuous, the particle undergoes no abrupt change of position (θ_3), and, by (4), no abrupt change in velocity $(d\theta_3/dt)$, since θ_2 is to be continuous: that is to say, no impulsive forces are operative.

Let us assume that the temperature (θ_4) *of the external atmosphere is constant.* Let us put

$$(6) \qquad \theta_4 = \frac{C}{K_2}, \quad f(t) = C.$$

The roots of the characteristic equation of (1), namely

$$(7) \qquad \lambda_1 = \tfrac{1}{2}(-K_1 + \sqrt{K_1^2 - 4K_2}), \ \lambda_2 = \tfrac{1}{2}(-K_1 - \sqrt{K_1^2 - 4K_2}),$$

are seen, by (5), to be both real and negative. Thus when $\epsilon = 1$ or when $\epsilon = 0$, the oscillations given by (3) are of the dead-beat type—that is to say, there is at most one maximum or minimum of θ_3 as long as ϵ maintains the same value. The centres of the dead-beat oscillation are

$$(8) \qquad \begin{cases} \theta_3 = \dfrac{K_3 + C}{K_2} \text{ when } \epsilon = 1; \\[2ex] \theta_3 = \dfrac{C}{K_2} \text{ when } \epsilon = 0. \end{cases}$$

These positions have simple interpretations, mechanically and thermally. Mechanically, they represent the equilibrium positions of the particle with and without the disturbing force K_3, provided that these positions lie in each case in the appropriate region. Thermally, the former represents the highest steady temperature $\bar{\theta}$ at which the burner, in continuous operation, can maintain the room, and the latter represents the steady temperature of the room when the burner does not operate at all—namely, the external temperature θ_4.

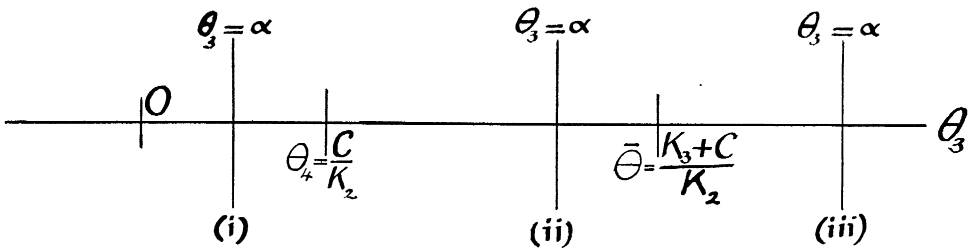

Fig. 1.

We may consider three cases, shown in Fig. 1., α_1 and α_2 being situated close to α:

(i) $\alpha_1 < \alpha_2 < \theta_4$: here the thermostat is set for a temperature below the external temperature.

(ii) $\theta_4 < \alpha_1 < \alpha_2 < \bar{\theta}$: here the thermostat is set for a temperature intermediate between the external temperature and the highest steady temperature to which the burner can heat the room.

(iii) $\bar{\theta}<\alpha_1<\alpha_2$: here the thermostat is set for a temperature above the highest steady temperature to which the burner can heat the room.

Cases (i) and (iii) are of little interest, practically or theoretically. In Case (i), it is easily seen that after at most one stationary value of θ_3, there is a dead-beat motion to θ_4, to which temperature the room tends steadily. Case (iii) is similarly discussed, and we may state the rather obvious conclusions:

In Case (i) the temperature of the room ultimately tends steadily to the external temperature θ_4.

In Case (iii) the temperature of the room ultimately tends steadily to the highest steady temperature $\bar{\theta}$ at which the burner can maintain it.

4. *The general nature of the oscillations.*

Let us turn to the interesting and practical Case (ii) where

$$\theta_4 < \alpha_1 < \alpha_2 < \bar{\theta}, \quad \theta_4 = C/K_2, \quad \bar{\theta} = (C + K_3)/K_2. \tag{9}$$

It is easy to see in a general way how the system behaves. Let us suppose, for definiteness, that initially $\theta_3<\alpha_1$. We start, then, with a dead-beat motion to $\theta_3=\bar{\theta}$. When $\theta_3=\alpha_2$ is encountered, this motion is replaced by a dead-beat motion to $\theta_3=\theta_4$. This carries θ_3 back across the position $\theta_3=\alpha_1$, and the original type of motion is resumed. It is clear that *the system oscillates across the thermostatic range $\alpha_1<\theta_3<\alpha_2$, and there is an infinite number of oscillations.* If we plot θ_3 against t, the graph will be a sinuous continuous curve, with a continuous first derivative, crossing the strip $\alpha_1<\theta_3<\alpha_2$ an infinite number of times. The second derivative has a discontinuity every time the curve crosses $\theta_3=\alpha_1$ from above or $\theta_3=\alpha_2$ from below.

To study the oscillations, it is convenient to introduce a change of variable suggested from mechanics. Let us write

$$v = \frac{d\theta_3}{dt}, \tag{10}$$

so that

$$\frac{d^2\theta_3}{dt^2} = v\frac{dv}{d\theta_3}. \tag{11}$$

Our differential equation (3) transforms into

$$v\frac{dv}{d\theta_3} + K_1 v + K_2\theta_3 = \epsilon K_3 + C; \tag{12}$$

this has the integrals

$$\left[v - \lambda_1\left(\theta_3 - \frac{\epsilon K_3 + C}{K_2}\right)\right]^{\lambda_1} = A\left[v - \lambda_2\left(\theta_3 - \frac{\epsilon K_3 + C}{K_2}\right)\right]^{\lambda_2}, \tag{13}$$

where λ_1, λ_2 are given by (7) and A is an arbitrary constant.

Let us now consider a graphical representation of the history of the system on a plane in which θ_3, v are taken as rectangular coordinates. This is, in fact, the "q, p" representation of the dynamical problem. Each possible history of the system is represented by a curve, which is continuous (since θ_3 and v are continuous), but which has discontinuities in direction on crossing $\theta_3=\alpha_1$ with $v<0$, and on crossing $\theta_3=\alpha_2$ with $v>0$. The sense of description of a curve is given by (10), which shows that since $dt>0$, θ_3 increases or decreases according as v is positive or negative. Equation (13) (with the appropriate values of ϵ) gives a singly infinite family of curves, covering the plane; the strip $\alpha_1<\theta_3<\alpha_2$ is covered twice.

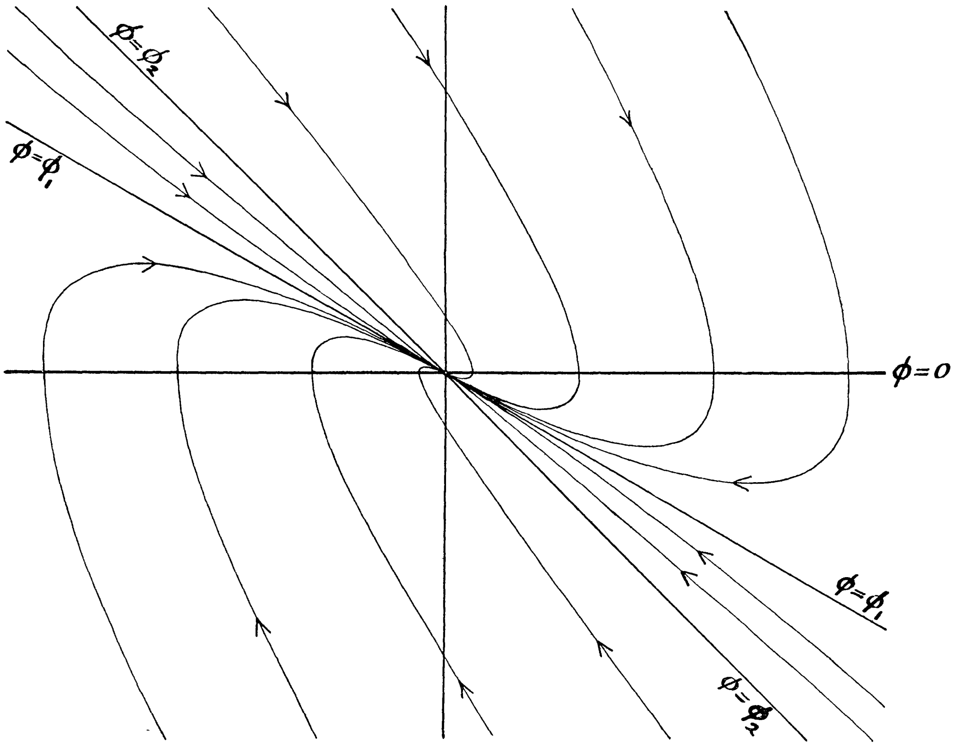

Fig. 2.

It is convenient to think of the complete families, given by substituting $\epsilon=1$ and $\epsilon=0$ in (13). The actual family would be given by drawing these complete families and then blotting out the part of the family $\epsilon=1$ for which $\theta_3>\alpha_2$, and the part of the family $\epsilon=0$ for which $\theta_3<\alpha_1$.

The family $\epsilon=1$ may be obtained from the family $\epsilon=0$ by a translation through a distance K_3/K_2 in the direction of the axis of θ_3. Either family may be obtained from a single member of it by a magnification with respect to the pole, which for $\epsilon=1$ is at $\theta_3=(K_3+C)/K_2$, $v=0$ and for $\epsilon=0$ at $\theta_3=C/K_2$, $v=0$. In

polar coordinates ρ, ϕ, with the pole as origin and the axis of θ_3 as initial line, the equation of either family is

$$(14) \qquad \rho^{\lambda_1-\lambda_2} = B[\sin(\phi - \phi_2)]^{\lambda_2}[\sin(\phi - \phi_1)]^{-\lambda_1},$$

where B is an arbitrary constant and

$$(15) \qquad \tan\phi_1 = \lambda_1, \quad \tan\phi_2 = \lambda_2.$$

The lines $\phi=\phi_1$, $\phi=\phi_2$ (which lie in the second quadrant, since λ_1, λ_2 are negative) also belong to the family: they are touched by the other curves of the family at $\rho=0$ (for $\phi=\phi_1$) and at $\rho=\infty$ (for $\phi=\phi_2$). The curves meet the axis of θ_3 orthogonally, except that the pole, where they are tangent to $\phi=\phi_1$. The curves[1] are plotted in Fig. 2, the sense of description being shown by the arrows.

Fig. 3 shows the nature of the oscillations about $\theta_3=\alpha$ in the case of ideal control ($\alpha_1=\alpha_2=\alpha$).

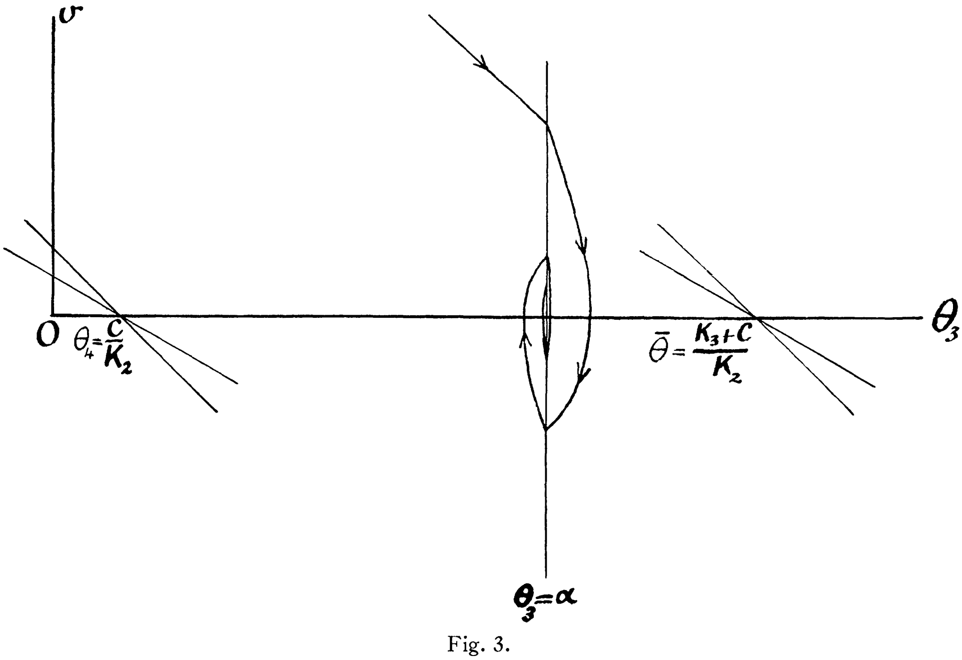

Fig. 3.

5. *The general nature of the ultimate oscillations.*

It is natural to expect that the temperature of the room tends, with indefinitely increasing time, to some definite temperature or to some simple type of oscillation. Indeed, we expect, in the case of ideal control, that the temperature tends to coincidence with the thermostatic temperature. In the case of the

[1] Although Figs. 2 and 3 are actual plottings, the values of the constants ($\phi_1=150°$, $\phi_2=135°$) were selected with a view to securing clear figures rather than a good approximation to conditions likely to arise in practice.

practical control, it is more difficult to see beforehand what the ultimate state will be.

Let us consider first the case of ideal control ($\alpha_1 = \alpha_2 = \alpha$). Let us suppose that the system passes from lower temperatures to the thermostatic temperature ($\theta_3 = \alpha$) at the point A (Fig. 4), passes on to a maximum temperature (say

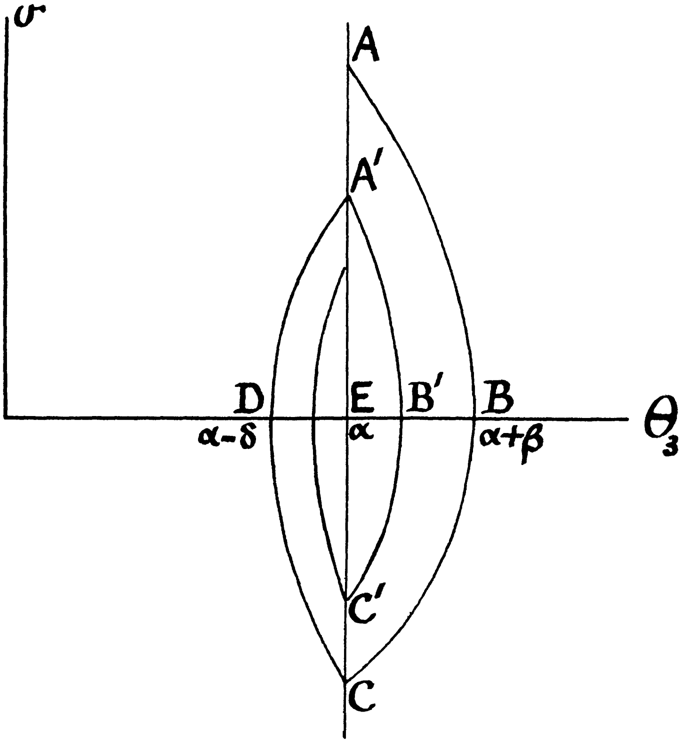

Fig. 4.

$\theta_3 = \alpha + \beta$) at B, and falls again to the thermostatic temperature at C. The burner does not operate during this process, and hence the displacement of the representative point in our diagram is governed by the equation

$$v \frac{dv}{d\theta_3} + K_1 v + K_2 \theta_3 = C. \tag{16}$$

If v_1, v_2 denote the two values of v corresponding to a general value of θ_3 between α and $\alpha + \beta$, v_1 corresponding to the arc AB and v_2 to the arc BC, we have from (16)

$$\frac{1}{2} \frac{d}{d\theta_3}(v_1^2 - v_2^2) + K_1(v_1 - v_2) = 0, \tag{17}$$

and hence we find

$$AE^2 - CE^2 = 2K_1 \int_{\alpha}^{\alpha+\beta} (v_1 - v_2) d\theta_3. \tag{18}$$

Since $v_1 - v_2 > 0$, we have $AE > CE$, where AE, CE represent positive numbers. When the burner comes into operation at C, and the system describes the arc CDA', it can be shown in precisely similar manner that $CE > A'E$. Therefore $AE > A'E$. Thus, in each complete oscillation, the points where the broken curve cuts $\theta_3 = \alpha$ approach the point E steadily, one from above and the other from below. These two sequences converge to limit points, which, obviously, either coincide at E or are equidistant from it. It is easy to see that the latter possibility must be ruled out. For let us suppose that there are limit points L, M such that $LE = ME$: the curve of the system passing through L in the positive sense will then cut ME internally. But in the history of the system there will be a curve which starts from a point on $\theta_3 = \alpha$ indefinitely close to L. If this initial point is sufficiently close to L, the curve will cut the lower half of $\theta_3 = \alpha$ above the point M. This is inconsistent with the hypothesis that M is the limit of the points on the lower half of $\theta_3 = \alpha$. Thus we see that the points of intersection with $\theta_3 = \alpha$ converge steadily to the point where that line cuts $v = 0$: in other words, *under ideal control the temperature of the room oscillates about the thermostatic temperature with steadily decreasing amplitude, and tends to it in the limit.*

Let us now consider the case of practical control ($\alpha_2 > \alpha_1$). Let us suppose that θ_3 increases up to α_2, and the burner ceases to operate at A in Fig. 5. The system then runs down the curve ABC, cutting $v = 0$ at B (say $\theta_3 = \alpha_2 + \beta$) and $\theta_3 = \alpha_1$ at C. The system then passes up CDA' cutting $v = 0$ at D ($\theta_3 = \alpha_1 - \delta$, say). For ABC we have the differential equation (12) with $\epsilon = 0$, and for CDA' the same equation with $\epsilon = 1$. Let us write

$$h = \alpha_2 - \alpha_1. \tag{19}$$

To each point A on the upper half of the line $\theta_3 = \alpha_2$ there corresponds a point A', obtained after a complete oscillation. From our theory of the ideal control, we know that, if the height of A above the line $v = 0$ is large compared with h, then A' will lie below A. On the other hand, if the distance of A from $v = 0$ is small compared with h, then a simple consideration of the general nature of the curves shows that A' will lie above A. Hence, as we move A down the line $\theta_3 = \alpha_2$, it overtakes its correspondent A', and thus there is a position of A (say $\bar{A}$) which coincides with its correspondent A'. Through this point there passes a closed curve, having corners where it meets $\theta_3 = \alpha_2$ and $\theta_3 = \alpha_1$: this represents a steady oscillation.

Now suppose the system starts at A, with a fairly large value of v, so that the corresponding A' lies below A. Following on the curve of the history after A', we get a spiral which (by the nature of the curves) can never cut itself. Thus the corner on $\theta_3 = \alpha_2$ moves steadily downwards: it cannot approach $v = 0$

indefinitely, because we have seen that it would commence to move up, which it cannot do, since the spiral does not cut itself. Thus the corner tends to the limit $\bar{A}$, and ultimately the motion of the system approaches asymptotically the closed circuit through $\bar{A}$. That, then, is the ultimate nature of the oscillation.

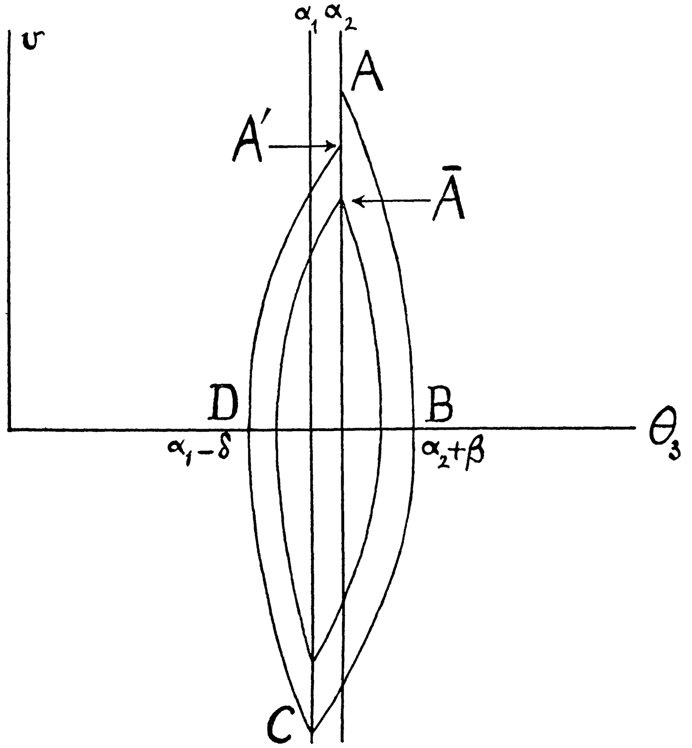

Fig. 5.

Let us assume that h is small; it is clear then that v_A will be small, when we approach the steady oscillation. We see from (18) that the loss of v-amplitude on passing from A to the point where ABC cuts $\theta_3=\alpha_2$ is of the order of β, which is of the order of $v_A{}^2$. There is a loss of the same order on passing from C along CDA' to the intersection with $\theta_3=\alpha_1$. In the steady oscillation this double loss of amplitude must be compensated by a gain in passing twice through the region $\alpha_1<\theta_3<\alpha_2$. This gain is of the order of

$$h\left(\frac{dv}{d\theta_3}\right)_A,$$

which, by (12), is seen to be of the order of h/v_A. Thus $v_A{}^2$ and h/v_A are of the same order. *Hence the v-amplitude of the steady oscillation is of the order of $h^{1/3}$, and hence the θ_3-amplitude is of the order of $h^{2/3}$.*

6. *Quantitative results concerning ultimate oscillations.*

For the case of ideal control, we wish to investigate (i) the limit of the periodic time of the small oscillations about the thermostatic temperature; (ii) the law of decay of amplitude of these oscillations.

For the case of practical control, we wish to investigate (i) the amplitude of the steady oscillation to which the system tends; (ii) the periodic time of this oscillation, and the parts of this time during which the burner is, and is not, in operation; (iii) the law of approach to the steady oscillation.

Up to a certain point, we can carry out these investigations in common, by treating the case of practical control ($h>0$), and then putting $h=0$ to get the case of ideal control.

We deduce from (12) the following values corresponding to $v=0$:

$$\frac{d\theta_3}{dv} = 0, \quad \frac{d^2\theta_3}{dv^2} = \frac{1}{\epsilon K_3 + C - K_2\theta_3}, \quad \frac{d^3\theta_3}{dv^3} = \frac{2K_1}{(\epsilon K_3 + C - K_2\theta_3)^2}. \tag{20}$$

If we write

$$L = K_2\alpha_2 - C > 0, \quad L' = K_3 + C - K_2\alpha_1 > 0, \tag{21}$$

we have

$$\begin{cases} \text{on } ABC(\epsilon = 0): \theta_3 = \alpha_2 + \beta - \dfrac{1}{2}\dfrac{v^2}{L + K_2\beta} + \dfrac{1}{3}\dfrac{K_1 v^3}{(L + K_2\beta)^2} + \cdots, \\ \text{on } CDA'(\epsilon = 1): \theta_3 = \alpha_1 - \delta + \dfrac{1}{2}\dfrac{v^2}{L' + K_2\delta} + \dfrac{1}{3}\dfrac{K_1 v^3}{(L' + K_2\delta)^2} + \cdots. \end{cases} \tag{22}$$

Applying these equations to the points (A, C) and (C, A'), we have

$$\begin{cases} \beta = \dfrac{1}{2}\dfrac{v_A^2}{L + K_2\beta} - \dfrac{1}{3}\dfrac{K_1 v_A^3}{(L + K_2\beta)^2} + \cdots, \\ h + \beta = \dfrac{1}{2}\dfrac{v_C^2}{L + K_2\beta} - \dfrac{1}{3}\dfrac{K_1 v_C^3}{(L + K_2\beta)^2} + \cdots, \\ \delta = \dfrac{1}{2}\dfrac{v_C^2}{L' + K_2\delta} + \dfrac{1}{3}\dfrac{K_1 v_C^3}{(L' + K_2\delta)^2} + \cdots, \\ h + \delta = \dfrac{1}{2}\dfrac{v_{A'}^2}{L' + K_2\delta} + \dfrac{1}{3}\dfrac{K_1 v_{A'}^3}{(L' + K_2\delta)^2} + \cdots. \end{cases} \tag{23}$$

Now β, δ are of the order of v_A^2, and hence their omission from the denominators causes only an error in terms of the order of v_A^4. Hence, to the order of v_A^3, we have

$$(24)\quad \begin{cases} 2L\beta = v_A^2 - \dfrac{2}{3}\dfrac{K_1}{L}v_A^3, & 2Lh + 2L\beta = v_C^2 - \dfrac{2}{3}\dfrac{K_1}{L}v_C^3, \\ 2L'\delta = v_C^2 + \dfrac{2}{3}\dfrac{K_1}{L'}v_C^2, & 2L'h + 2L'\delta = v_{A'}^2 + \dfrac{2}{3}\dfrac{K_1}{L'}v_{A'}^3. \end{cases}$$

Hence we deduce

$$(25)\qquad v_{A'}^2 + \frac{2}{3}\frac{K_1}{L'}v_{A'}^2 = 2L'h + v_C^2 + \frac{2}{3}\frac{K_1}{L'}v_C^3,$$

or, remembering that h is of the order of $v_A{}^3$, and $v_A > 0$, $v_C < 0$, $v_A{}' > 0$,

$$(26)\qquad v_{A'}^2 = 2L'h + v_C^2 + \frac{4}{3}\frac{K_1}{L'}v_C^3.$$

Also

$$(27)\qquad v_C^2 = 2Lh + v_A^2 - \frac{4}{3}\frac{K_1}{L}v_A^3,$$

and hence

$$(28)\qquad v_{A'}^2 = 2h(L + L') + v_A^2 - 4K_1(L'^{-1} + L^{-1})v_A^3/3,$$

which gives, for the decrease of amplitude in an oscillation,

$$(29)\qquad -dv_A = v_A - v_{A'} = \tfrac{2}{3}K_1(L'^{-1} + L^{-1})v_A^2 - h(L + L')v_A{}^{-1}.$$

For the steady oscillation, $dv_A = 0$: this gives for the v-amplitude $(\bar{v})$ of the steady oscillation, approximately,

$$(30)\qquad \bar{v}^3 = \frac{3h}{2K_1}\frac{L + L'}{L^{-1} + L'^{-1}} = \frac{3h}{2K_1}LL',$$

and for the corresponding θ_3-amplitudes, by (24),

$$(31)\qquad \beta = \frac{\bar{v}^2}{2L} = \frac{1}{2L}\left(\frac{3h}{2K_1}LL'\right)^{2/3}, \quad \delta = \frac{\bar{v}^2}{2L'} = \frac{1}{2L'}\left(\frac{3h}{2K_1}LL'\right)^{2/3}.$$

As for the periodic time of the steady oscillation, or an oscillation adjacent thereto, we have

$$(32)\qquad t(ABC) = \int_A^C (d\theta_3/v), \quad t(CDA') = \int_C^{A'} (d\theta_3/v).$$

In the former we use the approximate value given by (22),

$$(33a)\qquad d\theta_3 = -\,vdv/L,$$

and in the latter

$$(33b)\qquad d\theta_3 = vdv/L'.$$

Thus

$$t(ABC) = (v_A - v_C)/L, \quad t(CDA') = (v_{A'} - v_C)/L'. \tag{34}$$

But we have approximately

$$v_A - v_C = 2v_A, \quad v_{A'} - v_C = 2v_A, \tag{35}$$

so that we may write

$$t(ABC) = 2v_A/L, \quad t(CDA') = 2v_A/L'. \tag{36}$$

Thus in the steady oscillation the periodic time is

$$\tau = 2(L^{-1} + L'^{-1})\bar{v}, \tag{37}$$

and

$$\frac{t(\bar{A}BC)}{t(CD\bar{A})} = \frac{L'}{L}. \tag{38}$$

It will be well to restate the results obtained in an interpretable form:

In the case of practical thermostatic control, with the operating temperatures α_1, α_2, where $h(=\alpha_2-\alpha_1)$ is positive but small, the ultimate state is one of steady oscillation through a temperature range of the order of $h^{2/3}$. The ratio of the temperature amplitudes above and below the thermostatic range is approximately

$$\frac{\beta}{\delta} = \frac{L'}{L} = \frac{\bar{\theta} - \alpha}{\alpha - \theta_4}, \tag{39}$$

where θ_4 is the (steady) temperature of the external atmosphere, $\bar{\theta}$ the maximum temperature to which the room can be raised by continuous operation of the burner, and α is the thermostatic temperature (that is, the mean of the approximately equal temperatures α_1, α_2). These amplitudes are

$$\left\{\begin{aligned} \beta &= \frac{1}{2K_2(\alpha - \theta_4)}\left\{\frac{3hK_2^2}{2K_1}(\alpha - \theta_4)(\bar{\theta} - \alpha)\right\}^{2/3}, \\ \delta &= \frac{1}{2K_2(\bar{\theta} - \alpha)}\left\{\frac{3hK_2^2}{2K_1}(\alpha - \theta_4)(\bar{\theta} - \alpha)\right\}^{2/3}, \end{aligned}\right. \tag{40}$$

where K_1, K_2 are constants of the system, defined in (5).

In a steady oscillation the ratio of the time during which the burner is idle to that during which it is in operation is

$$\frac{t(\bar{A}BC)}{t(CD\bar{A})} = \frac{L'}{L} = \frac{\bar{\theta} - \alpha}{\alpha - \theta_4}, \tag{41}$$

the same as the ratio of the amplitudes, and the periodic time of the steady oscillation is

$$\tau = 2(\bar{\theta} - \theta_4)\left\{\frac{3h}{2K_1K_2(\alpha - \theta_4)^2(\bar{\theta} - \alpha)^2}\right\}^{1/3}. \tag{42}$$

Stress may perhaps be laid on the result concerning the relative times of operation and rest of the burner on account of its simplicity. Let us point out its practical implications. Let us suppose that the external temperature is 10° and the thermostatic temperature is 70°: let us suppose that the burner in continuous operation would raise the room to 100°. (Actually, the water in the radiators would probably boil under continuous action of the burners: it is not essential to our theory that this "maximum possible" temperature should be physically realizable: it is merely a function of the constants of the system.) In this case the time during which the burner operates is twice the time during which it is idle, the ratio being (70–10)/(100–70). The temperature $\bar{\theta}$, it will be remembered, is given by

$$\bar{\theta} = \frac{K_3 + C}{K_2} = \frac{K_3}{K_2} + \theta_4;$$

here K_2, K_3 are permanent constants of the system. Thus

$$\bar{\theta} - \theta_4 = \frac{K_3}{K_2}, \tag{43}$$

a permanent constant of the system: it can therefore be found by one observation. Thus if on any occasion the values of the external temperature (θ_4) and the thermostatic temperature (α) are noted, and also the ratio of the times during which the burner is idle and operative respectively, we can compute from (41) the value of $\bar{\theta}$, and hence from (43) the value of K_3/K_2. We are then able to predict from (41) the ratio of times of idleness and operation for any assigned external temperature θ_4; the formula can be written in the form

$$\frac{t(\bar{A}BC)}{t(DC\bar{A})} = \frac{K_3}{K_2}\cdot\frac{1}{\alpha - \theta_4} - 1, \tag{44}$$

which, naturally, tends to infinity as θ_4 tends to α.

In the case of ideal control, we have seen that the amplitude of the oscillations tends to zero: it follows from (42), on putting $h=0$, that *the periodic time also tends to zero.*

It remains to answer the questions raised regarding the decay of the oscillations. Let us consider first the case of practical control ($h>0$). Writing in (29)

$$v_A = \bar{v}(1 + \xi), \tag{45}$$

where ξ is small, we have

$$\begin{aligned} - d\xi &= \{(4/3)K_1\bar{v}(L^{-1} + L'^{-1}) + h\bar{v}^{-2}(L + L')\}\xi \\ &= 3h\bar{v}^{-2}(L + L')\xi, \end{aligned} \tag{46}$$

for the decrease of ξ in a time τ, which is given by (37). Hence the law of decay is

$$\frac{d\xi}{dt} = -\frac{3h}{2\bar{v}^3}LL'\xi = -K_1\xi. \tag{47}$$

Thus ξ decreases according to the law

$$\xi = (\xi)_{t=0}e^{-K_1t}, \tag{48}$$

and v_A according to the law

$$v_A = \bar{v} + (v_A - \bar{v})_{t=0}e^{-K_1t}. \tag{49}$$

In the case of the ideal control ($h=0$), we return to (29), now in the form

$$-dv_A = \tfrac{2}{3}K_1(L^{-1} + L'^{-1})v_A^2, \tag{50}$$

this giving the decrease in v_A in time τ, given by (36) as

$$\tau = 2(L^{-1} + L'^{-1})v_A. \tag{51}$$

Thus v_A decays according to the law

$$\frac{dv_A}{dt} = -\tfrac{1}{3}K_1v_A, \quad v_A = (v_A)_{t=0}e^{-K_1t/3}, \tag{52}$$

and, by (29), β and δ decay according to the law

$$\beta = (\beta)_{t=0}e^{-2K_1t/3}, \quad \delta = (\delta)_{t=0}e^{-2K_1t/3}. \tag{53}$$

It is interesting to note that the rate of decay (in one case, towards steady oscillation and, in the other case, towards steady temperature) depends only on the constant K_1, given by (5). The greater K_1, the more rapid the decay. This is very natural in the mechanical analogue, since the frictional resistance is proportional to K_1. In the thermal problem, we see that if we are to increase the rate of decay, we are to increase k_{23}, k_{32}, k_{34}—one or more of them: k_{21}, which determines the rate at which the burner supplies heat to the water, has, strangely enough, nothing to do with it. Since k_{23}, k_{32} determine the rate at which heat flows from the radiators into the room, it is natural to expect an increase of rate of decay by increasing them. But it is strange to find that the rate of decay is increased by increasing k_{34}, that is, by decreasing the insulation between the room and the external atmosphere.

A CONVERGENCE TEST AND A REMAINDER THEOREM

By W. F. LIBBY, Berkeley, California

The specification of the terms of an infinite series necessarily involves a correspondence between the values and the ordinal numbers of the terms of the series. This paper is concerned with series for which this correspondence is an analytic one.

We define a *regular* series to be a series $\sum_{n=1}^{\infty} u(n)$ where $u(z)$ is a function of z which is analytic at infinity. Then, with each regular series, there is associated a set of constants $c_0, c_1, c_2, \cdots$ and a least index $L \geqq 1$ such that

$$u(n) = c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \cdots \qquad n \geqq L; \tag{1}$$

and conversely any series $\sum u(n)$ for which (1) holds is a regular series.

THEOREM 1. *If $\sum u(n)$ is a regular series, then*

$$\sum_{n=L+1}^{\infty}\left[u(n) - c_0 - \frac{c_1}{n}\right] = \sum_{n=L+1}^{\infty} \sum_{k=2}^{\infty} \frac{c_k}{n^k}$$

where the series on the left is an absolutely convergent simple series and the series on the right is an absolutely convergent double series.

To prove this theorem, we note first that the series $c_0 + c_1 L^{-1} + c_2 L^{-2} + \cdots$ converges and hence has uniformly bounded terms. Thus there is a constant M such that $|c_k| L^{-k} < M$ or $|c_k| < ML^k$, $k = 0, 1, 2, \cdots$. Therefore when $n > L$

$$\frac{|c_2|}{n^2} + \frac{|c_3|}{n^3} + \cdots < M\left\{\left(\frac{L}{n}\right)^2 + \left(\frac{L}{n}\right)^3 + \cdots\right\}$$

$$= M\frac{n}{n-L}\left(\frac{L}{n}\right)^2 \leqq M(L+1)L^2\frac{1}{n^2}$$

and absolute convergence of the double series in the right member follows from convergence of the series $\sum n^{-2}$. With this fact established, absolute convergence of the series in the left member and equality of the sums follow from well known theorems.

From Theorem 1 we obtain immediately the formula

$$\sum_{n=1}^{\infty}\left[u(n) - c_0 - \frac{c_1}{n}\right] = \sum_{n=1}^{L}\left[u(n) - c_0 - \frac{c_1}{n}\right] + \sum_{n=L+1}^{\infty} \sum_{k=2}^{\infty} \frac{c_k}{n^k}$$

and the following corollary: *In order that a regular series may be convergent, it is necessary and sufficient that $c_0 = c_1 = 0$.*

We now give a theorem which indicates the number of terms of certain regular series which it is sufficient to take to ensure that the remainder shall be less than a specified term.

THEOREM 2. *Any convergent regular series for which the coefficients c_n are real and non-negative has a value S given by*

$$S = \sum_{n=1}^{G+h} u(n) + \theta u(G) \tag{2}$$

where G is any integer $\geqq L$, h is any integer $\geqq G^2 - G$, and θ is a real constant depending on G and h for which $0 < \theta < 1$.

The theorem being trivial when $c_n=0$ for all n, we can suppose in our proof that $c_n\neq 0$ for at least one value of n; then $u(n)>0$ for all $n\geqq L$. Evidently $S>\sum_{n=1}^{G+h} u(n)$. Hence we can complete the proof of Theorem 2 by showing that

$$S < \sum_{n=1}^{G+h} u(n) + u(G). \tag{3}$$

But (3) is equivalent to the inequality $\sum_{n=G+h+1}^{\infty} u(n) < u(G)$ which, since convergence of $\sum u(n)$ implies $c_0=c_1=0$, can be written

$$\sum_{k=2}^{\infty} c_k\left(\sum_{n=G+h+1}^{\infty} \frac{1}{n^k}\right) < \sum_{k=2}^{\infty} c_k\left(\frac{1}{G^k}\right). \tag{4}$$

To establish (4) it is, since $c_k\geqq 0$, sufficient to show that

$$\sum_{n=G+h+1}^{\infty} \frac{1}{n^k} < \frac{1}{G^k} \qquad k = 2, 3, 4, \cdots. \tag{5}$$

On multiplying (5) by G^k, we see that (5) holds for $k=2, 3, 4, \cdots$ when and only when it holds for $k=2$. Hence it will be sufficient to show that

$$\sum_{n=G+h+1}^{\infty} \frac{1}{n^2} < \frac{1}{G^2}. \tag{6}$$

But

$$\sum_{n=G+h+1}^{\infty} \frac{1}{n^2} < \int_{G+h}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{G+h}.$$

Hence for (6) and therefore (3), it is sufficient to have $1/(G+h)\leqq 1/G^2$ or $h\geqq G^2-G$ and Theorem 2 is proved.

An interesting special case of Theorem 2 is that for which $G=1$; then we may take $h=0$ and $S\leqq 2u(1)$ follows.

Let us apply Theorem 2 to the following series with $L=1$

$$\sum_{n=1}^{\infty} \frac{8}{(4n-3)(4n-1)}$$

and check the result with the known sum, π. Taking $G=2$ and $h=4$, we find

$$S = \pi = 8\left[\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\frac{1}{195}+\frac{1}{223}+\frac{1}{483}+\frac{\theta}{195}\right]$$

so that $3.1416=3.184+8[\theta/195]$ or $\theta=.665$ approximately; thus $0<\theta<1$ as the theorem states.

The author is much indebted to Professor Agnew of Cornell University for valuable suggestions as to shortening and simplifying the paper. The proof of Theorem 2 was especially so simplified.

POWER SUMS OF ROOTS

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Let $r_1, r_2, \cdots r_n$ be the n roots of the equation $f(x) \equiv x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n = 0$, so that $a_h (h=1, 2, \cdots, n)$ is the sum of their products taken h at a time; and let s_k be the sum of their kth powers, where k is any number, not necessarily integral or even real.

The relation

$$s_w - a_1 s_{w-1} + a_2 s_{w-2} - \cdots + (-1) a_w w = 0$$

for w any positive integer $\leqq n$, is well known. Since $s_0 = n$, for $w = n$ this may be written more symmetrically

$$s_n - a_1 s_{n-1} + a_2 s_{n-2} - \cdots + (-1) a_n s_0 = 0.$$

More generally

$$F(w) \equiv s_w - a_1 s_{w-1} + a_2 s_{w-2} - \cdots + (-1)^n a_n s_{w-n}$$

is identically zero *for any* w. For let $F_i(w) = r_i^w - a_1 r_i^{w-1} + a_2 r_i^{w-2} - \cdots + (-1)^n a_n r_i^{w-n}$, so that $F(w) = \sum_{i=1}^n F_i(w)$. Then $F_i(w) = r_i^{w-n} f(r_i) = 0$ since r_i is a root of $f(x) = 0$. Therefore $F(w) = 0$.

Power Sums of Roots of the Cubic $x^3 + a_2 x - a_3 = 0$.

If $a_1 = s_1 = 0$, then $s_2 = a_1 s_1 - 2a_2 = -2a_2$; $s_3 = a_1 s_2 - a_2 s_1 + 3a_3 = 3a_3$; $s_4 = a_1 s_3 - a_2 s_2 + a_3 s_1 = -a_2 s_2 = s_2^2/2$. In general, the relation between s_2, s_3, and sums of higher integral powers of the roots assumes a strikingly simple and elegant form, expressed in the

Theorem: If $n = 3$ and $a_1 = s_1 = 0$, and w is a positive integer > 3; then

$$s_w = w \sum \frac{(p_2 + p_3 - 1)!}{p_2! p_3!} \left(\frac{s_2}{2}\right)^{p_2} \left(\frac{s_3}{3}\right)^{p_3},$$

where the summation is over all non-negative integral values of p_2 and p_3 such that $w = 2p_2 + 3p_3$. (It is evident that $p_2 + p_3 \geqq 1$.)

For example, for $w = 4$ the only possible values for p_2 and p_3 are $p_2 = 2$, $p_3 = 0$. Hence we have

$$s_4 = 4 \frac{1!}{2!0!} \left(\frac{s_2}{2}\right)^2 = \frac{s_2^2}{2},$$

a result already noted.

To prove the theorem, consider all partitions of w of the form

$$w = 2 + 2 + \cdots + 2 + 3 + 3 + \cdots + 3.$$

Permute the terms of each partition in all possible ways. Let A be the class of those permutations whose final term is 2, B the class of those whose final term is

3. For example for $w=8$, we have A : 2222, 332; B: 233, 323. For a given partition containing p_2 2's and p_3 3's, the number of permutations in A, keeping the final 2 fixed, is

$$\frac{(p_2 + p_3 - 1)!}{(p_2 - 1)!p_3!};$$

and the number in B, keeping the final 3 fixed, is

$$\frac{(p_2 + p_3 - 1)!}{p_2!(p_3 - 1)!}.$$

Since $a_1=0$, $s_w = -a_2s_{w-2}+a_3s_{w-3}$. Then s_{w-2} and s_{w-3} may themselves be similarly decomposed, and by successive reduction we ultimately obtain

$$s_w = \sum_A (-a_2)^{p_2-1}(a_3)^{p_3}s_2 + \sum_B (-a_2)^{p_2}(a_3)^{p_3-1}s_3.$$

But $-a_2=s_2/2$ and $a_3=s_3/3$, and hence

$$\begin{aligned} s_w &= \sum_A \left(\frac{s_2}{2}\right)^{p_2-1}\left(\frac{s_3}{3}\right)^{p_3} s_2 + \sum_B \left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3-1} s_3 \\ &= \sum_A \left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3} 2 + \sum_B \left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3} 3 \\ &= \sum \left[\frac{(p_2 + p_3 - 1)!2}{(p_2 - 1)!p_3!} + \frac{(p_2 + p_3 - 1)!3}{p_2!(p_3 - 1)!}\right]\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3}. \end{aligned}$$

And since $2p_2+3p_3=w$, we have finally

$$s_w = w\sum \frac{(p_2 + p_3 - 1)!}{p_2!p_3!}\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3}.$$

For example we find:

$$s_5 = 5s_2s_3/6, \qquad s_6 = s_2^3/4 + s_3^2/3,$$

$$s_7 = 7s_2^2s_3/12, \qquad s_8 = s_2^4/8 + 4s_2s_3^2/9,$$

$$s_9 = 3s_2^3s_3/8 + s_3^3/9, \qquad s_{10} = s_2^5/16 + 5s_2^2s_3^2/12,$$

$$s_{11} = 11s_2^4s_3/48 + 11s_2s_3^3/54, \qquad s_{12} = s_2^6/32 + s_2^3s_3^2/3 + s_3^4/27,$$

$$s_{13} = 13s_2^5s_3/96 + 13s_2^2s_3^3/54, \qquad s_{14} = s_2^7/64 + 35s_2^4s_3^2/144 + 7s_2s_3^4/81,$$

$$s_{15} = 5s_2^6s_3/64 + 25s_2^3s_3^3/108 + s_3^5/81, \text{ etc.}$$

It may be noted that these relations hold for $n>3$ if $s_1=a_1=a_4=a_5=\cdots=a_n=0$, since to the roots of $x^3+a_2x-a_3=0$ we merely add $n-3$ zero roots to obtain those of $x^n+a_2x^{n-2}-a_3x^{n-3}=0$.

A Symbolic Expansion for a Power Sum

The relation

$$s_w = a_1s_{w-1} - a_2s_{w-2} + \cdots + (-1)^{n-1}a_ns_{w-n}$$

may be written

$$s_w^{(0)} = a_1s_w^{(1)} - a_2s_w^{(2)} + \cdots + (-1)^{n-1}a_ns_w^{(n)},$$

where $s_w^{(h)}$ symbolically represents s_{w-h}; or, since w is a variable, in the equivalent form

$$s_w^{(k)} = a_1s_w^{(k+1)} - a_2s_w^{(k+2)} + \cdots + (-1)^{n-1}s_w^{(k+n)}. \tag{1}$$

Theorem: $s_w^{(0)}$ may be symbolically represented by

$$\left[a_1s_w^{(1)} - a_2s_w^{(2)} + \cdots + (-1)^{n-1}a_ns_w^{(n)}\right]^p$$

where p is any positive integer, and the superscripts in parentheses operate as exponents and are assigned their meanings only after expansion.

This may be proved by induction. It is true for $p=1$. It only remains to show that $S^p=S^{p+1}$, where

$$S \equiv \left[a_1s_w^{(1)} - a_2s_w^{(2)} + \cdots + (-1)^{n-1}a_ns_w^{(n)}\right].$$

Expanding S^p we have

$$S^p = \sum\frac{p!}{p_1!p_2!\cdots}a_1^{p_1}(-a_2)^{p_2}\cdots s_w^{(p_1+2p_2+\cdots)}, \tag{2}$$

where the summation is for all sets of non-negative p_i such that $\sum p_i=p$.

$$\begin{aligned} S^{p+1} &= S^pS \\ &= \sum\frac{p!}{p_1!p_2!\cdots}a_1^{p_1}(-a_2)^{p_2}\cdots s_w^{(p_1+2p_2+\cdots)}S, \end{aligned} \tag{3}$$

where the expansion is to be completed. But by (1), $s_w^{(p_1+2p_2+\cdots)}S$ gives $s_w^{(p_1+2p_2+\cdots)}$ after expansion, so that the completion of (3) will yield the same result as (2).

As an example of the theorem, for $n=2$, $p=1$, we have $s_w=a_1s_{w-1}-a_2s_{w-2}$; but also for $n=2$, $p=2$, we have $s_w=a_1^2s_{w-2}-2a_1a_2s_{w-3}+a_2^2s_{w-4}$; etc.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

RECURSION RELATIONS IN CERTAIN EXPANSION COEFFICIENTS

By R. S. UNDERWOOD, Texas Technological College

In the expansions of $\cos(n\theta)$ and $\sin(n\theta)/\sin\theta$ in terms of descending powers of the cosine, there are simple recursion relations by which the coefficients for successive n's may be computed almost as easily as the triangle of binomial coefficients may be constructed. It is proposed here to derive these relations and incidentally to obtain some interesting identities involving binomial coefficients, as (9) and (10) below.

Expanding $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$, and equating real and imaginary coefficients, we get the well-known equation

$$(1)\ \cos(n\theta) = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$(2)\ \frac{\sin(n\theta)}{\sin\theta} = \binom{n}{1}\cos^{n-1}\theta - \binom{n}{3}\cos^{n-3}\theta\sin^2\theta + \binom{n}{5}\cos^{n-5}\theta\sin^4\theta - \cdots$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}\,.$$

If we express $\cos(n\theta)$ in terms of descending powers of the cosine, we have, from (1) and (2)

$$(3)\qquad \cos(n\theta) = A_{n,0}\cos^n\theta - A_{n,1}\cos^{n-2}\theta + A_{n,2}\cos^{n-4}\theta - \cdots \pm A_{n,r}\cos^{n-2r}\theta \mp \cdots$$

$$(4)\qquad \frac{\sin(n\theta)}{\sin\theta} = B_{n,0}\cos^{n-1}\theta - B_{n,1}\cos^{n-3}\theta + B_{n,2}\cos^{n-5}\theta - \cdots \pm B_{n,r}\cos^{n-2r-1}\theta \mp \cdots;$$

where

$$(5)\qquad A_{n,0} = 1 + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}\,,$$

$$(6)\qquad A_{n,r} = \binom{n}{2r} + \binom{r+1}{1}\binom{n}{2r+2} + \binom{r+2}{2}\binom{n}{2r+4} + \cdots,$$

$$(7)\qquad B_{n,0} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1},$$

$$(8)\qquad B_{n,r} = \binom{n}{2r+1} + \binom{r+1}{1}\binom{n}{2r+3} + \binom{r+2}{2}\binom{n}{2r+5} + \cdots.$$

Comparing $A_{n,r}$ with the same coefficient derived by another method in Loney's Trigonometry, we have

$$(9)\qquad \binom{n}{2r} + \binom{r+1}{1}\binom{n}{2r+2} + \binom{r+2}{2}\binom{n}{2r+4} + \cdots = 2^{n-2r-1}\left[\binom{n-r}{r} + \binom{n-r-1}{r-1}\right]$$

which holds wherever the binomial coefficient is defined, including the case $n=2$, $r=1$ if we define

$$\binom{n}{0} = \binom{0}{0} = 1.$$

Similarly, we have, using $B_{n,r}$

$$(10)\qquad \binom{n}{2r+1} + \binom{r+1}{1}\binom{n}{2r+3} + \cdots = 2^{n-2r-1}\binom{n-r-1}{r}.$$

THEOREM: $A_{n,r} = 2A_{n-1,r} + A_{n-2,r-1}$.

Proof:

$$\begin{aligned}
\cos(n\theta) &= \cos\theta\cos(n-1)\theta - \sin\theta\sin(n-1)\theta \\
&= \cos\theta\cos(n-1)\theta - \left[(\tfrac{1}{2})\cos(n-2)\theta - (\tfrac{1}{2})\cos(n\theta)\right]
\end{aligned}$$

$$\begin{aligned}
(11)\qquad \therefore \cos(n\theta) &= 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta \\
&= 2\cos\theta\left[2^{n-2}\cos^{n-1}\theta - \cdots \pm A_{n-1,r}\cos^{n-2r-1}\theta \mp \cdots\right] \\
&\quad - \left[2^{n-3}\cos^{n-2}\theta - \cdots \mp A_{n-2,r-1}\cos^{n-2r}\theta \pm \cdots\right].
\end{aligned}$$

Equating the coefficients of $\cos^{n-2r}\theta$ in (3) and (11) we have the recursion formula of the theorem.

Similarly, writing

$$\begin{aligned}
\sin(n\theta) &= \sin\theta\cos(n-1)\theta + \cos\theta\sin(n-1)\theta \\
&= \sin\theta\left[2^{n-2}\cos^{n-1}\theta - \cdots \pm A_{n-1,r}\cos^{n-2r-1}\theta \mp \cdots\right. \\
&\quad \left. + \cos\theta\sin\theta\left[2^{n-2}\cos^{n-2}\theta - \cdots \pm B_{n-1,r}\cos^{n-2r-2}\theta\cdots\right],\right.
\end{aligned}$$ we have

THEOREM: $B_{n,r} = A_{n-1,r} + B_{n-1,r}$.

We may then easily write successive rows of the arrays

1				1			
2	1			2			
2^2	3			2^2	1		
2^3	8	1		2^3	4		
2^4	20	5		2^4	12	1	
2^5	48	18	1	2^5	32	6	
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

In the left array the upper element in each column is 1 and other elements of all columns except the first are found as illustrated by the examples: $8=2\cdot 3+2$, $5=2\cdot 1+3$, $18=2\cdot 5+8$, etc. In the right array, $4=1+3$, $12=4+8$, $6=1+5$, $32=12+20$, etc., where, in the first equation for example, 1 is above 4 and 1 and 3 are in corresponding positions in the two arrays.

From the sixth rows of the arrays, for example, we have

$$\cos(6\theta) = 2^5\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1,$$

$$\frac{\sin(6\theta)}{\sin\theta} = 2^5\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$$

THE CUBIC $x^2+y^2+z^2-2xyz=1$

By A. S. MERRILL, University of Montana

The theory of multiple correlation suggests the cubic appearing as the title of this discussion. The formula for the coefficient of multiple correlation with three variables is, in the usual notation,

$$r_{0.12} = \sqrt{\frac{r_{01}^2 + r_{02}^2 - 2r_{01}r_{02}r_{12}}{1 - r_{12}^2}}\,.$$

When this coefficient assumes its limiting values, ± 1, we have the relation

$$r_{01}^2 + r_{02}^2 + r_{12}^2 - 2r_{01}r_{02}r_{12} = 1,$$

or, in terms of x, y, and z,

$$x^2 + y^2 + z^2 - 2xyz = 1. \tag{1}$$

It can be shown by methods which are used in any first course in analytic geometry that plane sections of this cubic surface include all of the conic sections.

I. *The intersection with each coordinate plane is a circle.* For, replacing one of the coordinates, say z, by 0 in (1) we have

$$x^2 + y^2 = 1.$$

II. *The intersection with a plane parallel to any one of the coordinate planes at a distance of* 1 *is a straight line.* For, substituting 1 for z we have

$$x^2 + y^2 - 2xy = 0;$$

that is,

$$x - y = 0.$$

III. *The intersection with such a plane when the distance is less than* 1 *is an ellipse.* For, substituting c for z we have

$$x^2 + y^2 - 2cxy = 1 - c^2.$$

Transforming by a 45° rotation we obtain

$$\frac{x^2}{1 + c} + \frac{y^2}{1 - c} = 1.$$

IV. *The intersection with such a plane when the distance is greater than* 1 *is an hyperbola.* The proof is exactly as in III. The resulting equation represents an hyperbola since in this case $1-c$ is negative.

V. *The intersection with a plane which bisects the angle formed by a pair of co-ordinate planes is a parabola cut by a straight line perpendicular to its axis.* If we rotate the entire surface through 45° about the x axis, equation (1) becomes

$$x^2 + y^2 + z^2 - xy^2 + xz^2 = 1. \tag{2}$$

One of the planes in question now coincides with the xy-plane. The equation of the intersection is found by substituting 0 for z in (2). This gives

$$x^2 + y^2 - xy^2 = 1,$$

or

$$(x - 1)(y^2 - 1 - x) = 0.$$

The straight line is easily identified as one of the lines of (II) above. The size, shape, and position of the parabola are evident from its equation. The substitution $y=0$ in (2) gives the intersection made by a corresponding perpendicular plane. Its equation is

$$(x + 1)(z^2 + x - 1) = 0.$$

Evidently the two parabolas are oppositely oriented.

End points of parallel axes of successive parallel ellipses of III and vertices of corresponding hyperbolas of IV lie upon these parabolas. The straight line is the degenerate conic between the ellipses and hyperbolas. On each parabola, the points near the vertex are end-points of *minor* axes; farther out, there are end-points of major axes; and finally, vertices of the hyperbolas.

If x, y, and z are limited to values not greater than 1 (as would be necessary in the correlation theory), equation (1) represents a closed surface. A fairly satisfactory model can be constructed from wall board. Laminae of elliptical shape fastened together will produce the required solid. One axis of these laminae varies from 0 to $2\sqrt{2}$ while the perpendicular axis varies from $2\sqrt{2}$ to 0. The successive values depend upon the thickness of the wall board. Configurations upon this closed surface include the three circles of I, the family of ellipses of II, the six (partial) parabolas of V, and a regular tetrahedron whose vertices are the points (1, 1, 1), (1, −1, −1), (−1, 1, −1) and (−1, −1, 1).

On a Certain Transformation of Infinite Series[1]

By J. A. Shohat, University of Pennsylvania.

Introduction. The actual evaluation of the sum of a given convergent series is very often greatly hampered, if not rendered impossible, by the slowness of the convergence (example: $\pi/4=1-1/3+1/5-1/7+\cdots$). Euler, Markoff and others[2] have given practical methods for improving the convergence, which, however, are omitted even in many advanced courses on infinite series. Thus, it seems worthy of interest to give an extremely simple transformation of infinite series which, in case of convergence, often renders it more rapid, and thus the series in question becomes more useful for practical applications. In case of divergence, the same transformation often yields the asymptotic expression (for $n\to\infty$) of S_n—sum of its first n terms. We give concrete illustrations for both cases, for the latter by means of Stirling's formula.

I. *Transformation of* S_n. The transformation referred to above is:

$$S_n \equiv \sum_{i=1}^{n} u_i = \sum_{i=1}^{n-1} i(u_i - u_{i+1}) + nu_n, \tag{1}$$

an identity holding true for any $u_1, u_2, \cdots, u_n$ (special case of Abel's transformation). Thus we are led to consider two series:

$$(U)\ \sum_{n=1}^{\infty} u_n, \quad (V)\ \sum_{n=1}^{\infty} v_n \qquad (v_n = n(u_n - u_{n+1})).$$

We know that, if in the series $\sum_{n=1}^{\infty} a_n$, $\lim_{n=\infty} na_n$ exists and $\neq 0$, then the series diverges (the converse is not true). This, in conjunction with (1), leads at once to several interesting conclusions.

Theorem I. *If* $\lim_{n=\infty} nu_n = l$ *exists, then the two series* (U) *and* (V) *are either both convergent or both divergent, according as* $l=0$ *or* $l\neq 0$. *In case of convergence, the two series have the same sum.*

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[1] Read before the Pennsylvania Chapter of Pi Mu Epsilon, in October, 1932.

[2] Cf. K. Knopp, *Theorie und Anwendungen der Unendlichen Reihen*, 2nd ed. (1924), pp. 242–249.

THEOREM II. *If* (U) *and* (V) *both converge, then* $\lim_{n=\infty} nu_n = 0$; *hence* (U) *and* (V) *have the same sum.*

It is the latter property which makes the transformation (1) very valuable as the illustrations below indicate. The following theorem, not directly connected with the main line of the present discussion, is also of frequent use in the theory of infinite series.

THEOREM III. *If* $\{u_n\}$ *is a monotonic decreasing sequence and* (U) *is convergent, then necessarily* $\lim_{n=\infty} nu_n = 0$.

In fact, here (V) consists of positive terms; hence, it either converges or diverges to $+\infty$. In the first case our statement is proved by Theorem I; in the second case, (1) shows, since (U) converges, that $\lim_{n=\infty} nu_n = -\infty$, which is impossible.

2. *Illustrations in case of convergence.* (i) Consider the series

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{2}$$

Here

$$u_n = \frac{1}{n^2}, \quad v_n = n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{2}{(n+1)^2} + \frac{1}{n(n+1)^2}$$

$$\frac{\pi^2}{6} = 2\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} - 2$$

$$\frac{\pi^2}{6} = 2\cdot\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} - 2$$

$$\frac{\pi^2}{6} = 2 - \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}. \tag{3}$$

If we apply once more the transformation (1), we get:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = 3\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)^2}$$

$$= 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)^2} - \frac{3}{4}$$

$$\frac{\pi^2}{6} = \frac{13}{8} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)^2}. \tag{4}$$

We thus have transformed (2) *into more rapidly convergent series* (3) *and* (4), the remainders, i.e. the sums of all terms starting with the $(n+1)$st being respectively of order of magnitude n^{-1}, n^{-2}, n^{-3}. With little computation, we could repeat the same transformation again. We also notice that, using the series (3) *and* (4), we get numerical values *limiting* π *both from above and below.* Thus,

$$\frac{\pi^2}{6} > \frac{13}{8} + \frac{1}{2}\cdot\frac{1}{2^2\cdot 3^2}, \quad \frac{\pi^2}{6} < 2 - \frac{1}{4} - \frac{1}{2\cdot 9}$$

$$\frac{1}{6}\sqrt{354} < \pi < \frac{1}{6}\sqrt{366};\ 3.13\cdots < \pi < 3.18\cdots. \tag{5}$$

Hence, taking one term in (4) and two terms in (3), we obtain the value of π with an error not exceeding 0.05. It can be shown (making use of $\int_1^n x^{-2}dx$) that

$$\lim_{n=\infty} n\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right] = 1.$$

It follows that about *twenty* terms are needed in (2), if we wish to obtain π with the above accuracy.

(ii) Apply the transformation (1) to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}. \tag{6}$$

We get, if n is even: $n = 2m$,

$$S_{2m} = \sum_{i=1}^{2m-1}(-1)^{i-1}\left[1 + \frac{1}{4i^2-1}\right] - \frac{2m}{4m-1},$$

and a similar formula for S_{2m+1}. Letting $m\to\infty$, we thus obtain:

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \mp \cdots = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2-1}. \tag{7}$$

The series (7), being an alternating one, yields both upper and lower bound for for π. It converges far more rapidly than (6). If, for example, an accuracy of 0.01 for $\pi/4$ is desired, we need 50 terms in (6), and only 5 terms in (7).

3. *Illustration: some elementary summations.* It is, we believe, of some interest to show that the same transformation (1) yields, in many cases, with very little or no computation, the *precise expression* for certain $S_n = \sum_{i=1}^n u_i$. As an example, the arithmetic series $\sum_{i=1}^n i$ may serve. Here, by (1),

$$\sum_{i=1}^{n} i = -\sum_{i=1}^{n-1} i + n^2 = -S_{n-1}^{(1)} + n^2 = -S_n^{(1)} + n^2 + n;\ S_n^{(1)} = \frac{n(n+1)}{2}.$$

Similarly,

$$S_n^{(2)} \equiv \sum_{i=1}^{n} i^2 = -\sum_{i=1}^{n-1} i(2i+1) + n^3 = -2S_n^{(2)} - S_{n-1}^{(1)} + n^3 + 2n^2$$

$$S_n^{(2)} = \frac{1}{3}\left[n^3 + 2n^2 - \frac{(n-1)n}{2}\right] = \frac{n(n+1)(2n+1)}{6}.$$

In the same way we can evaluate $S_n{}^{(3)} \equiv \sum_{i=1}^n i^3$, $S_n^{(4)}, \cdots$. In general, if the precise expression of $\sum_{i=1}^n u_i$ is known, (1) gives at once the expression for

$\sum_{i=1}^{n} i(u_i - u_{i+1})$. For example, applying (1) to $\sum_{i=1}^{n} q^i [=(1-q^{n+1})/(1-q)]$, we get immediately:

$$\sum_{i=1}^{n} iq^{i-1} = \frac{1-q^{n+1}}{(1-q)^2} - \frac{(n+1)q^n}{1-q}.$$

Of course, we cannot expect such an elementary transformation as (1) to work successfully in all cases, as is shown by the series $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots$.

4. *Illustration in case of divergence.* We close our discussion by showing that

$$\lim_{n=\infty} \frac{n!}{n^{n+1/2}e^{-n}} \text{ exists, } = l, \text{ with } 0 < l < \infty. \tag{8}$$

[Stirling's formula gives for l the value $(2\pi)^{1/2}$]. Applying the transformation (1) to $\sum_{i=1}^{n} \log i$, we get:

$$\sum_{i=1}^{n} \log i = \sum_{i=1}^{n-1} i \log \frac{i}{i+1} + n \log n = -\sum_{i=1}^{n-1} i \log\left(1 + \frac{1}{i}\right) + n \log n. \tag{9}$$

Taking 3 terms in the expansion for $\log(1+x)$, we have:

$$\log\left(1 + \frac{1}{i}\right) = \frac{1}{i} - \frac{1}{2i^2} + \frac{\theta_i}{3i^3} \qquad (i \geqq 1;\ 0 < \theta_i < 1),$$

which, substituted in (8), leads to

$$\sum_{i=1}^{n} \log i = n \log n - (n-1) + \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i} - \frac{1}{3} \sum_{i=1}^{n-1} \frac{\theta_i}{i^2}. \tag{10}$$

Making use of the convergence of the series $\sum_{i=1}^{\infty} \theta_i/i^2$ and of the existence of $\lim_{n=\infty} (\sum_{i=1}^{n} 1/i - \log n)$ (which is readily established, by means of $\int_1^n dx/x$), we obtain from (10) immediately:

$$\lim_{n=\infty} \left(\sum_{i=1}^{n} \log i - n \log n - \tfrac{1}{2} \log n + n \right) \text{ exists, } = B, \text{ with } 0 < B < \infty,$$

which is but another way of writing (8).

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Caractéristiques des systèmes différentiels et propagation des ondes. By Tullio Levi-Civita. Paris, Librairie Felix Alcan, 1932. x+114 pages. 20 francs.

Mathématiques financières. By J. Dubourdieu. Paris, Armand Colin, 1932. 220 pages. 10.50 francs; bound, 12 francs.

The Principles of Financial and Statistical Mathematics. By Maximilian Philip. The College of the City of New York Series in Commerce, Civics, and Technology. New York, Prentice-Hall, Inc., 1932. xx+406 pages. $3.50.

A First Course in Calculus. By E. S. Crawley and P. A. Caris. New York, F. S. Crofts & Co., 1933. x+342 pages.

Introductory College Algebra, Revised. By H. L. Rietz and A. R. Crathorne. xii+306+24 pages. $1.76.

Analytic Geometry. By Frederick S. Nowlan. New York, McGraw-Hill Book Company, 1933. xii+296 pages. $2.25.

Gli elemente d'Euclide e la critica antica e moderna. Edited by Federigo Enriques. Libro X. Bologna, Nicola Zanichelli, 1932. 326 pages.

Foundations of the Theory of Algebraic Numbers. By Harris Hancock. Volume II, The General Theory. New York, The Macmillan Company, 1932. xxviii +654 pages. $8.00.

C. A. Bjerknes, Sein Leben und seine Arbeit. By V. Bjerknes. German translation by Else Wegener-Koppen. Berlin, Julius Springer, 1933. vi+218 pages. RM 8.60.

Numerology. By E. T. Bell. Baltimore, Williams and Wilkins, 1933. viii+188 pages. $2.00.

Vector Analysis. By H. B. Phillips. New York, John Wiley, 1933. viii+236 pages. $2.50.

The Universe of Science. By H. Levy. New York, The Century Company, 1933. xvi+224 pages. $2.50.

Mathematical Excursions. By Helen A. Merrill. Norwood, Mass., The Norwood Press, 1933. xiv+146 pages.

Arithmetic for Teachers. By Harriet E. Glazier. New York, McGraw-Hill, 1932. xvi+292 pages. $2.00.

Elementary Mathematical Analysis. Volume II. By Mayme I. Logsdon. New York, McGraw-Hill, 1933. x+188 pages. $1.75.

Number, the Language of Science. By Tobias Dantzig. Second Edition, Revised. New York, Macmillan, 1933. xiv+262 pages. $2.50.

(According to the preface, the revision has been confined to the correction of errors, the recasting of a few passages, and the addition of a page of bibliographical notes. This book was reviewed in the Monthly in March, 1931, issue, page 164.)

Scripta Mathematica

A new periodical published by Yeshiva College, Amsterdam Avenue and 186th Street, New York, N. Y.

The mathematical world suffered a distinct loss when the publication of *Bibliotheca Mathematica* was suspended in 1915. Its editor, Gustav Eneström,

shortly before his death, expressed the wish to David Eugene Smith that some way could be found to continue this journal in America. Acting on this suggestion, the Mathematical Association of America endeavored to secure a subsidy fund sufficient to insure the success over a period of years of such an undertaking. Failing in this attempt, the Association, late in 1928, undertook an international canvass to determine whether a sufficiently large advance subscription list could be obtained to guarantee the project. The result was very enthusiastic *moral* support but not enough subscription pledges to warrant going ahead. Possibly the impending world-wide financial crash was already casting its shadow before it.

But now, in the midst of the depression, Yeshiva College in New York City comes forward with financial support[1] for a new quarterly journal, *Scripta Mathematica*, "the first of a series of publications planned by the college" with the ideal of "learning for the sake of learning." This journal, under the editorship of Jekuthiel Ginsburg, professor of mathematics in Yeshiva College, is to be devoted "to the philosophy, history, and expository treatment of mathematics," and will be entirely international in its scope. High standards for this journal will be assured by the character of its editorial board which includes as associate editors, Raymond Clare Archibald, Cassius Jackson Keyser, Louis Charles Karpinski, Gino Loria, Vera Sanford, Lao Genevra Simons, and David Eugene Smith. Although it will be seen that this publication is not in any sense a continuation of the *Bibliotheca Mathematica*, which it is hoped will sometime be revived, it promises a series of historical articles which will be welcomed as supplementing those which frequently appear in the MONTHLY.

The two numbers already issued (September and December, 1932) show a wide variety of interest and appeal—to the scholar, to the casual reader in other fields as well as in mathematics, and to any intelligent person whose curiosity leads him to wonder about the difficulties encountered by the human race in developing such a fundamental subject as mathematics. Any light which can be thrown upon the age-long growing pains of mathematics by historical research is not only of interest on its own account but is likely to point the way by analogy to a sympathetic attitude toward the difficulties now experienced with mathematical studies by the children of the present generation. For this and many other reasons we welcome the advent in this country of a historical journal in the field of mathematics.

Space will not permit a critical review here of the many interesting topics discussed in these two numbers, but the reviewer hereby confesses to the surrender of his own cold and critical attitude and warmly recommends their careful reading to the whole constituency of this MONTHLY. Without prejudice to the other contributions, all of which seem very worthy, consider for example, the

[1] It goes without saying that Yeshiva College hopes for a liberal subscription list which, so far as possible, may diminish the subsidy required and thus release funds for other contemplated publications. The subscription price is three dollars per year, which is a very reasonable charge for a volume of more than 380 pages of high character and excellent makeup.

following random samples: "Gaspard Monge, Politician," by David Eugene Smith, an article based on Smith's manuscript collection now donated to Columbia University;[1] "The Meaning and the Bearings of Mathematics," two articles by Cassius Jackson Keyser, written in his own clear and convincing style so well known to the mathematical world: "A. L. Cauchy in the History of Analytic Geometry," by Gino Loria; "Jacob ben Machir's Version of Menelaus's Work on Spherical Trigonometry," a translation by Jekuthiel Ginsburg, parts (1) and (2) and to be continued;[2] "Rare Mathematical Books in the University of Michigan Library," by Louis C. Karpinski; "The Ancient Peruvian Abacus," by L. Leland Locke, author of the richly illustrated work on *The Ancient Quipu or Peruvian Knot Record*, published by the American Museum of Natural History; "The Concept of Infinity," by D. Mordukhai-Boltovskoy, the first of a series of historical notes on this topic, translated from the Russian by Ginsburg; "A German-American Algebra of 1837," by Lao Genevra Simons. All told there are twenty-one signed[3] articles in the two numbers, aside from several excellent Book Reviews and a Department of Notes and Queries conducted by Raymond Clare Archibald. Of this latter department the reviewer wishes to say that it had long been his desire to see such a department attached to, or connected with, this MONTHLY and edited by Archibald, than whom probably there is no one in America or in Europe who is better qualified to develop such a department. Already in the second number this department occupies eighteen pages, much of which is of absorbing interest. Under the heading, "*Bibliographia de Mathematicis*," Archibald has listed some forty names of mathematicians together with references concerning them to articles or books many of which would be difficult for the average reader to obtain or the existence of which in many cases might be entirely unknown to the casual observer. He proposes to extend this list and keep it as complete as possible in the succeeding issues of the journal. In this and many other ways this department will render a real service. Perhaps, after all, it is better so rather than as a supplement to the MONTHLY.

To Yeshiva College the reviewer would extend his congratulations, believing that the new journal will soon make an honorable name for itself in the mathematical world, and will fulfill a very useful mission.

H. E. SLAUGHT

Topology. By Solomon Lefschetz. American Mathematical Society Colloquium Publications, Vol. XII. New York, 1930. ix+410 pages. $4.50.

Since the appearance of Veblen's book[4] on topology in the Colloquium series

[1] This magnificent collection of original manuscript will afford a rich source of historical material for use of this journal for many years to come.

[2] In the first article Ginsburg's name, as translator and editor, is omitted and in the second article, (except in the Hebrew text) there is no indication that it is a continuation of the previous one, and, indeed, the two titles are not the same in the text though they are in the Table of Contents. This is an unfortunate confusion.

[3] Every article, long or short, even if only a note or query, is signed by the author, thus indicating complete individual responsibility.

[4] O. Veblen, *Analysis Situs;* now in second edition, 1931.

of the American Mathematical Society, there have appeared important new combinatorial theories, each of which relates to the manner in which, given a polygonal configuration or "complex" K, we define an algebraic *chain* of K. Of these the "mod m" theory was introduced by Alexander in 1926, and the "mod L" (where L is a subcomplex of K) and "mod (L, m)" by Lefschetz about 1927.

Along with the emergence of these new basic theories there have evolved a wider degree of application of the combinatorial method and a host of new and far-reaching results. As instances of these we have the application of the combinatorial method to the study of compact metric spaces; applications to algebraic geometry (whose topological side, for the case of functions of two or more variables, really owes its development to Lefschetz); the increasing generality of duality theorems and the investigation of their interrelations; extension of the topological theory of intersections of complexes; etc.

In view of the large amount of material that has thus accumulated since the appearance of the Veblen book, there would seem to be no question that a work such as the one under review will fill a distinct need, especially as it covers so completely the extensive literature. In the opinion of the reviewer it does not (and the author apparently did not intend that it should) displace the excellent introduction to the combinatorial method furnished by the Veblen Colloquium. To be sure, it is complete in itself, but it seems advisable for the beginner in combinatorial theory to read first the Veblen book (beginning as it does with the elementary treatment of the 1- and 2-dimensional cases and then proceeding to the n-dimensional case) and then to use the Lefschetz exposition as supplementary.

The book has an index and there is included a valuable bibliography. Numerous diagrams are given in the early part of the book. There seem to be few misprints; it is important to call attention, however, to the erroneous definition of "continuous transformation" on page 3, which apparently escaped detection in the proof reading.[1]

R. L. WILDER

Differential Equations from the Algebraic Standpoint. By J. F. Ritt. American Mathematical Society Colloquium Publications, Vol. XIV. New York, 1932. x+172 pages. $2.50.

Systems of differential equations, ordinary and partial, which are algebraic in the unknown quantities and their derivatives, furnish the subject of this book. Not much has been done heretofore with the algebraic theory of such systems. In their treatment it has been customary to assume certain canonical forms; but the methods which have been proposed for reduction to such forms are subject to limitations and contain no effective means for preventing the entrance of extraneous solutions. The theory of systems of algebraic equations is in a much more satisfactory state. The aim of the present book is "to bring to the theory

[1] See the list of corrections given in connection with the review by P. A. Smith, Bull. Amer. Math. Soc., vol. 37 (1931), in the footnote on pp. 647–648.

of systems of differential equations, which are algebraic in the unknowns and their derivatives, some of the completeness enjoyed by the theory of systems of algebraic equations." The point of view taken is that of the author's memoir in volume 32 of the *Transactions of the American Mathematical Society*. While the existing theory of algebraic manifolds has afforded a guide to the investigation, it has turned out, as one would have expected, that the essentially new phenomena involved have called for the development of new methods. While "many questions still remain for investigation," the author has nevertheless made a marked advance toward a satisfactory theory of algebraic differential equations.

R. D. CARMICHAEL

Das Hauptproblem der Äusseren Ballistik. By Kyrill Popoff. Leipzig, Verlagsgesellschaft M.B.H., 1932. xi+214 pages. RM 16.40.

From both the theoretical and also the experimental point of view, the subject of exterior ballistics is naturally divided into two parts:

(A) The translation of projectiles, considered simply as heavy particles, through a resisting medium (the atmosphere) while subject to the acceleration of gravity.

(B) The motions of projectiles considered as rotating bodies of appreciable dimensions.

From the theoretical point of view, the major difficulties presented by the subject of exterior ballistics are encountered in (B), for to the differential equations of translation are at once added the Eulerian equations that define the gyrations of projectiles and the conditions for their stability. From the experimental point of view, the problems of greatest difficulty and greatest value are also encountered in (B), for they include all problems relating to stability and accuracy of flight.

The problems included under (B) appear not to have been investigated, either theoretically or experimentally, until during the World War, and then only in England and the United States. The work of the English (Fowler, Gallop, Lock, and Richmond) is contained in Philosophical Transactions of the Royal Society of London (1920), and the theoretical part of the work done in America is set forth in my *New Methods in Exterior Ballistics* (The University of Chicago Press).

In spite of the fact that the book of Popoff appeared in 1932, it contains no reference whatever to the problems of ballistics included under (B).

Those parts of ballistics included under (A) are likewise of two kinds:

(a) The theory of the translation of projectiles under ideal conditions.

(b) The theory of the effects of abnormal conditions upon the flight of projectiles, and the development of methods of rapidly applying them.

Under (b) are included the theory of the effects of: (1) abnormal intitial velocities, (2) abnormal directions of fire, (3) abnormal weights of projectiles, (4) the direction and velocity of winds at various altitudes, (5) abnormal air

densities, (6) variations in the elasticity of the air, (7) variations in gravity, (8) the rotation of the earth, and (9) the curvature of the surface of the earth. In addition, there is included under (b) the theory of "weighting factors" by means of which the effects of abnormalities may be easily taken into account. Obviously, under modern conditions, these problems are of great practical importance, and theoretically they have been found to be of great interest. Yet Popoff's book contains no reference whatever to them.

It follows that Popoff treated only the flight of projectiles, considered as heavy particles moving under ideal conditions, which was the problem treated by Euler, Cauchy, Siacci, and other writers on ballistics up to the time of the World War. In fact, he reproduces most of the methods of earlier writers, though correcting their errors in logic and extending their results, often to a notable degree. Like earlier writers, he considers the problem first in an ideal homogeneous atmosphere, devoting nine chapters and 136 pages to this part. Then he treats the problem for an atmosphere decreasing in density with altitude, devoting six chapters and 72 pages to this part. Frequently, he follows earlier writers in resorting to various approximations to the functions involved in order to separate the variables in the differential equations, sometimes at the cost of using another parameter than the time as an independent variable, and consequently requiring an inversion to reduce the results to a form suitable for practical use. Yet he uses series in many instances, always with an attention to their convergence not found in the work of other Continental writers on ballistics.

From the practical point of view, the work of Popoff is of no particular significance. From the theoretical point of view, it nearly completes the work of those by whom he appears to have been influenced, and logically it is vastly superior to that of any of them. In this respect, it is really distinguished, though on page 14 he makes the strange remark that experiments indicate that the retardation as a function of the velocity is continuous and satisfies the Lipschitz condition. The Lipschitz condition includes continuity and obviously neither could ever be established by experiment. The property is *assumed*, and from that point on the logic is secure. There is no intention to make any capital of this slip, except as it illustrates the very slight connection of the author with the practical aspects of his problem, for this misstatement is quite exceptional in a work distinguished for its rigor in a field often filled with errors in reasoning.

F. R. MOULTON

Differential and Integral Calculus. By J. H. Neelley and J. I. Tracey. New York, The Macmillan Co., 1932. viii+496 pages. \$4.00.

This text, by well-known authors, covers the usual material of an elementary course in the calculus, including a brief introduction to ordinary differential equations. The presentation is clear throughout and, except in a few instances, accurate. Difficult proofs are avoided, references being given to more advanced works. Care is taken to indicate what assumptions are made in the proofs given.

The reader will notice the following points. The expression for the differential of arc is obtained by considering the velocity of a point moving along the arc. In problems involving summation leading to a definite integral, Duhamel's theorem or an equivalent device is avoided by the assumption that infinitesimals of higher order may be neglected. What is defined to be the double integral is really the iterated integral. The general solution of the linear second order differential equation with constant coefficients is not given, but the authors show how the solutions can be obtained in any particular problem by means of a substitution on the dependent variable. The proof of the fact that the general solution of a differential equation of order n contains n arbitrary constants is faulty since it implicitly assumes that the equation obtained by eliminating the constants must be the original equation. There are no applications of partial differentiation to space geometry.

The book contains three chapters which review plane and solid analytic geometry and which should prove very useful to the student. We must point out, however, that the asymptotes of the hyperbola are incorrectly obtained.

The typography and general appearance of the book are attractive. There are two obvious misprints which will cause no trouble (p. 333 and p. 435). Numerous worked out examples and complete sets of exercises follow each topic.

SAMUEL BOROFSKY

Mathematical Nuts. By Samuel I. Jones. Published by the author, Nashville, Tenn., 1932. x+340 pages. $3.50.

This book contains a variety of nuts for lovers of mathematics. The author informs his readers that there are many varieties of nuts and various ways of classifying them. He then quotes from the humorist, Ralph A. Parlette, a schoolmate of the reviewer, in his *It's Up to You*. "I hold in my hand this glass jar containing little white beans and big black walnuts. I mix them all up. Then I shake the jar. They un-mix. The walnuts go to the top and the little beans go to the bottom. This little jar is a picture of what is going on everywhere in this world all the time. The world is just a big jar of life. . . . All kinds of people are in the jar of life. The jar goes on shaking all the time. It never stops shaking. We are either Big Nuts or Little Nuts." It is hard to tell which is the more numerous. It depends upon one's point of view, we suppose.

The book contains nine sections: I Nuts for Young and Old; II Nuts for the Fireside; III Nuts for the Classroom; IV Nuts for the Math Club; V Nuts for the Magician; VI Nuts for the Professor; VII Nuts for the Doctor; VIII Nuts Cracked for the Weary; IX Nut Kernels.

Some years ago the author published his *Mathematical Wrinkles*, a book very favorably commended by educators and editors in both England and America.

In the preparation of *Mathematical Nuts*, the author has far overstepped his former efforts. The reviewer has never before seen anywhere such an array

of interesting, stimulating, and effort inducing material as is here brought together. The questions range from the very easy ones, such as "Express 3 by using three threes" to some very difficult ones requiring the Calculus.

Much valuable information may be gained by young and old alike in devoting some of their leisure time to cracking of these nuts.

B. F. FINKEL

Introduction to Trigonometry and Analytic Geometry. By Ernest Brown Skinner, New York, The Macmillan Company, 1932. xi+189 pages. $1.80.

The problem of shortening the road to the calculus for those students who enter college with some two and a half years of high school mathematics is one which concerns all college teachers. They must therefore be interested in the book under review which represents a carefully worked out contribution toward the solution of the problem.

An efficient and unified treatment covers the standard topics in Trigonometry and Analytic Geometry with sufficient brevity to bring them within the work of a single semester. Naturally, the necessary curtailment involves some sacrifices; for example, there is no occurrence of curves like the lemniscate and limaçon, nor is there any great variety of exercises in the geometry of three dimensions. Incidentally, while mentioning geometry of three dimensions, the reviewer would wish for the introduction and considerable illustration of the concept of "direction numbers," but perhaps this is only a matter of personal taste. The really essential material is adequately and often freshly treated, and exercises given in sufficient, yet not overwhelming, number. Flexibility and stimulation are provided by the treatment in the text of well-chosen applications to practical problems, and by a suggestive chapter (XII) on "Miscellaneous Topics," as well as by the occasional adoption of a sort of "higher standpoint" in the treatment of elementary matters.

The typographical errors, of which several were noted, are nearly all obvious, though the omission of parentheses in the bold-face formulas on page 64 might trouble students. And is it not a little dangerous to tell freshmen (page 171) that "homogeneous equations may be solved for the variables"? Perhaps the answer is that the author presupposes a *good* course in College Algebra, and has furthermore deliberately left a good deal to the teacher. In any case, adverse criticisms can be directed only at minor matters, and the text merits serious attention, especially in institutions where the pedagogical problem at whose solution it aims is acute.

B. P. GILL

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

The annual business meeting and banquet was held in the Hotel Morton in Atlantic City at noon on December 27, 1932. The attendance was seventeen, including the following sixteen members of the fraternity: J. H. Bushey, Marguerite D. Darkow, F. F. Decker, H. A. DoBell, M. Dresher, H. S. Everett, E. J. Finan, E. R. Hedrick, Louis Ingold, C. C. MacDuffee, H. F. MacNeish, Mina S. Rees, W. H. Roever, Lao G. Simons, L. L. Smail, F. M. Weida.

Pi Mu Epsilon of the University of Pennsylvania

The University of Pennsylvania Chapter of Pi Mu Epsilon has enjoyed the year 1931–1932 under the leadership of Dr. G. C. Chambers as director. The officers for the year were: Dr. H. M. Lufkin, Vice Director; Miss M. W. Bell, Secretary; Mr. P. A. Knedler, Treasurer; Dr. H. B. Evans, Chairman of Scholarship Committee; Dr. J. A. Shohat, Chairman of Program Committee.

During the year thirty-two new members were initiated, seventeen in December, and fifteen in April. This makes the total number of active members 252.

The meetings and programs were as follows:

October 16, 1931: "Some aspects of modern Physics" by Dr. E. E. Witmer.
November 20, 1931: "Logarithms" by Mr. B. B. August.
December 15, 1931: "Alignment charts" by Mr. O. Schuck.
January 15, 1932: "Pi and e" by Miss R. E. Cohen.
February 19, 1932: "The place of Mathematics in Physics" by Dr. W. F. G. Swann.
March 18, 1932: "Gyroscopes" by Mr. P. M. Field.
April 15, 1932: "Some interesting curves" by Miss E. K. Clark.
May 6, 1932: "Geometries" by Dr. W. M. Smith.

The February meeting, when Dr. W. F. G. Swann, director of the Bartol Research Foundation of the Franklin Institute, spoke and the May meeting when Dr. W. M. Smith of Lafayette College spoke, were open meetings.

It is the custom of the chapter to award two prizes annually for the best papers presented during the year by undergraduates. At the May meeting, the prizes, each $12.50, were awarded to Miss E. K. Clark and Miss R. E. Cohen.

During the year special problems were offered for solution by any undergraduate student. A prize of $5.00 was offered for the best solution. On February 19th, Mr. B. Miller received the prize for the first problem, and on April 15th, Mr. B. August was announced as winner of the prize for the second problem. The winners of these prizes presented their solutions at meetings of the fraternity.

The events for the year closed with the annual picnic held Saturday, May 21st at Castle Rock.

M. W. BELL, *Secretary*

Pi Mu Epsilon of Pennsylvania State College

Our chapter submits the following outline of events. The officers for the academic year 1931–1932 were: Coleman Herpel, '32, Director; Richard Baker, '32, Vice Director; E. M. Fry, '32, Secretary; Miss Gladys Quigg, Instructor in Mathematics, Treasurer; Dr. C. A. Rupp, Associate Professor in Mathematics, Librarian.

The meetings and programs were as follows:

October 21, 1931: "Signum functions" by Dr. C. C. Wagner.

November 10, 1931: Business meeting. Ten students were elected.

December 16, 1931: Initiation and banquet. About fifty members were present.

January 6, 1932: "Conditional convergence of infinite series" by Dr. T. C. Benton.

February 11, 1932: "Some eccentricities of mathematics" by E. Ross.

March 2, 1932: "The special Relativity Theory" by L. R. Turner; "Empirical equations" by H. L. VanVelzer.

April 14, 1932: "Conics from a projection standpoint" by Dr. T. Cohen.

April 28, 1932: Business meeting. Five students were elected. New officers were elected.

May 9, 1932: "Foundation of Point Set Theory" by Professor R. L. Moore.

May 20, 1932: Cabin party.

E. M. FRY, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of North Carolina College

The officers for the year 1931–1932 were: Edna Livingston, President; Katherine Nowell, Vice President; Blanche Fisher, Secretary-Treasurer; Eleanor Shelton, Chairman of the Program Committee; Miss Emily Watkins, Faculty Advisor.

The meetings and programs were as follows:

October 1931: "1931 as a centennial year in the history of Mathematics." "Birth of Hippocrates" by Jerrie Arthur; "Death of Democritus" by Lollie Boyd; "Publication of *Clavis Mathematics*" by Etta Lowry; "Thomas Harriot" by Audrey James; "Death of Briggs" by Inez Reeves; "Birth of Amedee Mannheim" by Lottie Kennedy; "Birth of James Maxwell" by Olga Frisard; "Cauchy's discovery" by Eddis Byers; "Death of Sophie Germain" by Shirley Hall.

November 1931: "The theory of limits" by Virginia Allen; "The star polygon" by Julia McLendon. The meeting closed with the singing of the club's song.

December 1931: "The solar system" by Lucile Joyner; "Best known planets" by Hazel Goodman. The last talk was illustrated by slides.

January 1932: "Possibilities of a mathematics major in the actuarial departments" by Miss Elizabeth Hall from Jefferson Standard Life Insurance Company; "Clerical work of insurance companies" by Miss Elizabeth Case of the Jefferson Standard Life Insurance Company.

February 1932: Initiation of new members. The program consisted of mathematical puzzles and games.

March 1932: The program consisted of a memorial to one of our first Professors of Mathematics, Miss Gertrude Whittier Mendenhall.

April 1932: "History of Pi" by Lois Siler; "History of e" by Margaret Brown; "The transcendence of Pi and e" by Etta Lowry.

May 1932: The program consisted of a playlet called "Mathematics in Nature" and the cutting of mathematical figures. At this meeting, officers for the academic year 1932–1933 were elected.

ELEANOR SHELTON, *Chairman of Program Committee*

The Mathematics Club of the University of Colorado

The officers for 1931–1932 were: Howard James, President; Doris Huddleston, Vice President; Helen Wirz, Secretary and Treasurer.

The primary purpose of the club is "to point out to everyone the interest and value of vigorous thinking." The club meets once every two weeks and approximately half the meetings are open to all students.

The meetings and programs were as follows:

October 8, 1931: Business meeting.

October 22, 1931: Open meeting. "Mathematical approximations and their applications to the Quantum Theory" by Mr. Louis Strait.

November 15, 1931: Open meeting. "Algebraic differentiation" by Mr. Richard Furr.

January 11, 1932: Open meeting. "The theory of numbers" by Professor Emeritus Ira M. DeLong

January 28, 1932: Closed meeting. "The Geometry resulting from the rejection of the parallel postulate" by Mr. Victor Reno.

February 18, 1932: Open meeting. "The fourth dimension" by Dr. Kempner.

March 3, 1932: Closed meeting. "Famous mathematicians" by Miss Betty Neville.

April 21, 1932: Closed meeting. Business meeting.

May 11, 1932: Closed meeting. Business and election of officers for the academic year 1932–1933.

JOHN R. LACHER, *Secretary*

The Mathematics Club of Rutgers University

The officers of the Rutgers University Mathematics Club for the year 1931–1932 were as follows: Clifton L. Hickok, President; Richard N. Coan, Vice President; Milton Friedman, Secretary-Treasurer. The annual election of officers occurs at the May meeting.

Membership in the Club is open to all students above the Freshman year whose major interest is in Mathematics. This year we had twenty-seven active members.

The following gives the list of the papers read at our seven regular meetings:

October 15, 1931: "Congruences" by R. N. Coan; "Number theory" by H. S. Grant; "Theorem of Appolonius" by J. H. MacDonough.

November 19, 1931: "The earliest mathematics" by E. P. Starke; "Development of demonstrative geometry in Greece" by C. L. Hickok; "Early developments in the Theory of Numbers" by M. Friedman.

December 17, 1931: "Contributions to arithmetic and algebra during the Renaissance" by L. S. Stout; "The development of the Theory of Equations" by W. Weisbrot; "The development of algebraic symbolism" by S. Fenichel; "There is a fifth degree equation which has no solution in radicals" by H. S. Grant.

February 18, 1932: "Descartes and Analytic Geometry" by I. Kaufmann; "Early Projective Geometry" by H. Geller; "The theory of probability" by J. Chernick; "To find directrices and foci without transformation of the general equation of the second degree in two variables" by C. R. Wilson.

March 17, 1932: "The beginnings of Calculus among the Greeks" by C. L. Hickok; "Some famous paradoxes of Zeno" by L. H. Bunyan; "Leibnitz and his calculus" by T. Raiser; "Newton and his calculus" by W. Ward.

April 21, 1932: "Contributions of Euler and Gauss" by R. N. Coan; "Contributions of Laplace and Cauchy" by G. Abbott; "Fractional differentiation" by H. B. Huntley.

May 19, 1932: Generalized algebras" by M. Friedman; "Introduction to Projective Geometry" by C. L. Hickok; "Orthocentric groups of points and counterpoints" by R. Morris.

In conjunction with the mathematics club of the New Jersey College for Women there are held annually two special meetings open to the public. The programs this year were:

December 4, 1931: "A finite geometry" by Professor Alonzo Church of Princeton University.

April 27, 1932: "The early days of the American Mathematical Society" by Professor Thomas Fiske of Columbia University.

MILTON FRIEDMAN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 31. *Proposed by W. R. Ransom, Tufts College.*

Point P is on the line AB, one N-th of the way from A to B, and point Q is on the line AC, one N-th of the way from A to C. PC intersects QB at R. Prove that the area of the triangle ABC is $N(N+1)/2$ times the area of the quadrilateral $APRQ$.

E 32. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Prove that $\int \sec x\, dx = -2i \tan^{-1} e^{ix} + C$, where $i = \sqrt{-1}$.

E 33. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

The points A and B are any two points not in the plane M. Find the locus of the point X, in M, such that the lines AX and BX make equal angles with M.

E 34. *Proposed by V. F. Ivanoff, San Francisco, California.*

Solve $X^{-X} = (-X)^X$ and justify the number of solutions.

E 35. *Proposed by W. B. Campbell, Rangoon, Burma.*

Show that, under appropriate conditions, the limit as n approaches minus one, of the integral $\int_a^b X^n dX$, is equal to the integral $\int_a^b X^{-1} dX$.

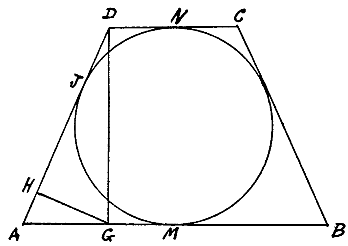
SOLUTIONS

E 11. [1932, 606] *Proposed by W. R. Ransom, Tufts College.*

This problem was offered by Mr. Francis of Exeter at a meeting of the Association of Teachers of Mathematics in New England in 1914.

Circumscribed about a circle is an isosceles trapezoid, $ABCD$, in which $DC < AB$, and $AD = BC$. Two perpendiculars are drawn; DG perpendicular to AB at G, and GH perpendicular to AD at H. Show that DA, DG and DH are the arithmetic, geometric, and harmonic means, respectively, between the pair of parallel sides AB and CD.

Solution by Theodore Lindquist, Michigan State Normal College.



From the diagram, $JD=DN=\frac{1}{2}DC$, and $AJ=AM=\frac{1}{2}AB$; so $DA=\frac{1}{2}(AB+DC)$. Furthermore, $DG^2=DA^2-AG^2=[\frac{1}{2}(AB+DC)]^2-[\frac{1}{2}(AB-DC)]^2=AB\cdot DC$, so that $DG=\sqrt{(AB\cdot DC)}$. Finally, from the similar right triangles, AGD and GHD,

$$DG/HD=DA/DG, \text{ so } 1/HD=DA/DG^2=\tfrac{1}{2}(AB+DC)/(AB\cdot DC)$$
$$=\tfrac{1}{2}[(1/DC)+(1/AB)].$$

Also solved by C. A. Barnhart, W. E. Buker, Mannis Charosh, L. S. Johnston and C. C. Richtmeyer.

E 12. [1932, 606] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Two coplanar right triangles, AOC and BOC, have the common hypotenuse OC. Using vector methods, express the vector OC in terms of the vectors OA and OB.

Solution by W. R. Ransom, Tufts College.

Represent the vectors OA, OB and OC by the letters A, B and C, respectively. The common perpendicular to A and B is $A\times B$, and the vector C can be expressed as $A+x(A\times B)\times A$, or as $B-y(A\times B)\times B$. Equating these and expanding the triple vector products, we get

$$A+x(BA^2-AA\cdot B)=B+y(AB^2-BB\cdot A).$$

Solving for x, substituting into the first expression for C, and reducing, we get

$$C=[A(A\cdot BB^2-A^2B^2)+B(A\cdot BA^2-A^2B^2)]/[(A\cdot B)^2-A^2B^2].$$

Also solved by Mannis Charosh, T. C. Esty and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would

also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3607. *Proposed by J. B. Reynolds, Lehigh University.*

A sphere of radius r and density k times that of water sinks from rest tangent at the top within a fixed sphere of radius R filled with water. Find the velocity of the sinking sphere when its center coincides with that of the fixed sphere. Find the acceleration of the sinking sphere.

3608. *Proposed by N. A. Court, University of Oklahoma.*

The vertex of a positive angle fixed in magnitude describes a given straight line, while the initial line constantly passes through a fixed point. Find the envelope of the second side.

3609. *Proposed by C. C. Carter, Bluffs, Illinois.*

Determine the area of a quadrilateral in terms of its sides, given in the order a, b, c, d, for which the diagonals are equal and at right angles. Is a solution possible if one of the sides, for example b, is not given?

3610. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Determine the position of a cylinder of revolution which cuts the vertical and horizontal planes in equal ellipses similar to a given ellipse.

3611. *Proposed by J. Rosenbaum, Milford, Conn.*

If each of the face angles at a vertex of a tetrahedron is a right angle, the square of the area of the face opposite that vertex is equal to the sum of the squares of the areas of the faces adjacent to that vertex.

Is the converse true?

SOLUTIONS

3516. [1931, 539]. *Proposed by Norman Miller, Queen's University.*

Prove that

$$\lim_{\substack{a_1\to a\\ a_2\to a\\ \cdots\\ a_{n+1}\to a}} \frac{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots a_1^{n-1} & a_1^r \\ 1 & a_2 & a_2^2 & \cdots a_2^{n-1} & a_2^r \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots a_{n+1}^{n-1} & a_{n+1}^r \end{vmatrix}}{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots a_1^n \\ 1 & a_2 & a_2^2 & \cdots a_2^n \\ \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots a_{n+1}^n \end{vmatrix}} = \frac{r(r-1)\cdots(r-n+1)}{n!}a^{r-n}.$$

Solution by J. M. Feld, Brooklyn College of the City of New York.

Let equation

$$c_0 + c_1x + \cdots + c_nx^n = x^r \quad (r > n+1) \tag{1}$$

have as $n+1$ of its roots $a_i(i=1, \cdots, n+1)$. We can thus obtain the system

$$c_0 + c_1a_i + \cdots + c_na_i^n = a_i^r \tag{2}$$

$$(i = 1, \cdots, n+1).$$

Solving (2) for c_n we find

$$c_n = f(a_1, a_2, \cdots, a_{n+1}) = \frac{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^r \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} & a_2^r \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{n-1} & a_{n+1}^r \end{vmatrix}}{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^n \end{vmatrix}}$$

As $a_i \to a$, $(i=1, \cdots, n+1)$, c_n approaches

$$\lim_{a_i \to a} f(a_1, a_2, \cdots, a_{n+1}) \quad (i = 1, \cdots, n+1).$$

Evidently this limit is the coefficient of x^n in (1) if the $n+1$ roots a_i all equal a. If (1) has $n+1$ equal roots, a, the equation obtained after differentiating (1) n times will be satisfied by $x=a$. Thus

$$n!c_n = r(r-1) \cdots (r-n+1)a^{r-n}$$

which proves the theorem

A Note by Otto Dunkel. Since the independent variables a_i are to approach a in any manner, there should be a definition of $c_0, c_1, \cdots, c_n$ for the cases where several or all of the arguments a_i are equal. If we then assume that the c_i's approach finite limits, the above method determines the limit value of c_n.

A proof of a generalization may be given as follows: Let $f(x)$ be a real function of the real variable x which possesses all derivatives up to and including the nth, $f^n(x)$, for an x-interval which contains $a, a_1, a_2, \cdots, a_{n+1}$; and let $f^n(x)$ be continuous in this interval. Denote the determinant in the denominator of the problem by V, and denote by U the determinant obtained by replacing a_i^n in the last column by $f(a_i)$. The generalization consists in replacing x^r of the problem by $f(x)$. Suppose first that no two a_i's are equal. We can then determine uniquely a set of functions of the a_i's, $c_0, c_1, \cdots, c_n$ such that

$$\phi(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n - f(x) \tag{3}$$

vanishes for the $n+1$ distinct values $a_1, a_2, \cdots, a_{n+1}$. The function $\phi(x)$ can be written as a determinant of order $n+2$. Hence the nth derivative of $\phi(x)$ is zero for $x=\theta$, where θ lies between the smallest and largest a_i. We have then

$$c_n = \frac{f^n(\theta)}{n!} = \frac{U(a_1, a_2, \cdots, a_{n+1})}{V(a_1, a_2, \cdots, a_{n+1})}. \tag{4}$$

Suppose that sets of the a_i's are equal, say a_1 occurs m_1 times, a_2 occurs m_2 times, etc., where $a_1, a_2, \cdots, a_t$ are distinct and $m_1+m_2+\cdots+m_t=n+1$. We assign a value to U/V for this case as follows: determine $\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_n$ so that the expression similar to (3) for $\phi(x)$ together with its derivatives up to and including the (m_1-1)th vanishes at $x=a_1$; similarly for $a_2, a_3, \cdots, a_t$. We then find as before

$$c_n = \frac{f^n(\bar{\theta})}{n!} = \frac{\overline{U}(a_1, a_2, \cdots, a_t)}{\overline{V}(a_1, a_2, \cdots, a_t)}, \tag{5}$$

where $\overline{V}$ is obtained from V by replacing in the latter certain of its rows by their derivatives. Also $\overline{V}$ is not zero as long as $a_1, a_2, \cdots, a_t$ are distinct. In the case of several or all a_i's equal we define the value of U/V as that of $\bar{c}_n$ or $\overline{U}/\overline{V}$. With this definition we have in all cases $U/V=f^n(\theta)/n!$, where θ lies between the smallest and largest a_i. Since $f^n(x)$ is continuous and θ approaches a in some manner, we have then

$$\lim_{a_1,a_2,\cdots \to a} \frac{U}{V} = \frac{f^n(a)}{n!}.$$

The above method was given by W. H. Echols in his paper, *On certain determinant forms and their applications*, in Annals of Math., vol. 6, 1892, pp. 105–126. In this paper the function denoted by $\phi(x)$ is written as a determinant of order $n+2$.

Complex Variables. Suppose that $f(z)$ is an analytic function of z within a circle with a as center and radius r, and let $a_1, a_2, \cdots, a_{n+1}$ be any values of z within this circle. Then in the determinant U we may develop each element a_i^m and $f(a_i)$ in series of powers of (a_i-a). Since we have only a finite number of multiplications and additions of absolutely convergent series we obtain an analytic function of $a_1, a_2, \cdots, a_{n+1}$ developed in a multiple power series. The form of the development may be readily seen by developing first only the last column of elements, $f(a_i)$. We obtain an infinite series

$$\sum A_j f^j(a)/j!,$$

where A_j is obtained by replacing in V the elements a_i^n of the last column by $(a_i-a)^j$. It is then clear that $A_j=0$ for $j<n$, and that $A_n=V$. From the form of V we have at once $V(a_1, a_2, \cdots, a_{n+1})=V(a_1-a, a_2-a, \cdots, a_{n+1}-a)$. We now develop the remaining elements of A_j and after certain simple reductions we find that A_j has the same first n columns as $V(a_1-a, a_2-a, \cdots, a_{n+1}-a)$,

but the $(n+1)$th column of A_j contains jth powers instead of nth powers. Thus A_j is divisible by V and the quotient is a homogeneous polynomial in the differences $a_1-a, a_2-a, \cdots$, of degree $j-n$. Hence

$$U = V\left[\frac{f^n(a)}{n!} + E\right],$$

where E is an analytic function of the a_i's. Hence the brackets give a definition of U/V for all cases. Since E approaches zero as the a_i's approach a in any manner, we obtain the same result as before for the limit of U/V.

In the given problem $f(x)=x^r$, and if r is an integer, positive, negative or zero, the problem is simplified. If r is a positive integer, U is divisible by V if no two of the a_i's are equal; and we define U/V as this polynomial quotient. The existence of the limit is then apparent, and it may be found as above or otherwise. If r is a negative integer, we exclude the case of $a=0$. We may write $U=(a_1a_2 \cdots a_{n+1})^rW$, where W is a determinant with $a_i^{-r}, a_i^{-r+1}, \cdots, a_i^{-r+n-1}, 1$ in the ith row. Again W is divisible by V if no two a_i's are equal, and we again obtain a definition of U/V as $(a_1a_2 \cdots a_{n+1})^rW/V$, where the factor W/V is a polynomial. The existence of a limit is again apparent.

If r is not an integer and a is not zero, x^r is analytic within a suitable circle with a as center which does not contain $x=0$, after selecting a determination of x^r for this circle.

Also solved by A. S. Householder, who gave the reference to Echols's work cited above.

3552 [1932, 300]. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

Prove that

$$\begin{vmatrix} e_{11} & 0 & 0 & \cdots & 0 & A_1 \\ e_{21} & e_{22} & 0 & \cdots & 0 & A_2 \\ e_{31} & e_{32} & e_{33} & \cdots & 0 & A_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ e_{n1} & e_{n2} & e_{n3} & \cdots & e_{n,n-1} & A_n \end{vmatrix}$$

$$= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s},$$

where (1) $e_{ij}=1$ if j is a divisor of i and $e_{ij}=0$ if j is not a divisor of i; (2) $p_i, p_j, \cdots$, are the distinct prime factors of n.

Solution by N. H. McCoy, Smith College.

In the given determinant, subtract from the n-th row each of the rows n/p_i, add each of the rows n/p_ip_j and so on, and denote the resulting determinant by D. Thus D is equal to the original determinant.

Let a_{nj}, $(j=1, 2, \cdots, n)$ denote the elements of the n-th row of D. Then if we let $e_{ij}=0(i<j)$, we may write

$$a_{nk} = e_{nk} - \sum e_{n/p_i,k} + \sum e_{n/p_ip_j,k} - \cdots + (-1)^s e_{n/p_1p_2\cdots p_s,k},$$
$$(k = 1, 2, \cdots, n-1),$$
$$a_{nn} = A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s}.$$

If k is not a divisor of n, clearly $a_{nk}=0$. Suppose that k divides n. Let $n=p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$, $k=p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, and suppose that $r_i<t_i$ for just m values of i. Then

$$a_{nk} = 1 - m + \binom{m}{2} - \cdots + (-1)^m = (1-1)^m = 0, \quad (k = 1, 2, \cdots, n-1).$$

Hence a_{nn} is the only non-vanishing element in the last row of D, and since $e_{ii}=1$, we have $D=a_{nn}$.

3556 [1932, 301]. *Proposed by the late Artemas Martin, Washington, D. C.*

A hollow sphere, external and internal radii R and r, rolls down an inclined plane in time t; after the cavity is half filled with water it rolls down the same plane in time t'. Determine the specific gravity of the sphere.

See the *Annals of Mathematics*, vol. 8 (1894), p. 104.

Solution by William Hoover, Columbus, Ohio.

Let M and k be the mass and radius of gyration of the hollow sphere; m, the mass of the water, l the length of the inclined plane; α, the inclination of the plane; x, the distance the center of the sphere has moved in the time t from the beginning of the motion; θ, the angle through which the sphere has rotated in this time. Assuming that there is no friction between the water and the inner surface of the sphere, we have in the second case the energy equation

$$\tag{1} \tfrac{1}{2}M(\dot{x}^2 + k^2\dot{\theta}^2) + \tfrac{1}{2}m\dot{x}^2 = g \sin \alpha(M+m)x,$$

or, since $\dot{x}=R\dot{\theta}$,

$$\tag{2} \begin{aligned} [M(R^2+k^2)+mR^2]\dot{x}^2 &= 2g \sin \alpha R^2(M+m)x, \\ [M(R^2+k^2)+mR^2]\ddot{x} &= g \sin \alpha R^2(M+m), \end{aligned}$$

where the second equation results from the derivative of the first. After integration we get

$$\tag{3} 2[M(R^2+k^2)+mR^2]l = g \sin \alpha R^2(M+m)t'^2,$$

observing that for $t=0$, $x=0$, $\dot{x}=0$, and that for $t=t'$, $x=l$.

For the first case of motion, we replace in (3) t' by t and set $m=0$. Hence after dividing the two results and thus eliminating l, g and $\sin \alpha$, we obtain

$$\tag{4} M\left[1-\left(\frac{t'}{t}\right)^2\right] = m\left[\left(\frac{t'}{t}\right)^2 - \frac{R^2}{R^2+k^2}\right].$$

We also have

$$k^2 = \frac{2}{5}\,\frac{R^5 - r^5}{R^3 - r^3}, \qquad M = \frac{4}{3}\pi\delta'(R^3 - r^3), \qquad m = \frac{2}{3}\pi\delta r^3,$$

where δ' and δ are the densities of M and m. These substitutions in (4) give the desired equation in δ'/δ.

3557 [1932, 359]. *Proposed by R. E. Moritz, University of Washington.*

Given a free flow of water from a vertical circular aperture, flowing partially full. The radius of the aperture is r, the depth of the water above the center of the aperture is a, the coefficient of entry is k. Required the rate of discharge under the action of gravity.

Solution by the Proposer.

Let dA represent the area of a cross-section of a differential element of flow, h its hydraulic head, then its velocity of discharge due to gravity is $v=\sqrt{2gh}$, where $g=32.2$, and the differential element of discharge due to the element of flow is $dQ=k\cdot v\cdot dA$.

If now (x, y) are the Cartesian coordinates of a point on the circumference of the aperture, referred to the center of the aperture as origin, and the horizontal direction for the direction of the x-axis, then

$$x^2 + y^2 = r^2, \quad dA = 2xdy, \quad v = \sqrt{2g(a-y)};$$

hence

$$dQ = k\sqrt{2g(a-y)}\cdot 2xdy = 2k\sqrt{2g}\cdot\sqrt{(a-y)(r^2-y^2)}\cdot dy$$

and

$$(1) \qquad Q = 2k\sqrt{2g}\int_{-r}^{a}\sqrt{(a-y)(r^2-y^2)}\cdot dy.$$

To evaluate this integral, let $x=\mathrm{sn}^{-1}[(y+r)^{1/2}(a+r)^{-1/2},\ k]$, where $k^2=(a+r)/2r$, then when $y=-r$, $x=\mathrm{sn}^{-1}(0,\ k)=0$, and when $y=a$, $x=\mathrm{sn}^{-1}(1,\ k)=K$; and

$$a - y = (a+r)\,\mathrm{cn}^2\,x, \quad r^2 - y^2 = 2r(a+r)\,\mathrm{sn}^2\,x\,\mathrm{dn}^2\,x,$$
$$dy = 2(a+r)\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,dx$$

$$(2) \qquad \int_{-r}^{a}\sqrt{(a-y)(r^2-y^2)}dy = 2(a+r)^2\sqrt{2r}\int_0^K \mathrm{cn}^2\,x\,\mathrm{sn}^2\,x\,\mathrm{dn}^2\,x\,dx,$$

$$= 2(a+r)^2\sqrt{2r}\int_0^K[\mathrm{sn}^2\,x - (1+k^2)\,\mathrm{sn}^4\,x + k^2\,\mathrm{sn}^6\,x]dx,$$

$$= \frac{2\sqrt{2r}}{15}[(a-3r)(r-a)K + 2(a^2+3r^2)E],$$

by the use of reduction formulas such as Peirce's formula 567. K and E are the complete Elliptic Integrals of the first and second class respectively. When (2) is substituted in (1) the final result is obtained.

3558 [1932, 359]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Of a plane triangle given one side, the length of the bisector of the opposite angle, and the altitude on the given side; state and prove the ruler-compass construction of the triangle.

Solution by Banesh Hoffmann, University of Rochester.

Construction. Set up the right triangle AKM having angle $AKM=90°$, $AK=$length of altitude and $AM=$length of bisector. Draw AN perpendicular to AM to meet MK in N and find the mid-point, O, of MN. Draw MR of length a ($2a=$length of base of required triangle) perpendicular to MK and mark off Q on MK such that $OQ=OR$ (Q nearer M than N). Make $QC=QB=a$ on MK; then ABC is the required triangle.

Proof. We must now show that angle BAC is actually bisected by AM. From the indicated construction, we have

$$QB^2 = QC^2 = RM^2 = RO^2 - OM^2 = OQ^2 - OM^2 = (OQ - OM)(OQ + OM)$$
$$= QM \cdot QN.$$

Therefore B, M, C, N is an harmonic set of points, and, since MAN is a right angle, AM and AN are the internal and external bisectors of angle BAC.

Note by the Editors. In the above construction it is assumed that the given bisector is the interior bisector and that it is greater than the given altitude. The solution is then unique. If the given bisector is the exterior bisector, Q should be taken nearer N than M. If the given altitude and bisector are equal, the above construction breaks down; but it can be replaced by an obvious and trivial one in this special case.

Also solved by Gertrude Blanch, W. R. Church, Roy MacKay, J. S. Miller, R. K. Morley, William Orange, A. Pelletier, H. D. Ruderman, and Paul Wernicke.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The faculty of the school of mathematics of the Institute for Advanced Study and the faculty of the department of mathematics at Princeton will issue the Annals of Mathematics under a board of editors selected from each of the two institutions.

Professor Herman Weyl, professor of mathematics at the University of Göttingen, and Dr. J. W. Alexander, professor of mathematics at Princeton, have been appointed professors in the Institute for Advanced Study.

Dr. Solomon Lefschetz, professor of mathematics at Princeton, has been appointed to the Henry Burchard Fine chair of mathematics to succeed Professor Oswald Veblen.

Mr. H. P. Fawcett has been appointed assistant professor of mathematics in the department of the University Laboratory High School at the Ohio State University.

Mr. F. C. Hall has been appointed an instructor at New York University.

Professor R. L. Börger, head of the department of mathematics at Ohio University, died December 26, 1932. He was a charter member of the Association.

Harlan Wilbur Fiske, of Kensington, Maryland, magnetician and chief of the magnetic land survey, department of terrestrial magnetism, Carnegie Institution at Washington, died December 26, 1932.

Ormond Stone, professor emeritus of the University of Virginia, was struck and instantly killed by an automobile January 17, 1933. With his passing the University of Virginia loses its oldest member, and astronomy a notable figure. He began his career as an astronomer at the Naval Observatory. From there he went to the Observatory at Cincinnati, and later he was called to the University of Virginia. Although primarily an astronomer, Professor Stone was very much interested in mathematics. He showed concrete evidence of this interest by founding the *Annals of Mathematics* and acting as its first editor. He was a charter member of the Mathematical Association.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

The Carus Mathematical Monographs

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of Number One is being brought out by the firm of Teubner in Leipzig and Berlin. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; "Projective Geometry," by Professor JOHN W. YOUNG.

The price of these Monographs is $1.25 per copy to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

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(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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NUMBER 5, MAY

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PASCAL'S HEXAGRAM WITH CERTAIN ELEMENTS IN MOTION

By W. H. BUNCH, University of Idaho

Notation. We shall refer to the 45 Pascal points as the P-points and the 60 Pascal lines as the p-lines. In all other cases capital letters will represent points and the corresponding lower case letters will represent their polar lines. Numbers will be used as subscripts and when no confusion results they will be used alone. Thus 12 may be the line joining T_1 and T_2, (12,45) may be a point determined by two lines, and (12,45)–(34,56) a line determined by two points. When the numbers are subscripts for the lower case letters, the above notations will have the dual meanings. We shall refer to the vertices of the Pascal hexagon as the T points and tangents to the conic at these points as the t-lines.

Let any one of the six T-points move along the conic and the other five remain fixed; then the following theorems may be established.

THEOREM I. *Fifteen of the 45 P-points will remain unchanged and 30 will move along straight lines determined by the fixed T-points.*

PROOF: Fifteen P-points are determined by the five fixed T-points; hence, they do not move. Six P-points lie on each line determined by any two T-points and the P-points are the intersections of all such lines. Five of these lines pass through each T-point, hence 30 P-points move along straight lines determined by the fixed T-points.

THEOREM II. *Every p-line passes through one and only one of the fifteen fixed P-points, and the other two P-points on each p-line move along two intersecting straight lines and describe perspective point rows upon them.*

PROOF: The three P-points on every p-line are formed by the intersection of the opposite sides of the hexagon, hence the two adjacent sides which pass through any one of the T-points must pass through two different P-points in each p-line. Therefore, two points on each line must move. The remainder of the theorem follows immediately.

THEOREM III. *The intersection of every two p-lines which do not intersect in a P-point describes a conic, which intersects the original conic in one of the fixed T-points and passes through two fixed P-points.*

PROOF: Let point 1 be any of the six T-points and 2, 3, 4, 5, and 6 be the other five. Let any two of these, say 5 and 6, be centers of projective pencils which generate the conic. Then the lines 15 and 16 are corresponding rays. Now let point 1 move along the conic and the lines 15 and 16 will describe perspective[1] point rows upon any two lines which intersect upon the conic excluding the line 56 which is a common ray of the two pencils. Let us choose, for ex-

[1] Lehmer, *Synthetic Geometry*, page 33.

ample, the lines 26 and 23. Then the points A and B (Fig. 1) will be corresponding points on two perspective point rows. Join point B to any fixed P-point P_1, and A to any other fixed P-point P_2. Consider P_1 and P_2 as centers of two pencils. Then the lines P_1B and P_2A will be two corresponding rays of two projective pencils and their intersection will be a point on another conic which passes through the points P_1 and P_2 and the intersections of the lines 26 and 23, that is, point 2 on the original conic. But all such points as A and B are identical with the 30 moving P-points. We need now only to show that one of the two moving points on every p-line through P_1 and one of the two on every p-line

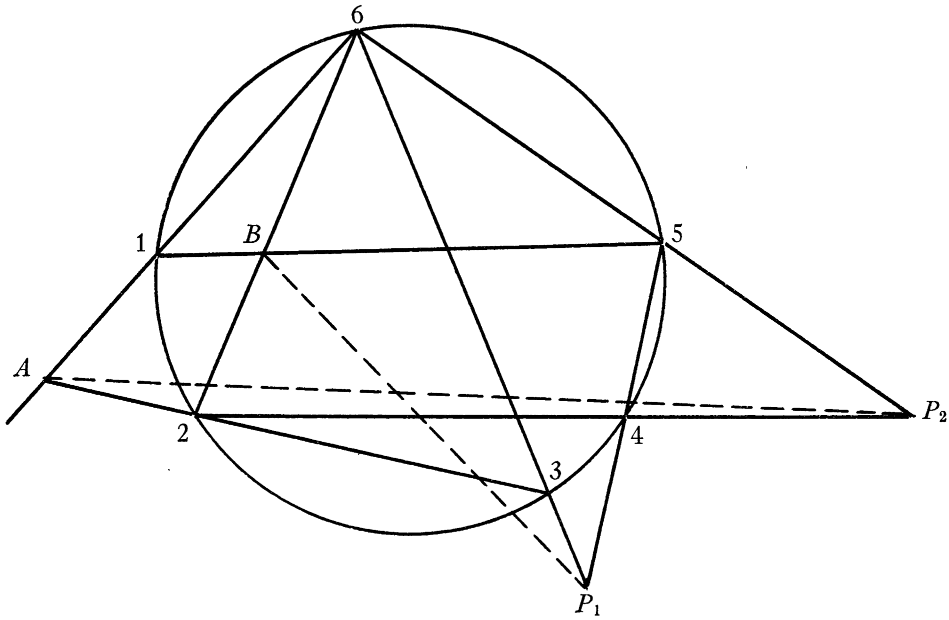

FIG. 1

through P_2 move along two lines which intersect on the conic or pass through two projective point rows on the same line or coincide upon the same line. Since the six T-points are used twice each in determining the three P-points on a p-line and only four fixed T-points are used once each in determining the fixed P-point on that line, all five fixed T-points are used in determining the two moving P-points on any p-line. And since not more than four fixed T-points are used in the moving lines 12 etc., which determine the four moving P-points on any two p-lines, it follows that one of the moving P-points on each of any two p-lines must move along a fixed line through the fifth fixed T-point. If they move upon the same line the points of intersection of that line and the conic are self corresponding points of the two point rows and either the two points describe two projective point rows upon that line or they describe two identical point rows in which case the two p-lines meet in a P-point upon that line.

THEOREM IV. *In the complete figure 1500 such conics exist.*

PROOF: The total number of intersections of the four p-lines through one fixed P-point with the four through another is 16. The total number of combinations of two points possible with 15 is 15.14/2. Hence the total number of intersections is 16.15.14/2 which is 1680. It was shown in Theorems I and III that when the intersection was in a P-point the result was a straight line. Hence 180 of these intersections form straight lines and 1500 form conics.

THEOREM V. *The 1500 conics are not all distinct, but 240 of them reduce to 80 distinct conics.*

PROOF: The proof of this theorem follows immediately from Steiner's Theorem and Kirkman's extension of Steiner's Theorem[1] which gives 80 points, in each of which there are three intersections of p-lines.

THEOREM VI. *Of the 105 combinations of two fixed points which serve as centers of projective pencils generating the 1500 conics, 15 combinations serve as centers for 12 conics each, 30 combinations for 14 conics each, and 60 combinations for 15 conics each.*

PROOF: Let the points 1, 2, 3, 4, 5, 6 be the six T-points. Then the opposite sides of the hexagon are 12 and 45, 23 and 56, and 34 and 61. The intersection of these opposite sides gives the three P-points on a p-line and we may choose the points in any order and select the numbers from the same position that we have above and still obtain a Pascal line.[2] Let us take any two orders—and it matters not which two we take, since we may interchange the numbers in any way so long as we draw from the proper positions—say $1\underline{23}5\underline{46}$ and $1\underline{45}2\underline{63}$. In the first, 23,46 will be a fixed P-point and the other eight P-points lying on the four lines through this point necessarily are:

12,45	13,54	12,56	13,56
61,35	25,61	14,35	14,52

Similarly 45,63 is the fixed P-point in the other order and the other eight corresponding points are:

14,26	15,26	14,23	15,23
13,52	13,42	16,52	16,24

We may classify these sets into three groups. In the first group both fixed P-points in the set of two may be determined by the same four numbers. Four points on the conic determine three P-points and five points may have five combinations of four points each; therefore, this group contains 15 sets and we may show as above that each scheme has four moving P-points in common. In

[1] Salmon, *Conic Sections*, 1924 Edition, p. 380.

[2] Salmon, loc. cit., p. 379.

the second group we may draw the same numbers from one of the underscored positions in each scheme. In this case the two fixed P-points in a set lie on the same line through two fixed T-points. There are ten of these lines and three fixed P-points lie on each line; hence, there are 30 sets in this group and we may show as above that each set has two moving P-points common to each scheme. The third group may contain all other combinations and hence 60 sets. Then, since there are 16 intersections in all between the 4 p-lines from each of any two fixed points, and those which ocur in P-points give straight lines, there must be 15 sets which generate 12 conics each and 30 sets, 14 conics each. Since each set must contain at least one straight line and there are 60 sets and 60 straight lines yet unused, each of these 60 sets must each contain one straight line and hence 15 conics.

Corollary. It may easily be shown by a study of combinations that the four straight lines in a set, mentioned above, are the lines joining the four fixed T-points, which determine the two fixed P-points in the set, to the fifth point which is not so used and hence the four lines are always distinct and intersect in one of the four T-points. Also when all five fixed T-points are used to determine the two fixed P-points in the set, that the two points which are so used only once determine the only straight line in the set. Hence, the two straight lines in each of the 30 sets above are never distinct, but two intersections describe the same line.

THEOREM VII. *The loci of the 16 intersections of the four p-lines through each of any two fixed P-points in 90 sets out of the 105 intersect eight by eight*[1] *in two of the fixed T-points and four by four in four other points, two of which lie on a line joining one of the two T-points above with one of the two fixed P-points. The other two lie on a line determined by the second fixed T-point above and the second fixed P-point. They also intersect four by four in each of the other three fixed T-points.*

In the remaining 15 out of the 105 sets the 16 loci all intersect in one fixed T-point and four by four in each of the other four fixed T-points, and do not intersect four by four in other points.

PROOF: Consider first any two fixed P-points determined by all five of the fixed T-points, say 63,45 and 65,42 and the eight p-lines which pass through these four by four determining the 16 intersections in the set. They are:

$$1\left\{\begin{matrix}63,45\\ 61,52,\\ 32,14\end{matrix}\right. \qquad 2\left\{\begin{matrix}63,45\\ 61,24,\\ 15,23\end{matrix}\right. \qquad 3\left\{\begin{matrix}63,45\\ 14,62,\\ 25,13\end{matrix}\right. \qquad 4\left\{\begin{matrix}63,45\\ 13,24,\\ 26,15\end{matrix}\right.$$

$$5\left\{\begin{matrix}65,42\\ 15,32,\\ 63,41\end{matrix}\right. \qquad 6\left\{\begin{matrix}65,42\\ 12,63,\\ 51,34\end{matrix}\right. \qquad 7\left\{\begin{matrix}65,42\\ 61,34,\\ 53,12\end{matrix}\right. \qquad 8\left\{\begin{matrix}65,42\\ 52,41.\\ 61,32\end{matrix}\right.$$

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[1] Note: In 30 sets the eight loci are not all distinct but the locus of two intersections coincide upon the line determined by the two points. See Cor. Theorem VI.

When point 1 coincides with point 2 the intersections of lines 5 and 6 with 1, 2, 3, and 4 all coincide upon point 2. Hence, eight loci must pass through this point. Likewise when point 1 coincides with point 3 the intersections of lines 1 and 2 with 5, 6, 7, and 8 coincide upon point 3 and eight loci must pass through point 3. These are the two points used once only in determining the two fixed P-points in the set. Now, let point 1 coincide with a fixed point which is used twice, say point 4, and the only intersections coinciding with point 4 are the lines 2 and 4 with 6 and 7; hence four loci intersect at point 4 and likewise we may show that four intersect at 5 and 6 each. Now, let point 1 coincide with point 3 again and the lines 5, 6, 7, and 8 coincide and are determined by the fixed P-point 65,42 and point 3, but lines 3 and 4 cut this line in two points other than point 3. Hence the loci intersect 4 by 4 on two points on that line other than the point 3. Similarly, if point 1 coincides with point 2 the lines 1, 2, 3, and 4 coincide and are determined by the fixed P-point 63,45 and point 2 and lines 6 and 7 cut this line in two points other than the point 2. Hence upon this line there are two points in which the loci intersect 4 by 4. Since we may choose the points 1, 2, 3, 4, 5, 6 without respect to order we may interchange points as we please in the above scheme with no restriction except that the same change be carried through one p-line. But even so, only two cases are possible. First, in 90 sets out of the 105 the two fixed P-points in each set will be determined by all five fixed T-points, two being used once only and three twice each as above and in these the interchange of points will not disturb the above reasoning. In the second case there are 15 sets (see theorem VI) in which the two fixed P-points are determined by only 4 T-points, each used twice and one point not used at all. We now examine the intersections in this case. We may take as our eight lines:

$$1\begin{cases}63,45\\61,52,\\32,14\end{cases}\quad 2\begin{cases}63,45\\61,24,\\15,23\end{cases}\quad 3\begin{cases}63,45\\14,63,\\25,12\end{cases}\quad 4\begin{cases}63,45\\13,24,\\26,15\end{cases}$$

$$5\begin{cases}65,43\\16,24,\\52,31\end{cases}\quad 6\begin{cases}65,43\\15,24,\\62,31\end{cases}\quad 7\begin{cases}65,43\\15,23,\\62,41\end{cases}\quad 8\begin{cases}65,43\\16,23.\\52,41\end{cases}$$

Now, let point 1 coincide with point 2 and the lines 1, 2, 3, and 4 coincide and are determined by the fixed P-point 63,45 and the point 2. Hence all 16 loci intersect in point 2, and since none of the first four lines intersect the last four except in the point 2, under this arrangement the four points above on these two special lines in which the loci intersected 4 by 4 do not exist in this case. We may show as before by letting point 1 coincide with 3, 4, 5, and 6 respectively that the loci intersect 4 by 4 in these points.

Interchanging of points in this case simply gives our 15 sets, or changes the scheme back to the first case. Hence the theorem is established.

THEOREM VIII. *On the line joining a fixed P-point to a fixed T-point, which is not used in determining the P-point, there lie two points through each of which*

twelve distinct conics pass and eight points through each of which eight conics pass. There are fifteen of these lines passing by threes through each fixed point on the conic.

PROOF: Due to symmetry we need to investigate only one line. We shall take the line joining the P-point 36,25 to point 4 on the conic. Since four p-lines coincide upon this line when point 1 coincides with point 4, there must be four Steiner points on this line. They are:[1]

$$S_1\begin{cases}36,25,41\\45,61,23,\\21,34,56\end{cases}\qquad S_2\begin{cases}36,25,41\\15,64,23,\\42,31,56\end{cases}\qquad S_3\begin{cases}25,36,41\\31,54,26,\\46,12,35\end{cases}\qquad S_4\begin{cases}25,36,41\\61,24,35.\\43,15,26\end{cases}$$

When point 1 coincides with point 4, S_1 coincides with S_2 and S_3 with S_4. There are also twelve Kirkman points on this line, but an entirely similar investigation shows that two of these coincide with S_1 and two with S_3 and the remaining eight coincide with point 4 on the conic. When two points on the conic coincide the number of Pascal lines reduces to thirty, the number of permutations of six points on a curve taking two at a time and two alike. By making a study of these similar to the proof of Theorem VII, it may easily be shown that these thirty lines are not all distinct but six of them coincide by twos with the three lines connecting point 4 to the P-points determined by points 2, 3, 5, and 6. Twelve others coincide by threes on the four lines joining point 4 to points 2, 3, 5, and 6. The other twelve are distinct and since the three columns of S_1 and S_2 become identical two of these intersect at S_1 and similarly two at S_3. Since all the Steiner and Kirkman points on the line in question are located, the remaining eight of the twelve lines must intersect this line in eight points. We now have three lines intersecting at S_1, one containing four p-lines and two containing two p-lines each, hence there are twenty intersections of p-lines at this point. But there are also two Steiner and two Kirkman points there (see theorem V); hence, there are twelve distinct conics intersecting in this point and likewise twelve at S_3. In each of the other eight points two lines intersect, one containing four p-lines and one containing two p-lines. Hence at each of these points eight conics intersect. In Theorem VII it was shown that fifteen of these lines exist, three passing through each fixed point on the conic.

It is of interest to note that the last eight of the twelve lines mentioned above intersect by twos in four S-points lying two on each of the other two lines through point 4 on which four p-lines coincide.

THEOREM IX. *If we should relate six tangents to a conic as we have six points on a conic, then a line joining two intersections of four of the five fixed tangents determines a point on the fifth fixed tangent at which two lines intersect each tangent to twelve distinct conics and eight lines each tangent to eight conics. There are fifteen of these points lying by threes on each of the five tangents.*

[1] The notation used here is interpreted thus: the first and second columns, the first and third columns, and the second and third columns give the 3 Pascal lines which meet in a Steiner point.

Since this is the exact dual of Theorem VIII, we shall not give a separate proof.

A few correspondences in the above investigation suggest certain theorems on the complete pentagon. Consider the three p-lines intersecting in a Steiner point,

$$\begin{cases} 36,25,41 \\ 45,61,23. \\ 12,34,56 \end{cases}$$

When 1 coincides with 4, four p-lines coincide upon the position of the p-line determined by the first two columns. This line is determined by the point 36,25 and point 4. Likewise the first and third columns determine a p-line. When 1 coincides with 6 four p-lines coincide and their position is fixed by the point 45,23 and the point 6. The second and third columns determine a similar set through the points 34,56 and 2. Although these three groups of four p-lines do not take these positions at the same time, but only as 1 coincides with 4, 6 and 2 respectively, the fact that the three lines that give rise to these positions meet in a Steiner point suggests that these three lines might be concurrent. Also since four p-lines coincide on each position it seems that there should be four of these points on each line. Hence, the following theorem is suggested.

THEOREM X. *The fifteen lines i formed by joining each of the five vertices of a pentagon in turn to the three points determined by lines joining the other four vertices, intersect in threes in twenty points G in addition to the vertices and four G-points lie on each i-line.*

PROOF: We shall use five T-points as the vertices in order to keep the notation uniform. Then in the two triangles whose sides are respectively (32), (21), (15), and (45), [(45,13)–(25,34)], (34) the corresponding sides meet in three points (23,45), (34,51) and the intersection of [(13,45)–(34,52)] with (2,1). But by Pascal's theorem we may show that the line joining the first two of these meets t_4 in the third, hence the three lines joining corresponding vertices meet in a point. But these are the three i-lines,

$$\begin{array}{l} (31,45)-2 \\ (23,51)-4. \\ (25,34)-1 \end{array}$$

The notation shows that if 3 and 1 be exchanged or 4 and 5 or both these changes be made at the same time, that the first line is unchanged. Hence four of these G-points lie on each i-line and since the numbers may be assigned to the points without respect to order the proof is complete.

We shall now state the dual of theorem X for reference in the following discussion.

THEOREM XI. *The fifteen points I formed by the intersection of each of the five sides of a pentagon in turn with three lines determined by joining the intersections*

of the other four sides, lie in threes on twenty lines g in addition to the sides and four g-lines pass through each I point.

Dual Properties: If we let the five fixed T-points be the vertices of the pentagon of theorem X and the corresponding t-lines the sides of the pentagon of theorem XI, we may develop an interesting cycle of pole and polar relationships among the lines through a T-point and the points on these lines.

The fifteen i-lines are now determined by joining in the proper way the fifteen fixed P-points to the five fixed T-points. The three P-points determined by a given four T-points determine the three i-lines through the fifth T-point, say T_n. But these three P-points are the vertices of a self-polar triangle; hence, the sides of this triangle determine the three I-points on t_n. The fifteen P-points determine 105 lines, thirty of these are formed each by a given two T-points, fifteen are the sides of the five sets of self-polar triangles and are formed each by a given four T-points. The remaining sixty are determined by using all five T-points and are therefore of the form (13,52)–(45,23). We shall call these the k-lines and their poles the K-points. It was shown in theorem VIII that these k-lines intersect by twos in thirty S-points lying by twos on each i-line. Hence, the s-lines are the lines of theorem IX which are each tangent to twelve conics. We may determine a certain set of six S-points using the method of theorem VIII from the six Steiner points:

$$S_1\begin{cases}36,25,41\\45,61,23,\\12,34,56\end{cases}\quad S_2\begin{cases}36,45,12\\52,61,43,\\14,23,56\end{cases}\quad S_3\begin{cases}25,36,41\\31,54,26,\\46,12,35\end{cases}$$

$$S_4\begin{cases}36,45,12\\25,31,46,\\14,26,35\end{cases}\quad S_5\begin{cases}45,23,16\\26,41,35,\\13,65,42\end{cases}\quad S_6\begin{cases}13,45,26\\56,32,14.\\24,16,35\end{cases}$$

When the moving vertex coincides with a fixed T-point, say point 4, S_1 and S_3 lie on i_1; S_2 and S_6 lie on i_3; and S_4 and S_5 lie on i_2 where the three i-lines pass through point 4. (See Figure 2.) Steiner has shown[1] that the three groups of two, S_1, S_2 and S_3, S_4 and S_5, S_6 are each divided harmonically by the conic. Hence S_1 lies on s_2 and S_2 lies on s_1.[2] Now, through S_1 pass k_1, k_2, i_1, s_2 and twelve conics. (See Figure 2.) Similarly, on s_1 which is tangent to twelve conics lie K_1, K_2, I_1, and S_2. A similar relationship exists among the two S-points in each of the other two groups, tying the six S-points, the six s-lines, the three I-points and the three i-lines together in an interesting figure. There are five of these sets in the complete figure. Figure two shows a cycle of these relationships.

Since the six points on the conic may be grouped into six sets of five points: theorem X and a generalization of the discussion following theorem XI give,

THEOREM XII. *The forty-five Pascal points lie by twos on 360 lines k, sixteen through each Pascal point, which intersect by twos in 180 points S. The S-points*

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[1] Salmon, loc. cit., p. 380.

[2] Salmon, loc. cit., p. 148.

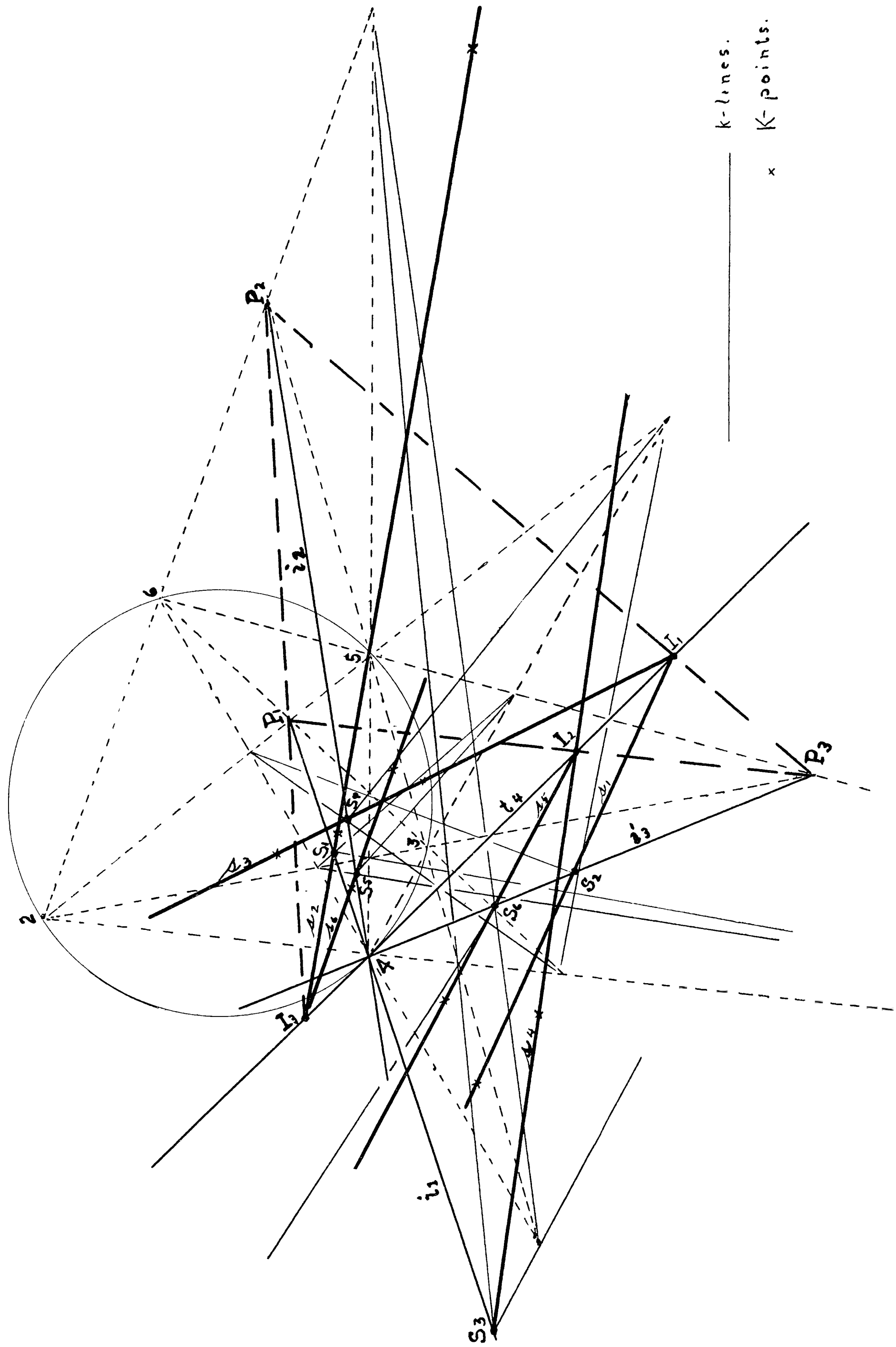

FIG. 2

lie by twos on 90 lines i. The i-lines intersect in groups of fifteen in each vertex of the hexagon and by twos in each of the forty-five Pascal points and by threes in 120 points G which lie in sets of four on each line. The S-points also lie in 90 sets of two which are harmonically divided by the conic.

Construction: The fifteen i-lines through a given vertex of the hexagon are determined by joining the vertex to the fifteen Pascal points determined by the remaining five vertices.

The sixteen k-lines through a given Pascal point are determined by joining the given point to the sixteen Pascal points determined by using three of the vertices of the hexagon used in the given point and one vertex not used in the given point. Thus one k-line through 36,45 would pass through 34,62.

NON-ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

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1. From the theory of plane quaternions $a+bi+cj$ two types of plane vector analysis have arisen. The one interprets $a+bi$ as a vector w in the complex plane and carries on its operations according to the algebra of quaternions. The other interprets $bi+cj$ as a real vector $\mathbf{w}$ and decomposes the quaternion product $\mathbf{w}_1\mathbf{w}_2$ into its scalar and vector parts, which are thereafter used separately as "products."

To indicate the interpretation used, we shall denote complex and real vectors by italic and bold-face letters respectively. Thus the plane vector whose components are u, v may be written as

$$w = u + vi \text{ or } \mathbf{w} = u\mathbf{i} + v\mathbf{j}.$$

In the first case

$$w_1w_2 = (u_1u_2 - v_1v_2) + (u_1v_2 + u_2v_1)i; \tag{1}$$

in the second

$$\mathbf{w}_1\mathbf{w}_2 = -(u_1u_2 + v_1v_2) + (u_1v_2 - u_2v_1)\mathbf{k}. \tag{2}$$

In Gibbs' notation

$$u_1u_2 + v_1v_2 = \mathbf{w}_1\cdot\mathbf{w}_2, \quad u_1v_2 - u_2v_1 = \mathbf{k}\cdot\mathbf{w}_1\times\mathbf{w}_2.$$

Let $w' = u - vi$ denote the conjugate of w; then from (1) and (2) we find that

$$w_1'w_2 = \mathbf{w}_1\cdot\mathbf{w}_2 + i\mathbf{k}\cdot\mathbf{w}_1\times\mathbf{w}_2, \quad w_1w_2' = \mathbf{w}_1\cdot\mathbf{w}_2 - i\mathbf{k}\cdot\mathbf{w}_1\times\mathbf{w}_2,$$

so that

$$\mathbf{w}_1\cdot\mathbf{w}_2 = \tfrac{1}{2}(w_1w_2' + w_1'w_2), \tag{3}$$

$$\mathbf{k}\cdot\mathbf{w}_1\times\mathbf{w}_2 = \tfrac{1}{2}i(w_1w_2' - w_1'w_2). \tag{4}$$

From (3) and (4) we have the conditions

$$w_1 w_2' + w_1' w_2 = 0, \quad w_1 w_2' - w_1' w_2 = 0$$

for the perpendicularity and parallelism respectively of the complex vectors w_1, w_2.

We next introduce the operators

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} \text{ and } \nabla = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

in the real and complex planes, and their conjugates ∇'. Then if $\phi(x, y)$ is a real scalar point function,

$$\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y};$$

and in view of (3) and (4),

$$\text{(5)} \qquad \operatorname{div}\boldsymbol{w} = \nabla\cdot\boldsymbol{w} = \tfrac{1}{2}(\nabla w' + \nabla' w),$$

$$\text{(6)} \qquad \boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{w} = \boldsymbol{k}\cdot\nabla\times\boldsymbol{w} = \tfrac{1}{2}(\nabla w' - \nabla' w)i,$$

$$\text{(7)} \qquad \nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\nabla'\phi.$$

It is often advantageous to introduce the variables

$$z = x + iy, \quad z' = x - iy$$

instead of x, y in the complex plane. Thus to any function $w(x, y)$ there corresponds uniquely a function $w(z, z')$. Then we find that

$$\frac{\partial w(z, z')}{\partial z} = \tfrac{1}{2}\nabla' w(x, y), \quad \frac{\partial w(z, z')}{\partial z'} = \tfrac{1}{2}\nabla w(x, y),$$

whence the operational identities

$$\text{(8)} \qquad \frac{\partial}{\partial z} = \tfrac{1}{2}\nabla', \quad \frac{\partial}{\partial z'} = \tfrac{1}{2}\nabla.$$

Therefore on changing variables from x, y to z, z' in (5), (6), (7), we obtain:

$$\text{(5)}' \qquad \operatorname{div}\boldsymbol{w} = \frac{\partial w'}{\partial z'} + \frac{\partial w}{\partial z}$$

$$\text{(6)}' \qquad \boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{w} = \left(\frac{\partial w'}{\partial z'} - \frac{\partial w}{\partial z}\right)i$$

$$\text{(7)}' \qquad \nabla^2\phi = 4\frac{\partial^2\phi}{\partial z\partial z'}\cdot$$

We are now in possession of a dictionary that will paraphrase the theorems for

real vector functions $\boldsymbol{w}$ into the corresponding theorems for non-analytic functions w in the complex plane.

2. We first seek to paraphrase the theorem that the derivative of $\boldsymbol{w}$ in the direction of the unit vector $\boldsymbol{t}$ is

$$\frac{d\boldsymbol{w}}{ds} = \boldsymbol{t} \cdot \nabla \boldsymbol{w}, \tag{9}$$

where we assume the existence and continuity of the partial derivatives involved. Let $\boldsymbol{t} = e^{i\theta}$; then from (3) we have

$$\frac{dw}{ds} = \tfrac{1}{2}e^{i\theta}\nabla' w + \tfrac{1}{2}e^{-i\theta}\nabla w.$$

Since $dz = e^{i\theta}ds$ in the direction $\boldsymbol{t}$,

$$\frac{dw}{dz} = \tfrac{1}{2}\nabla w e^{-2i\theta} + \tfrac{1}{2}\nabla' w. \tag{10}$$

Thus dw/dz will be independent of θ when and only when $\nabla w = 0$, a condition that is precisely a synthesis of the Cauchy-Riemann equations. In view of (8), (10) may also be written in the simple form[1]

$$\frac{dw}{dz} = \frac{\partial w}{\partial z'}e^{-2i\theta} + \frac{\partial w}{\partial z} \cdot \tag{11}$$

From (11) we see that if the vector dw/dz is drawn from a point, then as θ varies from 0 to 2π its end-point travels twice in the opposite direction about a circle of center w_z and of radius $|w_{z'}|$—the *Kasner circle.*[2] Moreover if $w_z w_{z'} \neq 0$ and

$$w_z = ae^{i\alpha}, \quad w_{z'} = be^{i\beta},$$

$$\frac{dw}{dz} = be^{i(\beta - 2\theta)} + ae^{i\alpha};$$

hence $|dw/dz|$ attains its maximum value $a+b$ when $\theta = \frac{1}{2}(\beta - \alpha)$ and its minimum value $|a-b|$ when $\theta = \frac{1}{2}(\beta-\alpha) + \frac{1}{2}\pi$. These are the principal directions of dw/dz at the point z.[3]

3. We next consider certain vector fields w of particular importance in mathematical physics.

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[1] G. Calugaréano, *Sur les fonctions polygènes d'une variable complexe*, Comptes rendus, vol. 186 (1928), p. 930.

[2] E. Kasner, *General theory of polygenic functions*, Proc. Nat. Acad. of Sciences, vol. 14 (1928), p. 75.

[3] Tissot, *Sur les cartes géographiques*, Comptes rendus, vol. 49 (1859), p. 673. Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de mathématiques, Ser. 9, vol. 2 (1923), p. 335.

The Irrotational Field: rot $\boldsymbol{w}=0$. Then $\boldsymbol{w}$ is expressible as the gradient of a scalar point function:

$$\boldsymbol{w} = \nabla\phi \text{ where } \phi = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{w}\cdot d\boldsymbol{r}.$$

In the complex plane, therefore, $w(z, z')$ is irrotational when and only when

$$\frac{\partial w}{\partial z} - \frac{\partial w'}{\partial z'} = 0;$$

then there exists a real function $\phi(z, z')$ such that

$$w = \frac{\partial \phi}{\partial z'}, \quad \text{and} \quad \phi = \int_{z_0}^{z} (w'dz + wdz').$$

The field lines are the orthogonal trajectories of the curves $\phi=\text{const}$.

The Sourceless Field: div $\boldsymbol{w}=0$. Then $\boldsymbol{w}$ is expressible as

$$\boldsymbol{w} = \boldsymbol{k}\times\nabla\phi \text{ where } \phi = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{w}\times\boldsymbol{k}\cdot d\boldsymbol{r}.$$

In the complex plane, therefore, $w(z, z')$ is sourceless when and only when

$$\frac{\partial w}{\partial z} + \frac{\partial w'}{\partial z'} = 0;$$

then there exists a real function $\phi(z, z')$ such that

$$w = i\frac{\partial \phi}{\partial z'} \quad \text{and} \quad \phi = i\int_{z_0}^{z} (w'dz - wdz').$$

The field lines are the curves $\phi=\text{const}$.

We summarize these results in the

THEOREM. *If $\phi(z, z')$ is a real function with continuous partial derivatives, then $\phi_{z'}$ and $i\phi_{z'}$ give respectively an irrotational field with lines orthogonal to $\phi=const.$, and a sourceless field with the lines $\phi=const.$*

The above results give a simple method for decomposing a plane vector function into an irrotational and a sourceless part. For if

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \quad \text{rot}\,\boldsymbol{w}_1 = 0, \quad \text{div}\,\boldsymbol{w}_2 = 0,$$

there exist real functions ϕ, ψ such that

$$w = \frac{\partial\phi}{\partial z'} + i\frac{\partial\psi}{\partial z'} = \frac{\partial}{\partial z'}(\phi + i\psi).$$

Hence the vector

$$\phi + i\psi = \int w(z, z')dz' \quad (z \text{ constant})$$

is determined to an arbitrary additive $f(z)$; the decomposition is therefore unique.

If the vector field $w(z, z')$ is both irrotational and sourceless, rot $\boldsymbol{w}=0$ implies that

$$w = \frac{\partial\phi}{\partial z'}; \text{ then } w' = \frac{\partial\phi}{\partial z}, \quad \operatorname{div}\boldsymbol{w} = 2\frac{\partial^2\phi}{\partial z\partial z'} = 0.$$

Hence ϕ is a real harmonic function, w is an analytic function of z', and w' is an analytic function of z.

THEOREM. *If w is an analytic function of z', the field is both irrotational and sourceless. The conjugate of an irrotational and sourceless field vector is an analytic function of z.*

4. Finally let us paraphrase the integral theorems in the plane:

$$\int \operatorname{div}\boldsymbol{w}d\sigma = \oint \boldsymbol{n}\cdot\boldsymbol{w}ds = \oint \boldsymbol{k}\cdot\boldsymbol{w}\times d\boldsymbol{r}, \tag{12}$$

$$\int \boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{w}d\sigma = \oint \boldsymbol{w}\cdot d\boldsymbol{r}, \tag{13}$$

which hold when $\boldsymbol{w}$ and $\nabla\boldsymbol{w}$ are continuous. Here $\boldsymbol{n}$ is a unit external normal to the closed curve bounding the area over which the left-hand integrals are taken; and $\boldsymbol{k}$ is a unit vector normal to the plane in the sense of progression of a right-handed screw turned in the direction of the circuit integrals on the right.

Using (3), (4), (5), (6), the paraphrases of (12) and (13) are

$$\int (\nabla w' + \nabla' w)d\sigma = i\oint (wdz' - w'dz), \tag{14}$$

$$i\int (\nabla w' - \nabla' w)d\sigma = \oint (wdz' + w'dz). \tag{15}$$

Multiply (14) by i and add to (15); then

$$i\int \nabla w'd\sigma = \oint w'dz,$$

and on replacing w' by w, we have

$$\oint wdz = i\int \nabla wd\sigma = 2i\int \frac{\partial w}{dz'}d\sigma. \tag{16}$$

When w is analytic over the area, $\oint wdz=0$; thus (16) is a generalization of

Cauchy's Integral Theorem. With $w = z'$ in (16) we obtain the usual circuit integral for a plane area; and with $w = zz'$ we obtain expressions for the first moment of a plane area as a circuit integral.

From (16) we may deduce the generalization of Cauchy's integral formula,

$$f(\zeta) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-\zeta}\,dz - \frac{1}{\pi}\int \frac{\partial \bar{f}/\partial z'}{z-\zeta}\,d\sigma,$$

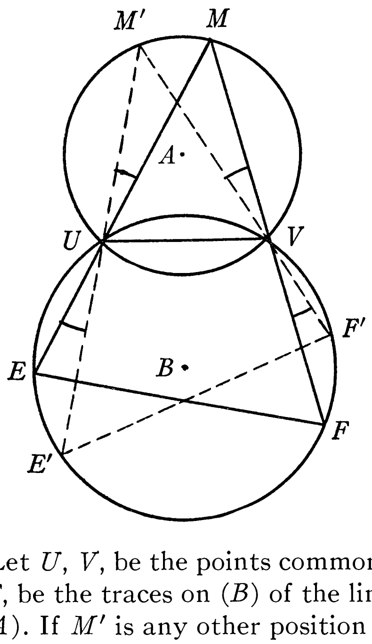
where $f(z) \equiv \phi(x, y) \equiv \bar{f}(z, z')$.

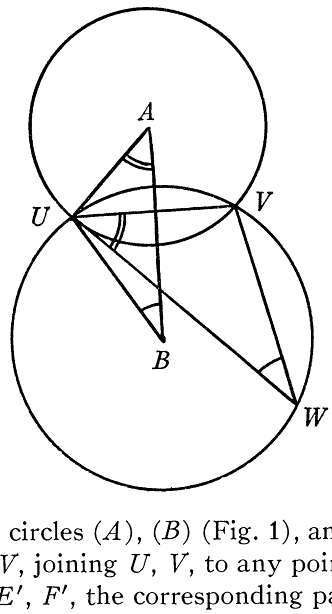
ON TWO INTERSECTING SPHERES

By N. A. COURT, University of Oklahoma

I. Introduction. The object of this paper is to extend to two intersecting spheres some properties of two intersecting circles.[1] For the convenience of the reader I shall reproduce here those properties of the circles which will be referred to in what follows.

(a) *The two lines joining the two (real) points of intersection of two circles to a variable point of one of these circles determine in the other circle a chord of constant length.*





Let U, V, be the points common to the two circles (A), (B) (Fig. 1), and let E, F, be the traces on (B) of the lines MU, MV, joining U, V, to any point M of (A). If M' is any other position of M, and E', F', the corresponding pair of points on (B), it is readily seen that the arcs EE', FF', are equal, and that EF, $E'F'$, are the diagonals of an isosceles trapezoid; hence the proposition.

[1] Nathan Altshiller-Court. *Sur deux cercles secants.* Mathesis, vol. 39 (1925), p. 453.

(b) When the point M coincides with U (Fig. 2), the point F also coincides with U, the line MUE becomes the tangent to (A) at U, and the chord EF is the segment UW intercepted on this tangent by the circle (B).

(c) Let A, B, denote the centers of (A), (B), (Fig. 2). The two triangles AUB, UVW, are equiangular, hence $UW:AB = UV:AU$; or, if $AU=a$, $AB=d$, $UV=c$, and $UW=y$, then $ya=cd$. This relation gives the length of the constant chord $EF=y$ of the above theorem (a).

II. THEOREM. *The variable cone having for base the (real) circle of intersection of two spheres and for vertex a variable point of one of these spheres cuts the other sphere again along a circle whose radius is constant.*

The cone Σ having for base the common (real) circle (S) of the two given spheres (A), (B), and for vertex any point M of (A) penetrates into (B) along (S), hence Σ emerges from (B) along another circle (T).

The principal plane of Σ is determined by the line MS joining M to the center S of (S) and by the perpendicular at S to the plane of (S); hence the principal plane of Σ contains the centers A, B, of (A), (B), and cuts these spheres along great circles. This principal plane cuts (S) along the principal diameter UV, and the principal elements MU, MV, of Σ meet (B) again in the ends E, F, of a diameter of (T).[1] The proposition thus follows immediately from the case of the plane (see Ia).

III. If $AU=a$, $AB=d$, $UV=c$, and $EF=y$, we have (see Ic)

$$ya = cd. \tag{1}$$

We have thus a simple expression for the length y of the diameter EF of (T) in terms of the fixed elements of the two given spheres.

When the vertex M of the cone Σ coincides with a point L of the circle (S), the cone degenerates, and (T) becomes the circle of intersection of the sphere (B) with the tangent plane to (A) at L; the formula (1) gives the diameter y of this circle.

The segment BT joining B to the center T of the circle (T) is equal to the distance h of B from the plane of (T). If we denote by b the radius of (B) we obtain from the right triangle BTE and the formula (1),

$$h^2 = (4a^2b^2 - d^2c^2):4a^2. \tag{2}$$

If the sphere (A) is bisected by the sphere (B), then $c=2a$, and $d^2=b^2-a^2$, and the formulas (1), (2), give

$$y = 2d, \quad h = a.$$

The reader may formulate verbal statements expressing these results.

[1] Rouché et Comberousse. *Traité de géométrie*, vol. II, pp. 223–224. Gauthier-Villars. Paris, 1900.

IV. Consider the cone Σ' having (S) for its base and its vertex on the sphere (B). The formula (1) gives for the diameter x of the circle of intersection of Σ' with (A),

$$xb = cd;$$

hence

$$cd = yb = xa, \text{ or } x:y = a:b.$$

Thus: *The two cones having for base the common circle of two spheres and for vertices any two points on the two spheres cut these spheres again along two circles whose radii are proportional to the radii of the spheres on which the circles are situated.*

As a limiting case we have: *The two tangent planes to two spheres at a point common to these spheres cut the spheres along two circles whose radii are proportional to the radii of the spheres on which the circles are situated.*

V. If the sphere (A) and the circle (S) remain fixed while the sphere (B) varies describing the coaxal pencil having (S) for basic circle, we have, by (1),

$$y:d = c:a,$$

hence: *The cone passing through the basic circle of a coaxal pencil of spheres and having its vertex on a given sphere of the pencil cuts the spheres of the pencil again along circles whose radii are proportional to the distances of the centers of the respective spheres from the center of the given sphere.*

VI. When the vertex M of the cone Σ lies on (S) the preceding proposition takes the following form: *The tangent plane to a sphere of a coaxal pencil at a point of the basic circle of the pencil cuts the other spheres along circles whose radii are proportional to the distances of the centers of the respective spheres from the center of the first sphere.*

The special case of the last two propositions when (S) is a great circle on (A) is left for the consideration of the reader.

VII. *If p, q, are the squares of the radii of two spheres, t the square of their line of centers, and s the square of the radius of their common circle, we have*

$$4st = 4pq - (p + q - t)^2. \tag{3}$$

If u is the power, with respect to either of the two given spheres (P), (Q), of the trace X of the radical plane of (P), (Q), on their line of centers PQ, we have

$$u = XP^2 - p = XQ^2 - q,$$

and hence

$$XP^2 + XQ^2 = 2u + p + q. \tag{4}$$

On the other hand,

$$XP^2 - XQ^2 = p - q,$$

and we have, both in magnitude and in sign,

$$(XP + XQ)(XP - XQ) = p - q,$$

or

$$(XP + XP + PQ)(QX + XP) = p - q.$$

Hence we have, both in magnitude and in sign,

$$2PX \cdot PQ = PQ^2 + (p - q);$$

and, similarly,

$$2QX \cdot QP = QP^2 + (q - p).$$

Multiplying these two equalities and putting $PQ^2 = t$, we have, both in magnitude and in sign,

$$4PX \cdot XQ \cdot t = t^2 - (p - q)^2. \tag{5}$$

Multiplying (4) by $2t$ and adding to (5) we have, after simplification,

$$4ut = (p + q - t)^2 - 4pq.$$

Now u and s are equal in magnitude and opposite in sign, hence the announced relation (3).

In the above proof no use has been made of points on either sphere, hence the formula (3) is valid for any two spheres whose centers are real and the squares of whose radii, as well as the square of the radius of their common circle, are either positive, or negative.

If the spheres (P), (Q), are orthogonal, we have $t = p + q$, and (3) becomes $4(p+q)s = 4pq$ or[1]

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

VIII. The formula (3) applied to the spheres (A), (B), (see II, III) becomes

$$c^2d^2 = 4a^2b^2 - (a^2 + b^2 - d^2)^2; \tag{6}$$

hence, using the formula (1),

$$y^2a^2 = 4a^2b^2 - (a^2 + b^2 - d^2)^2. \tag{7}$$

We have thus the value of the diameter $EF = y$ in terms of the radii of the given spheres and of the length of their line of centers.

From the formulas (6), (2), we have

$$h = (a^2 + b^2 - d^2) : 2a. \tag{8}$$

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[1] Nathan Altshiller-Court. *On five mutually orthogonal spheres.* Annals of Mathematics, vol. 30 (1929), p. 614.

IX. If (A), (B), are orthogonal, we have $a^2+b^2=d^2$, and (7) gives $y=2b$; hence:

The cone passing through the common circle of two orthogonal spheres and having its vertex on one of these spheres cuts the second sphere along a great circle.

This result also follows from (8), since h in this case becomes zero.

The same formula (8) shows that the tangent plane to one of two orthogonal spheres at a point common to the two spheres passes through the center of the second sphere, as was to be expected.

X. As the vertex M of Σ varies on (A) the plane of the circle (T) remains at a fixed distance from the center B of (B), hence the plane of (T) envelopes a sphere (C) concentric with (B). The sphere (C) is tangent to the planes which touch (A) at the points along (S) (see III), i.e., (C) is inscribed in the cone which circumscribes (A) along (S). Stated in this form the proposition admits of the following projective generalization:

Given two quadric surfaces (A), (B), intersecting along two conics (S), (S'), the cone projecting one of these conics, say (S), from a variable point of one of the surfaces, say (A), cuts the second surface (B) along a conic whose plane envelopes a third quadric (C) tangent to (B) along the conic (S'). Moreover (C) is inscribed in the cone circumscribing (A) along the conic (S).

THE DEVELOPMENT OF THE FUNDAMENTAL CONCEPTS OF INFINITESIMAL ANALYSIS

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The mathematical interests of the ancient Greeks were primarily centered in geometry and geometrical constructions. As early as the fifth century B.C. they found themselves confronted with three problems which they soon realized could not be solved by the use of the unmarked straightedge and compasses alone.[1] These problems were the trisection of an arbitrary angle, the quadrature of the circle, and the duplication of the cube (or Delian problem).

The problem of squaring the circle has, throughout the ages, attracted interested attention. Among the first to attempt a solution was Antiphon (c. 430 B.C.), whose method was that of inscribing a regular polygon and exhausting the area by doubling the number of sides. By applying this method to both inscribed and circumscribed polygons, Archimedes (c. 225 B.C.) was able to calculate that the value of π lies between 3 1/7 and 3 10/71. Among other methods devised for squaring the circle were those using the quadratrix of Hippias (c. 425 B.C.), and the spiral of Archimedes.

[1] Sir Thomas Heath, *A History of Greek Mathematics*, Clarendon Press, Oxford, 1921, Vol. I, p. 219.

The problem of trisecting an angle "presumably arose from attempts to continue the construction of regular polygons after that of the pentagon had been discovered."[1] Out of the efforts to arrive at this construction the development and use of the conchoid by Nicomedes (c. 180 B.C.) and the use of conics by Pappus (c. 300 A.D.) are probably the most interesting.

"Eratosthenes (c. 230 B.C.) in his letter to Ptolemy III, relates that one of the tragic poets introduced Minos on the stage erecting a tomb for his son, Glaucus; and then, deeming the structure too mean for a royal tomb, he said 'Double it, but preserve the cubical form' . . . and continues: 'Later (in the time of Plato), so the story goes, the Delians, who were suffering from a pestilence, being ordered by the oracle to double one of their altars, were thus placed in the same difficulty.'"[2] Out of the research on the Delian problem we have the method of Archytas (c. 400 B.C.) in which he employed a right cone, a cylinder, and a tore of zero inner diameter; the probable discovery and use of conic sections by Menaechmus (c. 350 B.C.);[3] the use of the conchoid by Nicomedes and the invention and use of the cissoid by Diocles (c. 180 B.C.).[4]

Thus we see that out of the research on these three famous geometric problems of antiquity there developed many new curves whose properties were unknown. The interest in the properties of these curves, and curves of a similar nature, has been one of the strongest of mental stimuli in the development of the methods of infinitesimal analysis.

While Antiphon had made direct use of the method of exhaustion in his efforts at the quadrature of the circle, the discovery of this method is attributed by some to Hippocrates of Chios (c. 460 B.C.) and by others to Eudoxus (c. 370 B.C.). In the mathematical researches attributed to these men are found the first attempts to employ a technique that was in any way based upon a concept of infinitesimals or the application of a limiting process. The method of exhaustion may thus be considered as a form of infinitesimal analysis whose fundamental hypotheses are contained in the two propositions:

"(1) If A and B be two magnitudes of the same kind, of which A is greater, and there be taken from A more than its half (or any other fraction) and from the remainder more than its half (or any other fraction) and so on, the ultimate remainder will be less than B."

"(2) Let there be two magnitudes P and Q, both of the same kind and let a succession of magnitudes X_1, X_2, X_3, etc., be each nearer and nearer to P, so that any one X_r, shall differ from P less than half as much as its predecessor differed. Let Y_1, Y_2, Y_3, etc., be a succession of magnitudes similarly related to

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[1] Heath, op. cit., p. 235.

[2] G. J. Allman, *Greek Geometry from Thales to Euclid*, Longmans, Green & Co., 1889, p. 85.

[3] Heath, op. cit., pp. 251–255.

[4] D. E. Smith, *History of Mathematics*, Ginn and Co., 1923, Vol. I, p. 118.

Q, and let the ratios $X_1 : Y_1$, $X_2 : Y_2$, etc., be all the same with each other and the same as $A : B$. Then $P : Q :: A : B$."[1]

Archimedes, probably the greatest mathematician of all antiquity, and one of the greatest in the history of the world, was an original investigator in problems relating to the quadrature of curvilinear plane figures and the quadrature and cubature of curved surfaces. "These investigations, . . . , actually (in the words of Chasles) 'gave birth to the calculus of the infinite conceived and brought to perfection successively by Kepler, Cavalieri, Fermat, Leibniz, and Newton.' He performed in fact what is equivalent to *integration* in finding the area of a parabolic segment, and of a spiral; the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of second degree."[2] He first discovered the solution of the quadrature of the parabola through application of principles of mechanics but later confirmed his result by applying the method of exhaustion. This latter method consisted in exhausting the area of a segment of a parabola by means of a series of triangles inscribed "in the recognized manner," i.e., the vertex of the triangle inscribed in a segment being the point of contact of the tangent drawn parallel to the base of the segment. By this process he was able to express the value of the parabolic area in the form $A(1+\frac{1}{4}+(\frac{1}{4})^2+(\frac{1}{4})^3+ \cdots)$ where A was the area of the original inscribed triangle. Having obtained this series Archimedes, instead of passing to the limit, used the method of *reductio ad absurdum* to determine its value.

Although Archimedes made several outstanding contributions to Stereometry, particularly his work on *Spheroids and Conoids*, interest in this field of endeavor did not remain alive. This lack of interest was due, in part, to the influence of the Athenian school, in part, to Apollonius (c. 225 B.C.) whose work on conics drew attention in that direction, but, for the most part, it was due to the inflexibility and lack of generalization of the method of exhaustion. It remained for the imagination of later races to conceive of other methods by the aid of which further advancement might be possible.

The years of Roman domination of the civilized world and the subsequent Dark Ages were not notably fruitful in mathematical advancement since the Romans were not given to abstract thinking and did not encourage it. The political and social condition of the western world did well to keep the germ of knowledge alive, it certainly did not furnish a healthy field for its development and growth. The more or less complacent acquiescence to the will of the Church was also a factor in preventing progress in the fields of mathematics and the sciences. It was not until the Renaissance that interest in mathematical research of any significance began to be revived.

In Germany occurred the first really successful revolt from the ecclesiastical domination that had prevented fruitful scientific development. Among the

[1] J. Gow, *A Short History of Greek Mathematics*, Cambridge University Press, 1884, pp. 171–172.

[2] Heath, op. cit., Vol. II, p. 20.

leaders in this movement we find Johannes Kepler (1571–1630) who was the first to introduce into geometry the concepts of infinitely small and infinitely great quantities. He proposed the atomistic theory of geometric figures, which states that they can be considered as composed of an infinite number of indivisible units; thus, lines are aggregates of points, surfaces of lines, and solids of surfaces. Extending this idea further, a sphere may be considered as the sum of cones which are in their final form coincident with the radii of the sphere. Likewise, a cone may be considered to be the sum of triangles formed by passing planes through the vertex of the cone. These schemes bear a very close resemblance to the methods used by Archimedes and also to the method of summation later used in the integral calculus.

A great deal of Kepler's original research was incited by an interest in the work of Archimedes on solids. In his *Spheroids and Conoids*, Archimedes had created many solids whose volumes he had determined; these solids were conceived as generated by the revolution of conics, spirals, and circular segments about their respective axes. Kepler published in his *Nova Stereometria* an extension of this work on solids. This extension was accomplished by means of considering the solids generated by the revolution of these curves about a chord, a tangent, or even an extraneous line. In this way he added ninety new solids to those proposed by Archimedes, the simplest of which he solved. The more complex ones, however, remained as a test of the ingenuity of later mathematicians. The second chapter of this book dealt with the question of the proper method of gauging the contents of a wine barrel, and this lead him into a discussion on cross-sectional areas in which we find the foundation of the rule on maxima and minima.[1] By estimating the circumference of an ellipse whose axes are $2a$ and $2b$ to be $\pi(a+b)$, he anticipated the problem of rectification of curves; another problem also anticipated by Kepler was that of inverse tangents.[2]

Cavalieri (1598–1647) was lead by Kepler's method of integration into the development of his method of indivisibles. "In order to prove his fitness for the Chair of Mathematics at the University of Bologna he submitted, in 1629, the manuscript of his famous work, *Geometria Indivisibilibus Continuorum Nova quadam ratione promota*, which he published in 1635."[3] He considered geometrical magnitudes to be composed of indivisible elements which could be formed by continually slicing with parallels, and then used the ratio of the sum of these indivisible elements to a known constant. A good example of this method may be seen in the relation between the sum of the indivisible elements of a triangle as compared to the area of a parallelogram with the same base and altitude. Cavalieri considered the lines that fill a triangle to be in arithmetic progression

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[1] J. E. Montucla, *Histoire des Mathématiques*, Henri Agone, Paris, 1802, Tome 2, p. 31.

[2] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, B. G. Teubner, Leipzig, 1913, Band 2, p. 829.

[3] D. E. Smith, *Source Book in Mathematics*, McGraw-Hill Book Co., 1929, p. 605.

beginning with the apex 1 and going to the base n. The sum of these numbers is $n(n+1)/2$, but each line of the parallelogram is n and consequently its area is n^2, and the ratio of these two quantities is $\frac{1}{2}$ since n is infinite.

Although Cavalieri, by means of his method of indivisibles, was able to solve most of the solids that Kepler had proposed as well as the theorem of Pappus with respect to a volume generated by the revolution of a plane figure about an axis, it was criticized as being unscientific. If a line has no width, how could any number of them be found so as to fill a plane figure? If a plane has no thickness how could it be possible that even an infinite number might form a solid? Whether scientific or not the method exerted an enormous influence on the period that followed and served as a method of integration for forty or fifty years.

The claim of Cavalieri to the method of indivisibles was contested by friends of Roberval (1602–1675). Mersenne (1588–1648) had, in 1629, proposed the quadrature of the cycloid and, in 1634, Roberval sent him a solution of this problem in which he employed the method of indivisibles. Walker[1] lends support to Mersenne's statement "that Roberval's method of indivisibles was worked out between 1628 and 1634." She further points out, however, that there seemed to be no controversy between the two mathematicians themselves.[2] According to Chasles, Roberval was much more rigorous in the application of the method than was Cavalieri.[3]

Although not contributing directly to the methods of infinitesimal analysis the influence of Descartes (1596–1650) is by no means to be overlooked nor minimized. Descartes's greatest discovery was his coördinate geometry. This did not consist merely of applying algebra to geometry, for others, beginning with Archimedes, had used algebra in connection with geometry. Descartes systematized the method, introduced the notion of variables and constants into geometry, conceived of curves as generated by a moving point, referred these curves to two lines perpendicular to each other and represented them by equations involving two variables, the relation of these variables being determined by the distances from the two lines of reference. It was this notion of expressing curves by algebraic equations that made possible the step from geometry to analysis and thus paved the way for calculus.

In his *Arithmetica Infinitorum*, published in 1655, Wallis (1616–1704) gave a treatment of quadratures that summed up the ideas of integral calculus, without the mechanism, as we have it today. In this treatise, he assumed the principle of continuity first advanced by Kepler and extended the idea to include the analytic concepts advanced by Descartes. Like other mathematicians of his

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[1] Evelyn Walker, *A Study of the Traité des Indivisibles of Gilles Persone de Roberval*, Teachers College Columbia University, Contribution to Education #446, Bureau of Publications, Teachers College, N. Y., 1932, p. 143.

[2] Evelyn Walker, op. cit., p. 15.

[3] M. Chasles, *Aperçu Historique sur l'Origine et le Développement des Méthodes en Géométrie*, M. Hayez, Bruxelles, 1837, p. 61.

time, he considered geometric plane figures "as if" composed of an infinite number of lines, but was careful to say, that really these lines were parallelograms of infinitesimal width.

Pascal's (1623–1662) contribution to the theory of integration was embodied in a series of letters on certain problems concerning the properties of the cycloid. He employed practically every artifice known at that time, in particular, may be noted his summation of sines, the evaluation of "ungulae"[1] which had been suggested by St. Vincent (1584–1667), and the triangular and pyramidal sums. He called triangular sums "indivisibles" with respect to simple sums and pyramidal sums "indivisibles" with respect to triangular sums.[2] Pascal employed the method of Cavalieri with eminent success. His researches, according to d'Alembert, closely approach the integral calculus, and form the connection between the methods of Archimedes and Newton.[3]

Up to this point the narrative has been confined to the introduction to the integral calculus. Along with this evolution of the theory of integration, there developed out of problems concerning maxima and minima some of the concepts that finally led to the differential calculus. The problem of tangents was an inheritance from the ancients. In the early part of the seventeenth century interest in the problem was revived. Three different view points of this period were:

(1) tangents are secants whose two points of intersection with the curve coincide;

(2) tangents are prolongations of the sides of an inscribed polygon of an infinite number of sides;

(3) tangents are the direction of a curve at any particular moment.

The third view point was that of Roberval who recognized the value of his idea and at the same time felt its limitations. He succeeded in applying it to only a few curves; that he failed to make a general application was due to his inability to devise a scheme that was not dependent on a special property of the curve in question. The method bears a remarkable analogy to Newton's method of fluxions.

Fermat's (1601–1665) method was dependent on the principle of maxima and minima, first enunciated by Kepler, that in the neighborhood of a maximum value of a variable its increment becomes evanescent. For the determination of a maximum, he employed a rule which may be stated as follows: To determine any maximum value of a function $f(x)$, substitute $x+e$ for x, and, from the resulting expression, subtract $f(x)$, divide by e and eliminate the remaining e's. It is readily seen that this method is essentially that of the differential calculus. For this reason Lagrange attributed the discovery of the calculus to

[1] Ungulae denoted hoof-shaped solids, such as the frusta of cylinders or cones cut off by planes that are not parallel to one another.

[2] M. Marie, *Histoire des Sciences Mathématiques et Physiques*, Gauthiers-Villars, Paris, Tome IV, 1884, p. 193.

[3] W. W. R. Ball, *A Short History of Mathematics*, The Macmillan Co., 1893, p. 288.

Fermat. M. Poisson challenged this in the following words: "But the calculus consists of an ensemble of rules for finding immediately the differential of all functions, rather than the use that can be made of these infinitely small variations to solve such or like species of problems; and under this definition the creation of the differential calculus does not go back of Leibniz, author of the algorithm and of the notation which has so generally prevailed since the origin of calculus and to which infinitesimal analysis is principally responsible for its progress."[1]

Fermat also solved various problems of integration employing an ingenious application of arithmetic and geometric means. His demonstrations were rigorous but, even as in his method of tangents, they lacked a generality sufficient to affect the general trend of the progress of the science.

Isaac Barrow (1630–1677) extended and systematized the methods used by Fermat by introducing along with the e, another infinitesimal, a. He made use of the a and e somewhat as Leibniz used the dy and dx. Child is of the opinion that Barrow's method was not an extension of Fermat's, that the two methods were discovered independently, and that it was a mere coincidence that both men used the same letter e as the increment of x. He is also quite positive in giving Barrow the credit for the original concept of the characteristic triangle which Leibniz used so effectively at a later date.[2] All of Barrow's work was geometrical, his rule being devised primarily for the purpose of constructing tangents. The rule, however, is more important than that for which it was created, since by using it, he was able to perform genuine differentiations and integrations, though in a geometric form. In fact, Barrow created a calculus, similar in principle to the calculus of Leibniz and Newton, but applicable only to geometry.

Sir Isaac Newton (1642–1727) was a pupil of Barrow's and was greatly influenced by his mathematical efforts and those of Wallis. His first essay was the result of a study of Wallis's *Arithmetica Infinitorum*. Wallis had been unable to expand binomials with fractional exponents and consequently had been unable to apply his methods to the area of a circle, since this involved the expansion of the expression $y=(1-x^2)^{1/2}$. Newton, by a scheme of interpolation, had found the required series and effected the quadrature. Later he discarded interpolation and discovered the binomial theorem for all real exponents, by the aid of which he found expressions in the form of infinite series for areas and arcs of curves.[3]

Newton termed his plan of interpolation and the binomial theorem as his first and second methods. His third method, which was a new and independent system of infinitesimal analysis, he called the method of fluxions. It seems that

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[1] Chasles, op. cit., p. 63.

[2] J. M. Child, *The Geometrical Lectures of Isaac Barrow*, Open Court Publishing Co., 1916, pp. 13–14.

[3] Ball, op. cit., p. 328.

he had given serious thought to this method in 1665 and by the following year had formulated it into a rather definite system, but that he did not communicate with any of his friends about it until 1669 when he released the secret to his friend and former teacher, Isaac Barrow. In 1676, in response to a request from Leibniz for information on the inverse problem of tangents, he wrote that he was in possession of a general method of solution, but, in accordance with the spirit of the times, concealed it in a cipher, which when deciphered contained not only a statement of the problem but also a hint as to its solution. The statement bore the following meaning "given any equation whatsoever, involving fluent quantities, to find the fluxions, and vice versa."[1]

The term, fluxion, indicates motion. The idea of the fluxional calculus developed from the concept that a geometrical magnitude was the result of continuous motion of a point, line, or plane. This motion, speaking of plane curves, could be considered, when referred to coördinate axes, as the resultant of two motions, one in the direction of the X-axis and the other in the direction of the Y-axis. The velocity of the X-component and the Y-component were called "fluxions" by Newton. He represented them by $\dot{x}$ and $\dot{y}$. The velocity of a point is represented by an equation involving the fluxions $\dot{x}$ and $\dot{y}$; reciprocally, the arc is the "fluent" of the velocity of a moving point, so the x and y values are the fluents of the fluxions $\dot{x}$ and $\dot{y}$. The problem of fluxions is, thus, dual in nature:

"1. The length of the space described being continually given; to find the velocity of the motion at any time proposed.

"2. The velocity of the motion being continually given; to find the length of the space described at any time proposed."[2]

Fluxions were not considered as infinitesimals, but "moment of a fluxion" was the term introduced to denote an "indefinitely small" quantity. "If the moments of x be represented by the product of its celerity $\dot{x}$ into an indefinitely small quantity (that is by $\dot{x}o$), the moment of y will be $\dot{y}o$, since $\dot{x}o$ and $\dot{y}o$ are to each other as $\dot{x}$ and $\dot{y}$."[3] These moments are thus seen to be infinitely small increments and are analogous to the infinitesimals of Leibniz. In the application of this method to the solution of problems the final step always involved the rejection of infinitely small quantities as of no value in comparison with the other terms. Newton felt that the rejection of terms, no matter how small, always introduced a source of error in the results, and proceeded to build his theory of fluxions on a basis of rates, which involved a concept of limits.

In the introduction to the *Quadratura Curvarum* appeared an illustration of this second method of fluxions, in which he found the fluxion of x^n to be nx^{n-1}.[4]

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[1] A. DeMorgan, *Essays on the Life and Works of Newton*, Open Court Publishing Co., 1914, p. 25.

[2] F. Cajori, *A History of Mathematics*, The Macmillan Co., 1919, p. 193.

[3] Cajori, op. cit., pp. 194–195.

[4] G. H. Graves, *Development of the Fundamental Ideas of the Differential Calculus*, Mathematics Teacher, 1910, Vol. 3, p. 86.

"Let a quantity x flow uniformly and let it be required to find the fluxion of x^n. During the time in which x, by flowing, is becoming $x+o$, the quantity x^n becomes $(x+o)^n$; that is, by the method of infinite series,

$$x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \cdots$$

and the increments o and

$$nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \cdots$$

are to each other as 1 and

$$nx^{n-1} + \frac{nn-n}{2}ox^{n-2} + \cdots .$$

Now let the increment vanish and the last ratio will be 1 to nx^{n-1}."

In the same treatise Newton connected this idea of fluxions with the problem of tangents. He considered a tangent as the limiting position of a secant as may be seen from his statement: "If the points (of intersection with the curve) are distant from each other by an interval however small, the secant will be distant from the tangent by a small interval. That it may coincide with the tangent and the last ratio found, the two points must unite and coincide altogether."[1]

Newton always considered fluxions as finite velocities, which were determined by the ratios of nascent or evanescent quantities. Feeling the logical difficulties involved in such a concept, he attempted to overcome them by his notion of prime and ultimate ratios which he developed in the *Principia:* "The ultimate ratios in which quantities vanish, are not really the ratios of ultimate quantities, but limits toward which the ratios of quantities, decreasing without limit, always approach; and to which they can come nearer than any given difference, but which they can never pass nor attain before the quantities are diminished indefinitely."[2]

In contrast to Newton who had received his early mathematical training under such a capable mathematician as Isaac Barrow, Leibniz (1646–1716) was a self-taught mathematician. However, he had the good fortune to make the acquaintance of the eminent physicist, astronomer, and mathematician, Christian Huygens (1629–1695) who advised him to study the works of Pascal, Cavalieri, and Descartes.

While studying Cartesian geometry Leibniz became interested in the problems concerning direct and inverse tangents. In his approach to these prob-

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[1] Graves, loc. cit.

[2] Smith, *A Source Book in Mathematics*, p. 618, translation from the *Principia* by Professor Evelyn Walker of Hunter College.

lems he drew the characteristic triangle, an infinitely small triangle included between differences of the ordinates, differences of the abscissae, and infinitely small arcs of the curve. In a tract written in 1673 he made two rather significant statements: "(1) There is a direct connection between the inverse method of tangents and quadratures; (2) The matter will be most accurately investigated by tables of equations."

By the side of, and linked closely with, the progress of the differential calculus of tangents was developed the science of integration. From an early conceived notion of the relation of quadratures to the "differences of applied lines," Leibniz arrived at a direct connection between them. Starting with the Cavalierian concept that areas are but the summation of lines, he first created $\int$, the symbol for summation (or integration). Reasoning that since summation increased the power, or dimension, so would differencing decrease the same, he progressed to the true meaning of differences and introduced the symbol, d, for the process of taking differences. Henceforth integration, in place of being that of which differentiation was the contrary calculus, became the inverse of differentiation.

In an undated manuscript, probably written about 1680, Leibniz showed the power of his new symbolism. Wallis, Pascal, and others had stated that areas might be obtained by the summation of an infinite number of rectangles whose altitudes were infinitesimals, but in their solutions had proceeded as if these rectangles were straight lines in arithmetic progression. Leibniz recognized the presence of this infinitestimal altitude by including it in the expression for an area. He wrote, "I represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae . . . I represent in my calculus the area of a figure by $\int y\, dx$, or the sum of the rectangles contained by each y and the dx that corresponds to it." Then he proceeded to link his method with that of Cavalieri by saying, "here, if the dx's are taken equal to one another, the method of Cavalieri is obtained."[1] Leibniz had arrived, in a manner far superior to any he had anticipated, at his second objective, namely that "the quadratures of all figures follow from the inverse method of tangents, and thus the whole science of sums and quadratures can be reduced to analysis, a thing that nobody even had hopes of before."[2] He had thus created a calculus dependent on the calculus of differences, which he first termed *Calculus Summatorius* and later *Calculus Integralis.*

The calculus of differences of Leibniz did not differ in principle from the method of tangents used by Barrow. There was, however, the important difference that whereas Barrow was interested primarily in geometry and applied his method only to that end, Leibniz was able to treat algebraic expressions by his method, geometry simply being one of the fields in which applications could

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[1] J. M. Child, *The Early Mathematical Manuscripts of Leibniz,* Open Court Publishing Co., 1920, p. 138.

[2] Child, ibid., pp. 60–61.

be made. Each gave a set of rules for differentiation and integration, the former geometric, and the latter algebraic. Leibniz termed his set of rules an "algorithm." It was this algorithm, along with the notation for differentiation and integration, that marked his work as a distinct step forward in infinitesimal analysis.

A great dispute as to the priority of the discovery of the infinitesimal calculus arose between Newton and Leibniz. Friends of the two men took sides until finally the dispute became so severe that it influenced the development of the subject for many years to follow. For the most part, the mathematicians of England followed the Newtonian idea while those of the continent followed Leibniz's point of view.

On the continent, the two Bernoulli brothers, Jacques (1654–1705) and Jean (1667–1748), Marquis de l'Hôpital (1661–1704), and d'Alembert (1717–1783) were among the most prominent of the early contributors to the development of these newly proposed methods of infinitesimal analysis. In Great Britain a Scotchman by the name of Craig (1656–1718) published several treatises in the form of books and contributions to the *Philosophical Transactions* (London) in which he made use of the notation and method suggested by Leibniz. It is said that, in preparing one of these manuscripts for publication, he submitted it to Newton to read and that this manuscript was influential in causing Newton to allow his method of fluxions to be made public in the 1693 edition of Wallis's algebra.[1] The priority dispute influenced Craig to use the fluxional notation in his later papers. Other followers of Newton were De Moivre (1667–1754), Taylor (1685–1731), MacLaurin (1698–1746), and Cotes (1682–1716). With the exception of Taylor and MacLaurin they all used fluxions as infinitely small quantities, thus grafting Newton's notation onto the calculus of Leibniz.

The two methods were not without their critics. Most prominent among these were Nieuwentijt (1654–1718) and Rolle (1652–1719) on the continent and Berkeley (1684–1753) in Great Britain. Nieuwentijt's objections were somewhat philosophical, being; (1) the inaccuracy of rejecting infinitely small quantities; (2) the doubtful existence of differentials of higher order; and (3) the lack of universality of the method in that it had not been applied to other than algebraic expressions. Leibniz, in answer, skillfully met the first objection, only partially and rather poorly the second, and completely ignored the third.[2] Rolle's objections were largely computational and personal. Berkeley's objections were largely metaphysical, being inspired by the attitude of certain mathematicians who had complained of the incomprehensibility of religion. In his remarkable book, *The Analyst*, he attempted to show that the calculus was much more incomprehensible, basing his argument on what he considered to be four weaknesses of the subject, namely the dropping of terms, the concept of

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[1] F. Cajori, *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, Open Court Publishing Co., 1919, pp. 38–39.

[2] Child, op. cit., p 145.

velocity, the higher order of fluxions, and the ratio of evanescent increments. Cajori refers to the publication of *The Analyst* as "the most spectacular event of the century in the history of British mathematics."[1] Whatever may have been the intent of the author in this publication, the results were exceedingly beneficial to the calculus in that it provoked the clarification of some of its concepts.

We are indebted to Euler (1707–1783) not so much for his philosophy of the calculus, though he introduced a few new ideas with reference to the ratios of increments, but more for his two treatises *Institutiones Calculi differentialis* (1755) and *Institutiones Calculi integralis* (1768–1770). These two works, combined, form a complete text in which was summed up all that was then known on the subject, and the first text to treat the subject matter of the calculus in a general and systematic manner.

The principles of the calculus were, during the latter half of the eighteenth century, still involved in philosophical difficulties which were somewhat serious in nature. In the sub-title to his *Théorie des Fonctions Analytiques*, Lagrange (1736–1813) states that this work contains "the principles of the differential calculus, freed from any consideration of infinitesimals, of vanishing elements, of limits and fluxions, and reduced to the algebraic analysis of finite quantities." In this statement he has summed up the fundamental philosophical points of contention. In the introduction he proceeds to elaborate upon these difficulties, finally somewhat summarizing his statements in these words: "But this method (the method of fluxions) has, as that of limits of which it is only an algebraic translation, the great inconvenience of considering the quantities in the state, so to speak, in which they cease to exist, for, although we can conceive of the ratio of two finite existing quantities, that ratio does not offer to the mind a clear and precise idea of what happens when both terms become zero at the same time."[2]

Landen (1719–1790) an English geometrician, in a work entitled "The residual analysis a new branch of the Algebraic art" and published in London in 1764, proposed a purely analytic method to replace the method of fluxions. Landen proposed that "instead of the infinitesimal or null differences of varying quantities, different values of these quantities should be used, and they should be equated only after the vanishing factor had been caused to disappear through division."[3] Although this put the differential calculus on a more rigorous foundation, it caused the subject to lose a great deal of its simplicity and ease of operation.

Lagrange in commenting on his own contribution said "in a memoir . . . the purpose of which was to present the analogy between differentials and positive powers, and between integrals and negative powers, I stated that the theory

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[1] Cajori, ibid., p. 57.

[2] J. L. Lagrange, *Théorie des Fonctions Analytiques,* Courcier, Paris, 1813, p. 4.

[3] Lagrange, loc. cit.

of development of functions in series contained the real principles of the differential calculus, freed from any consideration of infinitesimals or limits, and, by this method, I proved Taylor's theorem, which is fundamental to the theory of series, and which up to this time, had been established only by means of the calculus, or by the consideration of infinitesimal differences."[1] In this development of functions in series, Lagrange called the first term the *primitive function*, and the different functions of the same variable, which appeared as multipliers in subsequent terms of the series, he called *derived functions* since they depended only on the primitive function and not on the undetermined increment used to effect the expansion. He expressed surprise that Newton had missed this approach to the calculus, since it was he who had first made any progress in the study of series.

Though Lagrange's method of developing the calculus succeeded in abstracting the notion of function, mathematicians soon detected that his arguments were fallacious. He had sought to define his *derived functions* in terms of coefficients in infinite series, concerning the convergence of which he knew nothing, and his proof that $f(x+h)$ could always be expanded in a series of ascending powers of h was by no means rigorous.[2]

With the researches of A. L. Cauchy (1789–1857) we have the culmination of the efforts to place the methods of infinitesimal analysis on a sound metaphysical basis. He made the first rigorous attempt to establish Taylor's theorem and thus to substantiate Lagrange's efforts to abstract the notion of function and place the calculus on an analytic basis. He also introduced a new method of considering limits and a new theory of continuity of functions, each of which helped in the exposition of the fundamental principles of the calculus.

In the meantime the "Analytical Society" had been formed at Cambridge, England, in 1813 and through the untiring efforts of the society the Leibnizian or differential notation dy/dx was introduced to replace the Newtonian, or fluxional, notation $\dot{y}$.

Thus we see that by the end of the first quarter of the nineteenth century infinitesimal analysis had reached the point of development where its fundamental principles and methods were on a sound metaphysical basis, its notation fairly well systematized and standardized, and its simplicity and facility of operation quite generally appreciated. Varied and interesting were the avenues of thought that were to become open sesames to future mathematicians under the influence of this new powerful tool of mathematical research.

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[1] Lagrange, ibid., p. 5.

[2] Cajori, op. cit., p. 257.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A PROBLEM IN GEOMETRICAL PROBABILITY

By C. H. FISCHER, University of Minnesota

A problem[1] recently solved in this Monthly was stated as follows: A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends will project over the edge of the table?

As is usual in the case of problems of this type, the solution is dependent upon certain basic assumptions; in this case, upon the assumptions made as to the manner in which the rod was placed upon the table. The published solution was obtained under the assumptions that perhaps appeal most to one's intuition, namely, that the direction of the rod is immaterial, and that the probability that the center of the rod will lie in any given region is proportional to the area of that region. A further study of this problem was made because of its resemblance to the well known Bertrand's Paradox, which asks for the probability that a chord drawn at random across a circle be at least as long as the side of the inscribed equilateral triangle. Czuber[2] gives six illustrative solutions using different basic assumptions as to what shall be considered as "equally likely" in drawing a chord at random across the circle, and he points out that further solutions could thus be obtained, each correct upon the basis of its own hypotheses.

In the present problem the rod may be considered as lying along a chord of the circle. An additional variable is necessary here, since the position of the center of the rod along the chord must also be considered. The variables of the problem may be chosen in various ways. For example, the angle the rod makes with a fixed line in the plane may be selected as the first variable, and the coordinates of the center of the rod as the other two variables. Another possible selection might find the first point of intersection of the rod (produced) with the circle, chosen as one variable, the second intersection as the second variable, and the position of the center of the rod along this line as the third variable. Other selections could be made. Let us examine the results obtained under a few particular sets of assumptions, making use of probability functions instead of using purely geometric methods.

1a). First, consider the assumptions of the published solution, as cited above. Choose rectangular coordinates, with the circle given by $x^2+y^2=r^2$, and

[1] Problem 3539, this MONTHLY, vol. 40 (1933), pp. 60–61.

[2] Czuber, *Wahrscheinlichkeitsrechnung*, (1908), vol. 1, pp. 106–109.

the rod parallel to the x-axis. The probability function of the distance of the center of the rod from the y-axis is $4(r^2-x^2)^{1/2}/\pi r^2$. That of the distance of the rod from the x-axis is $(r^2-x^2)^{-1/2}$. Then the probability that neither end of the rod will project over the edge of the table is given by

$$\frac{4}{\pi r^2}\int_0^{(r^2-a^2)^{1/2}}\int_0^{(r^2-y^2)^{1/2}-a} dxdy.$$

The expressions for the other two probabilities may be set up in the same manner. The results, as published, are:

A). The probability that neither end of the rod will project over the edge of the table is:

$$\frac{2}{\pi}\text{ arc cos}\frac{a}{r} - \frac{2a}{\pi r^2}(r^2 - a^2)^{1/2}.$$

B). That one end of the rod will project:

$$\frac{4a}{\pi r^2}(r^2 - a^2)^{1/2} - \frac{a}{\pi r^2}(4r^2 - a^2)^{1/2} - \frac{4}{\pi}\text{ arc cos}\frac{a}{r} + \frac{4}{\pi}\text{ arc cos}\frac{a}{2r}\cdot$$

C). That both ends of the rod will project:

$$\frac{a}{\pi r^2}(4r^2 - a^2)^{1/2} - \frac{2a}{\pi r^2}(r^2 - a^2)^{1/2} + \frac{4}{\pi}\text{ arc sin}\frac{a}{2r} - \frac{2}{\pi}\text{ arc sin}\frac{a}{r}\cdot$$

1b). Now assume that the position of the first point of intersection of the rod (produced) and the circle is immaterial. Select one such point. Assume that the probability that the center of the rod will lie in any given region is proportional to the area of that region. Choose polar coordinates, with the circle given by $\rho = 2r\sin\theta$. Then the probability function of the angle made by the rod and the x-axis is $4\pi^{-1}\sin^2\theta$. That of the distance of the center of the rod from the origin is $(2r\sin\theta)^{-1}$. Then the probability that neither end of the rod will project over the edge of the table is given by

$$\frac{2}{\pi r}\int_{\text{arc sin}\,a/r}^{\pi/2}\int_a^{2r\sin\theta-a}\sin\theta d\rho d\theta.$$

The expressions for the other probabilities may be set up in the same manner. The results are identical with those of 1a, as the basic assumptions, though somewhat different in form, are essentially equivalent.

2). Assume that the direction of the rod is immaterial. Let the circle be given by $x^2+y^2=r^2$. Then, for any fixed direction, assume that all distances of the rod from the x-axis, chosen parallel to the rod, are equally likely. This probability function is given by $1/r$. For any fixed distance from the x-axis, assume further that all distances of the center of the rod from the y-axis are equally likely. This probability function is given by $(r^2-y^2)^{-1/2}$. Then the

probability that neither end of the rod will project over the edge of the table is given by

$$\frac{1}{r}\int_0^{(r^2-a^2)^{1/2}}\int_0^{(r^2-y^2)^{1/2}-a}(r^2-y^2)^{-1/2}dxdy.$$

It will be noticed that we are integrating over the same area as in $1a$, but with a different integrand. The expressions for the other two probabilities may be set up in the same manner. The results, listed in the same order as those under $1a$, are

A). $$\frac{1}{r}\left\{(r^2-a^2)^{1/2}+a\cdot\arcsin\frac{a}{r}-\frac{\pi a}{2}\right\}.$$

B). $$\frac{1}{r}\left\{2r+\frac{\pi a}{2}-2a\cdot\arcsin\frac{a}{r}-2(r^2-a^2)^{1/2}\right\}.$$

C). $$\frac{1}{r}\left\{(r^2-a^2)^{1/2}-r+a\cdot\arcsin\frac{a}{r}\right\}.$$

3). Assume that the position of the first point of intersection of the rod with the circle is immaterial. Let the circle be given by $\rho=2r\sin\theta$. Then assume that all positions of the second intersection are equally likely, or the equivalent of this, that all angles made by the rod and the x-axis are equally likely. The probability function of this angle is $2/\pi$. Then assume that for any fixed angle all distances of the center of the rod from the origin, measured along the chord, are equally likely. This probability function is given by $(2r\sin\theta)^{-1}$. Then the probability that neither end of the rod will project over the edge of the table is given by

$$\frac{1}{\pi r}\int_{\arcsin a/r}^{\pi/2}\int_a^{2r\sin\theta-a}\csc\theta\,\rho d\theta.$$

Here we are integrating over the same area as in $1b$, but with a different integrand. The expressions for the other two probabilities may be set up in the same manner. The results, listed in the same order as those under $1a$, are

A). $$1-\frac{2}{\pi}\arcsin\frac{a}{r}+\frac{2a}{\pi r}\log\frac{r-(r^2-a^2)^{1/2}}{a}.$$

B). $$\frac{4}{\pi}\arcsin\frac{a}{r}-\frac{4}{\pi}\arcsin\frac{a}{2r}+\frac{2a}{\pi r}\log\frac{2r-(4r^2-a^2)^{1/2}}{a}-\frac{4a}{\pi r}\log\frac{r-(r^2-a^2)^{1/2}}{a}.$$

C). $$\frac{4}{\pi}\arcsin\frac{a}{2r}+\frac{2a}{\pi r}\log\frac{r-(r^2-a^2)^{1/2}}{2r-(4r^2-a^2)^{1/2}}-\frac{2}{\pi}\arcsin\frac{a}{r}.$$

Analytically, cases 2 and 3 are as simple as case 1. If probability is defined as the limit of relative frequency, one might inquire into the possibility of testing these solutions experimentally. If an attempt is made to design mechanical devices for placing the rod upon the table in such a manner that the assumptions of 1, 2, and 3, respectively, will hold, we find little difficulty in designing such a device for case 1. Considerable difficulty is encountered with the remaining two cases, however, and the device would be either quite complicated mechanically, or else involve a process of two steps. This investigation helps to demonstrate the reasonableness of the basic assumptions of the published solution.

It might be of some interest to compute the numerical values of these probabilities for some special cases. If $r=5$ and $a=4$, for example, these probabilities are, approximately, as given in the following table:

	Case 1	Case 2	Case 3
Probability that neither end projects	.1037	.0854	.0565
Probability that one end projects	.8014	.5726	.5649
Probability that both ends project	.0949	.3420	.3786

Numbers Equal to the Sum of the Digits of Their Nth Powers

By G. E. Raynor, Lehigh University

On page 457 of the first volume of Dickson's *History of the Theory of Numbers* one finds the following statement: "Moret-Blanc proved that 1, 8, 17, 18, 26, 27 are the only numbers equal to the sum of the digits of their cubes." The writer has amused himself by finding similar results for powers other than cubes. The following table gives the only numbers N which are equal to the sum of the digits of their nth powers, for values of n up to 8.

n	N
1	1, 2, 3, 4, 5, 6, 7, 8, 9
2	1, 9
3	1, 8, 17, 18, 26, 27
4	1, 7, 22, 25, 28, 36
5	1, 28, 35, 36, 46
6	1, 18, 45, 54, 64
7	1, 18, 27, 31, 34, 43, 53, 58, 68
8	1, 46, 54

We shall illustrate the method by which these results were obtained for the

case $n=4$. As is evident, our method is very crude. It may interest readers of the Monthly to find a more refined process for extending the above table.

Let m be the number of digits in N. Then

$$10^{m-1} \leqq N < 10^{m} \tag{1}$$

and hence $N^4 < 10^{4m}$. It follows that the number of digits in N^4 is not greater than $4m$ and hence that the sum of these digits is not greater than $9\times 4m = 36m$. Now, by the first part of (1) $36m$ will be less than N if

$$36m < 10^{m-1} \tag{2}$$

and this in turn will be satisfied if

$$100m < 10^{m-1}$$

or

$$m < 10^{m-3}. \tag{3}$$

But $10 > e^2$, where e is the base of the natural system of logarithms, and hence if

$$m < e^{2(m-3)} = 1 + 2(m-3) + \cdots \tag{4}$$

(2) and (3) will be satisfied. But (4) is satisfied if

$$m \leqq 1 + 2(m-3)$$

which reduces to

$$m \geqq 5.$$

Also, by inspection we see that (2) is satisfied for $m=4$. Hence, it follows that a number which is equal to the sum of the digits of its 4th power cannot have more than three digits.

However, a number of three digits can have in its 4th power not more than twelve digits which can add up to not more $12\times 9 = 108$. Hence $N \leqq 108$. But $110^4 = 146{,}410{,}000$ and it is evident that the number less than this whose digits have the greatest sum is 99,999,999, the sum of the digits being 72. Hence $N \leqq 72$. By a similar process we find that no number $\leqq 72$ can have a 4th power the sum of whose digits is more than 64. Now running down a table of fourth powers[1] from 64 we find that the numbers 1, 7, 22, 25, 28, 36 satisfy our condition and hence are the *only* numbers which are equal to the sum of the digits of their fourth powers.

In closing we wish to raise the following question. Given an arbitrary positive integer n, does there exist an integer N, other than unity, such that the sum of the digits of N^n is equal to N?

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[1] Glover's Tables of Applied Mathematics give the first eight powers of all integers from 1 to 100.

A Suggestion Regarding Foreign Languages in Mathematics

By E. T. Bell, California Institute of Technology

While following a mathematical discussion in which English, French, and German were used by various speakers, I was struck by the remarkably small number of common, non-technical words sufficient to lubricate the technical terms and to present a mathematical argument intelligibly. The common words include conjunctions, prepositions, personal pronouns, a few nouns, and a handful of verbs, particularly the auxiliaries. Some of the common verbs, like "put," or "take," of course are used with uncommon meanings, but these idioms could be included in the technical vocabulary. For an argument in analysis, algebra, or the theory of numbers, it would seem that 300 common words are more than sufficient; for geometry the number may be higher. For a philosophical discussion dealing, say, with mathematical logic, much more would be necessary. To the 300 or so common words must be added the technical vocabulary of the subject discussed.

It should be possible, by sampling pages of foreign mathematical books and periodicals to form a reasonable estimate of the number of *common* words necessary for a given branch of mathematics. If the guess of 300 is anywhere near the truth, it would be worth someone's time to compile such common vocabularies for the languages in which mathematics is alive. By concentrating on the minimum lists, supplemented by dictionaries if necessary, a working competence in two or three languages might readily be acquired by reading minutely two or three fairly long papers of which the mathematical subject matter was already familiar to the reader.

The suggestion that minimum lists be compiled is not an attempt to sabotage the usual teaching of French, German, and Italian. It is recognized that without a fair mastery of these languages, mathematical education soon comes to a halt. But what about Dutch, the Scandinavian languages, and Russian? Much Russian work in the theory of numbers, for example, is practically buried for years to most readers, and little of it ever receives adequate translation or review.

It is granted that such a hand to mouth method of gaining a mathematical reading knowledge of a foreign language would lack polish, to say nothing of syntax. But for those who are not born polyglots, some expedient is demanded. And it may be recalled that Sophus Lie, in spite of continued residence and lecturing in Germany, never did master the niceties of die, der, das. He even went so far as to compliment a rising young American mathematician, whose German was as efficient and ungrammatical as his own, on the clarity with which he expressed his ideas. A vocabulary is a necessity; syntax is a luxury (for mathematicians).

I believe that a study of this question, resulting in minimum common and technical lists, would be of value to students of mathematics. If the work were well done, the resulting pocket "dictionary" should find a publisher without difficulty. So there may be something in it, in more ways than one.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

College Algebra. By H. P. Pettit and P. Luteyn. New York, John Wiley & Sons, 1932. 276 pages, 2 tables. $1.90.

Practically all of the many college algebras that have been written during the past several years devote the early chapters of the text to a review of elementary algebra. A most interesting feature of this book is that it has broken away from this conventional practice and treatment. The authors believe "that it is a pedagogical mistake to start the course in college algebra with an extensive *review* of elementary algebra" because it is discouraging and fails to arouse interest in the minds of the students. Therefore in the first two chapters they introduce new material, namely, functions, graphs, and the summation convention, but they also incorporate a considerable amount of review of elementary algebra.

Apart from the arrangement, much the same material and treatment is found in this text as in other college algebras. Four separate chapters, III, VIII, XV, and XVIII, on problem solving only are included in the book. The reviewer is disappointed in finding that probability is omitted from this text. It not only naturally follows permutations and combinations, but it is a subject which generally appeals very much to the interest of the students. The binomial theorem follows permutations and combinations and is developed by means of combinations, although it is again treated later by mathematical induction. The method of completing the square in developing the formula for the solution of the general quadratic equation differs from that usually found in college algebras. It is also interesting to note that no particular quadratic equations are solved by the method of completing the square before the development of the formula for the general quadratic by this method. Probably the latter would be better understood by the student if he applied the method first to a few particular equations. The introduction of the summation convention is a valuable addition to college algebra. Relatively a great deal more space (40 pages) is devoted to integral and fractional exponents, radicals, and exponentials, than is usually found in college algebras. This is a noteworthy feature of the book, and the student should complete the course with a better than average knowledge of that particular phase of college algebra.

Apparently a special effort has been made by the authors to connect the subject with the mathematics which is to follow; and, in the reviewer's opinion, the strongest feature of the text is the inclusion of numerous problems which not only have practical value but are very closely related to the other sciences. There are many problems dealing with automobiles, airplanes, motorboats,

taxes, stocks, investments, rates, mechanics, heat, and electricity, which should be of great interest to the student. Problems related to physics and chemistry are very frequently introduced. Possibly a greater use of bold faced type to emphasize the important theorems, etc., would add value to the book.

On the whole it is well written, and the student using it as a text should not only get a good knowledge of college algebra but at the same time learn much about the physical sciences with which mathematics is most closely related.

E. D. WELLS

A Short Course in Trigonometry. By James G. Hardy. New York, The Macmillan Company, 1932. ix+181 pages: and the *Macmillan Tables*, revised edition, edited by E. R. Hedrick, 1930, xx+142 pages. $2.25

The arrangement of the material in this trigonometry differs from the usual sequence in that the chapter on logarithms and logarithmic solution of triangles is relegated to a subordinate position which, in this instance, is the last chapter. This, we believe, is greatly to be desired.

It is the opinion of the reviewer that the author has achieved his professed aim of making "clear why each topic is taken up" by the care with which they are introduced. It should interest the thoughtful student to see *in the book* the meaning of such terms as "co-function" and "double-angle" of formula fame. The instructor should welcome the insistence upon non-memorization of those results easily obtained by the application of a few simple fundamental concepts.

The book has been carefully written. This is evidenced by the discussion of such troublesome points—to the student—as why $-x$ does not necessarily mean that x is negative, and the meaning of the symbol $\tan 90° = \infty$.

A generous supply of exercises has been provided and some answers. A feature of the answers is that it is always indicated whether four or five place tables have been used in the computation. The tables are more numerous than those usually found in textbooks on trigonometry, for they include tables of powers, roots, reciprocals and many others, fourteen in all.

We believe the book to be eminently teachable and one that the student can read with understanding.

C. A. NELSON

Mathematics of Relativity (Lecture Notes). By G. Y. Rainich. Ann Arbor, Edwards Brothers, Inc., 1932. 67 pages.

Professor Rainich's book should prove welcome both to those who have occasion to give a course in Relativity and to those who are being initiated into the subject. This work, as the title indicates, is patterned after the lecture notes of the author. It is, however, complete as a presentation of the mathematics of Relativity in that it offers a background adequate for independent work in the subject.

The book is divided into five chapters. The main purpose of the opening chapter is to build up on the assumptions of classical physics the complete ten-

sor. Here the notion of tensor is used only to identify a group of associated physical quantities. At the close of the chapter the central question is raised: do the differential relations satisfied by the complete tensor determine the fundamental physical quantities involved? Toward answering this question the remainder of the book is devoted. The second chapter opens with some fundamental facts of four dimensional geometry followed by a formal presentation of its axiomatic basis leading to the author's own "vector argument" treatment of tensors. The author then considers the invariance of the equations of physics, the role of the imaginary fourth coordinate, and the representation of curves in a coordinate system moving with respect to a given system. The third chapter is devoted to Special Relativity (a partial answer to the question of Chapter 1), attention being given to generalized mass, the Lorentz transformations, addition of velocities, motion of photons, and the field of an electron in motion.

In the fourth chapter, which is given to curved space, the central role of the Riemann tensor in its identification with the complete tensor is brought out. The representation of tensors in general co-ordinates, and the physical significance of covariant and contravariant representations as well as differentiations are here treated. At the close of the chapter geodesics are defined in curved space. The final chapter opens with a clear and concise analysis of the task of General Relativity and proceeds to answer further the question put in the first chapter. It presents in closing the three important results of General Relativity which have been verified by experiment, namely the precession of the perihelion of a planet, the effect on the path of light of the gravitational field through which it passes, and the "Red Shift" in a Solar Spectrum.

This work shows considerable care in preparation, the subject matter being organized into an organic whole following a plan outlined in the introduction. The author is to be thanked for having added to his original contributions this exposition of the mathematics of Relativity, which in virtue of the arrangement and selection of its subject matter, and the clarity and conciseness of its treatment, is the most readable text on the subject that has come to the attention of the reviewer. It is to be hoped that these lithotyped notes will be succeeded in the next edition by a printed volume.

S. B. LITTAUER

Freshman Mathematics. By H. L. Slobin and W. E. Wilbur. New York, Ray Long and Richard R. Smith, Inc., 1932. xviii+438 pages. $2.60

This text covers in three distinct parts the subjects of college algebra, trigonometry, and analytic geometry. The space devoted to these is almost exactly in the ratio 2:1:2. This arrangement should appeal to those who

(i) desire a single text book for the entire year,
(ii) prefer a more or less traditional topical arrangement,
(iii) believe that calculus should be postponed until the second year.

On these points there is, of course, a considerable division of opinion. We do not mean to say that the authors have not achieved a certain degree of unifica-

tion. Thus graphing in rectangular coordinates is first introduced in the study of the linear function in algebra. It is continued in the chapter on the quadratic function and again in the solution of simultaneous equations. The trigonometric functions are graphed in the section on trigonometry. The work in conic sections is definitely tied up with the chapter in algebra on the quadratic function, although separated from it by some two hundred pages.

Nearly half of the section on algebra is a review of school algebra. This is followed by chapters on the usual array of disconnected topics, logarithms, progressions, permutations and combinations, probability, mathematical induction, partial fractions. The function concept is emphasized. Functions are graphed as a means of exhibiting their behavior. The authors do not find the zeros of the quadratic function so that they can draw its graph. Rather, the graph is used as a means of determining the nature and even the magnitude of the zeros. (Incidentally the authors use root for zero.) This is a distinct advance over many existing texts. Students of calculus who regard the first and second derivatives chiefly as aids in curve sketching can hardly be said to have grasped the real significance of the subject. Some readers will not like the expression, "the function does not cut the x-axis" (p. 59).

The part on trigonometry does not require any special comment. As usual the student is asked to prove certain identities. The reviewer thinks that the purpose of this work would be far better served if the problem were to simplify given trigonometric expressions. This would not only develop initiative on the part of the student but prepare him for what he will later be called on to do.

Perhaps the greatest novelty of treatment occurs in the part on analytic geometry. The reader will look in vain for a chapter on the ellipse starting with the usual definition. He will find instead that the ellipse has been defined in the algebra as the curve whose equation is so and so. The other conic sections are similarly defined. There is a chapter on the properties of the conic sections.

The authors have included a chapter on empirical curves. This is preceded by a chapter which discusses the graphs of the needed functions. The book closes with a 22 page chapter on Solid Analytic Geometry.

There are numerous problems and answers are given to nearly all of them. We have noted very few misprints. The authors are careless in the matter of significant figures. We find for example on page 136, $\log_e 763=x$, $763=e^x=2.71828^x$, $\log_{10} 763=x \log_{10} 2.71828$, $2.8825=x(.4343)$, $x=6.6370$. Thus in one calculation 4, 5, and 6 significant figures are used. (Actually $\log_e 763=6.6373$ to the nearest fourth decimal place.)

Your reviewer thinks this text book should prove satisfactory for a course covering the subjects which it treats. It is our personal opinion, however, that some of the material ordinarily covered in algebra and analytic geometry should be reserved for later courses in these subjects. This would make it possible to devote a part of the freshman year to calculus.

C. A. Shook

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–32

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Washington

Although we have not been a chapter of Pi Mu Epsilon very long, this being our first annual report, our organization is one of three years' standing, having been organized in the fall of 1929 by a group of graduate students in the Mathematics Department. The purpose of this group of students, who called themselves the Junior Mathematics Club to distinguish it from the Faculty Research Club, was to give the members opportunity to present papers on various phases of Mathematics in which they are interested, and to acquaint the other members with these various branches.

The plan in regard to topics presented has been one of permitting the speaker his choice of topic, and the result has been extremely interesting and instructive, a variety of subjects having been chosen and offered to the group. Meetings are held every other week, and the members take turns being responsible for the programs. Visitors are welcomed at these meetings, so every audience is composed of interested undergraduates together with the members.

The meetings and programs were as follows:

October 20, 1931: "Problems in the calculus of variations with discontinuous solutions" by Horace C. Ayres.

November 10, 1931: "Discussion of the brachistochrone by the calculus of variations" by John A. Carlson.

November 24, 1931: "Invariants of a quadratic by tensor algebra" by Philip N. Carpenter.

December 8, 1931: "D'Alembert's theorem" by Helen M. Copenhagen.

January 18, 1932: "Solutions of the equation $x^y=y^x$" by Mary E. Haller.

February 11, 1932: "Some particular applications of mathematics to economics" by Myra Jensen.

February 25, 1932: "Selections from the calculus of variations" by Kenneth Knox.

March 10, 1932: "Invariants of projective differential geometry" by Earl Rex.

March 31, 1932: "Fourier series in engineering" by Archie Adams.

April 20, 1932: "Cubic curves" by Thomas Doyle.

May 4, 1932: "Coordinate systems—generalized coordinates" by Ingomar Hostetter.

May 18, 1932: "The theory of the harmonic analyzer" by John Carlson.

June 2, 1932: "A problem in the calculus of variations" Master's thesis by Kenneth Knox.

The installation of this club as a chapter of Pi Mu Epsilon took place March 5, 1932, with Dr. Lee McFarlan who was formerly a member of the chapter at the University of Missouri acting as installing officer. After the installation ceremony, a banquet was held in honor of the occasion; Dr. R. E. Moritz, head of our department, and Dr. R. M. Winger, Professor of Mathematics, were our guests, the latter presenting a delightful talk on "Mathematics and Idealism."

This year thirteen new members have been elected, six being initiated on April 6, 1932, and the remaining seven on June 2, 1932. The total membership in school at present is twenty-one. All of the members are faculty members or graduate students in Mathematics, with a few undergraduates who are exceptionally capable in their work. The policy of the group has always been to stress the scholastic side of the fraternity, and to minimize details of the organization, only such time as is absolutely necessary for efficiently carrying out our program being spent on business.

The officers for the Spring Quarter of the year were: Kenneth E. Knox, Director; Mary E. Haller, Vice Director; Helen M. Copenhagen, Secretary and Treasurer. The officers are elected every quarter. On Decoration Day, the group joined with the undergraduate Mathematics Honorary in a picnic at one of the parks near Seattle.

HELEN M. COPENHAGEN, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club, Delta Chi, of the University of Toledo

This club was organized three years ago in order that the students might have an opportunity to consider various topics which, although closely related to the classroom work, are not generally included in the standard courses. Also it endeavors to create a desire for individual study and research. All students who have taken or are taking calculus are eligible for membership. This year there were forty-six members.

At the regular meetings, which are held the third Thursday of each month, a student discusses and demonstrates some phase of mathematics which is of interest to the entire group. At the close of the meetings refreshments are served.

Professor Wayne Dancer is Faculty Advisor for the club. The officers for the year were: Mary Krepleever, President; Fern Walker, Vice-President; Zora Powlesland, Secretary-Treasurer.

The meetings and programs were as follows:

September 26, 1931: Steak roast at Maumee Canal Side-Cut Park. This served as a reunion for the old members and a welcome for the new.

October 8, 1931: "Rhombic dodecahedrons" by Edwin Strutton.

November 12, 1931: "Mathematical puzzles and curiosities" by Professor Wayne Dancer.

December 18, 1931: Christmas party at the home of Professor J. B. Brandeberry. The program was: "Line coordinates" by Kenneth Rossman. We also enjoyed a mathematical treasure hunt and other games and the exchanging of gifts.

January 21, 1932: "The history of Pi" by Merrill Seps; Contest on necessary and sufficient condition.

February 25, 1932: "Continued fractions" by Edith Ein; Roll Call—each member answering with the name and accomplishments of some mathematician whose name began with the same letter as his own.

March 17, 1932: St. Patrick's Day Party (Open meeting). The program was: "Various proofs of the theorem of Pythagoras" by Jane Kamke; Banjo Solos by Kenneth Rossman; Mathematical contests.

April 21, 1932: "Singular points" by Enzia Parks.

May 20, 1932: "Configurations" by Professor Dancer; "Applications of the sine curve" by Professor Brandeberry; Third annual banquet at L'Aiglon; Installation of new officers.

FERN WALKER, *Vice President*

The Mathematics Club of Wellesley College

During the year 1931–1932 the Wellesley College Mathematics Club was organized under the direction of the following officers: Claudia E. Jessup, President; Emily A. Neal, Vice-President; Ruth S. Ball, Treasurer, Barbara Alden, Secretary, Dorothy Reinman, Junior Executive, and Professor Mabel M. Young, faculty advisor. The club officers for the following year are elected by

a majority vote at the last meeting of the club held in April or May. They are elected by the club members, and any undergraduate, except Freshmen, interested in mathematics may become a member simply by paying the club dues of seventy-five cents.

The officers are nominated in conference by the present club officers, but the President and Vice-President are always selected from members of the coming Senior class, and the Junior Executive from the coming Junior class. The faculty advisor to serve for the following year is formally asked by the out-going President, but the office succeeds each year to a different member of the faculty. There were seventy-two active members enrolled in the club for the year 1931–1932.

The aim of the club is to stimulate the interest of students for mathematics by providing entertainment through mathematical recreations, the reading of mathematical papers on selected topics, mathematical songs and so on. This year six meetings were held, the dates arranged by the committee on the social schedule of the college. The programs were as follows.

The first meeting, October nineteenth, was a "get-acquainted" meeting for the benefit of the new members. An ingenious arrangement of mathematical equations was worked out by which each person was given a slip of paper with a component of an analytic equation written on it, and he was told to find the group in which his component belonged. Then the members of their respective equations introduced themselves to the rest of the club members. Professor Helen A. Merrill gave a survey report of the history of the Mathematics Club, and mathematical songs were sung.

The second meeting was held November twentieth. Claudia Jessup opened the meeting with a numerical problem by which she could tell the age of any member present, all of whom were satisfactorily mystified. Topics were then given by the club officers. Barbara Alden spoke on the *Impossibility of Trisecting an Angle;* Emily Neal explained *Pascal's Triangle* and demonstrated the formation of magic squares, Ruth Ball proved very convincingly that one equals zero, and two equals one, Dorothy Reinman entertained the club with a varied repertoire of mathematical puzzles, among which she proved that anyone's age was equal to Methuselah's, and that the diagonal of a one-hundred foot square was two hundred feet.

The third meeting of the club was held January twenty-second when a demonstration of some new mathematical models recently acquired by the Mathematics Department was conducted by members of the faculty. The models were made of strings to form ruled surfaces of the second order. Curves were made visible by a lantern which showed the intersection of other curves on lantern slides with the string models. The demonstration was interesting as well as instructive.

For the fourth and fifth meetings, February nineteenth and March eighteenth, a Mathematics Club contest was announced in which the club members could compete for excellence in giving certain topics selected by the club officers. A prize of a book entitled *Mathematical Wrinkles* was given to the winner.

At the first of the two meetings Edythe Fairbanks spoke on *Spirals in Art and Nature;* Constance Wall on *Infinity in Geometry* and Elizabeth Richardson on *Maria Agnesi and the Witch Curve Named After Her.* The club members voted in closed ballots for the speaker they enjoyed most, and judged on the four points: poise, clearness and correlation, the topic's interest in general, and its interest for mathematics students.

At the second meeting topics were given by Ann Dunham on *Lewis Carroll;* Mary Dean Clement on *Familiar Curves in Architecture* and Persis Bullard on *Modern Plane Geometry and Some of Its Properties.* The prize as announced at the last meeting of the club—the supper meeting by tradition—was awarded to Elizabeth Richardson.

The supper meeting was held April fifteenth and supper was served by the club officers. Then the officers for the year 1932–1933 were elected.

At most of our meetings we found that the singing of mathematical songs provided amusement and created an informal atmosphere which was very pleasant. Professor Merrill has made up several songs of her own and presented them to the Club, and one of the Mathematics Club

members, Barbara Trask, collaborated with Miss Merrill in making up the air and melody for a song which was adapted to words from a poem by Arthur Guitterman.

We have had a large attendance this year, an average of over half our members being present at the meetings.

BARBARA ALDEN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 36. *Proposed by B. H. Brown, Dartmouth College.*

Show that the thirteenth of the month is more likely to be Friday than any one of the other days of the week.

E 37. *Proposed by Arthur Haas, Thomas Jefferson High School, N. Y.*

In the following multiplication of a three-place number by a two-place number, all but three of the sixteen digits have been replaced by the letter x. Reconstruct the problem and show that the solution is unique.

$$
\begin{array}{rrrrr}
 & & x & x & x \\
 & & & x & x \\
\hline
 & & x & x & x \\
 & x & x & 4 & \\
\hline
x & x & x & 0 & 1
\end{array}
$$

E 38. *Proposed by J. R. Musselman, Western Reserve University.*

It is well known that the midpoints of the sides of any plane quadrilateral constitute the vertices of a parallelogram. Determine the most general conditions under which the parallelogram becomes (a) a rhombus, (b) a rectangle, and (c) a square.

E 39. *Proposed by W. R. Ransom, Tufts College.*

Obtain both roots of the equation

$$x^2 - 365.04x + 45.04 = 0$$

correct to four significant digits, by means of four-place logarithm tables.

E 40. *Proposed by Maud Willey, Long Beach, Mississippi.*

Let $C_i=0$, $(i=1, 2, 3)$ be the equations of three circles. Prove that the three circles, $\sum_{i=1}^{3}K_{ij}C_i=0$, $(j=1, 2, 3)$, have the same radical center as the three circles $C_i=0$.

Prove that the four spheres, $\sum_{i=1}^{4}K_{ij}S_i=0$, $(j=1, 2, 3, 4)$, have the same radical center as the four spheres whose equations are $S_i=0$, $(i=1, 2, 3, 4)$.

Prove that the n hyperspheres, $\sum_{i=1}^{n}K_{ij}S_i=0$, $(j=1, 2, \cdots, n)$ in space of $n-1$ dimensions, have the same radical center as the n hyperspheres $S_i=0$, $(i=1, 2, \cdots, n)$.

E 41. *Proposed by V. F. Ivanoff, San Francisco.*

A variable circular arc of constant length, l, has one end fixed in position and direction. Find the locus of its other end.

SOLUTIONS

E 16. [1933, 51] *Proposed by G. A. Yanosik, New York University.*

Prove that the envelope of the circles whose diameters run from points on a parabola to its focus, is the straight line tangent to the parabola at its vertex.

I. *Solution by Mannis Charosh, New Utrecht High School, Brooklyn, N.Y.*

Let the equation of the parabola be $y^2=8ax$. The coordinates of the focus F are $(2a, 0)$. The problem is now equivalent to proving that each of the given circles is tangent to the y-axis.

If the point P $(2b, 2c)$ is any point on the parabola, then $c^2=4ab$. The coordinates of M, the midpoint of FP, are $(a+b, c)$. The equation of the circle centered at M and with radius FM is therefore

$$\begin{aligned}[x-(a+b)]^2+[y-c]^2 &= [a-b]^2+c^2\\ &= [a-b]^2+4ab\\ &= [a+b]^2.\end{aligned}$$

If $x=0$, the resulting equation has the double root, $y=c$, so that this circle is tangent to the y-axis for each point P on the parabola.

Since each circle goes through the focus F, it too is part of the envelope.

II. *Solution by Simon Vatriquant, L'Athénée Royale D'Ixelles, Bruxelles.*

Let F be the focus of the parabola, V the vertex, and P any point on the curve. Let the tangents at P and V meet at T, which is thus the pole of VP. Then the diameter through T bisects VP, and also bisects FP at M (since it is parallel to the base VF of the triangle PVF). Since the projection of the focus of any parabola upon any tangent to that parabola, is a point on the tangent at the vertex, we know that FT is perpendicular to TP. Then the circle on the diameter FP is centered at M, goes through T, and is tangent to VT, since VM is a diameter of the parabola. Since the tangent to the parabola at V is thus

tangent to every one of the designated family of circles, it forms an envelope of that family.

Also solved by Theodore Lindquist, C. C. Richtmeyer, J. E. Thompson, R. N. Walter, and the proposer.

E 17. [1933, 51] *Proposed by Morgan Ward, California Institute of Technology.*

In Bierens de Haan's "Nouvelles Tables d'Intégrales Définies," Table 113, formula 1, page 162, it is stated that

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} = \frac{2\pi^2}{27}.$$

Show that this result is incorrect.

Solution by the proposer.

Since

$$\frac{1}{1+x+x^2} = 1 - x + \frac{x^3}{1+x+x^2},$$

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} = -\int_0^1 (1-x)\log x dx - \int_0^1 \frac{x^3 \log x dx}{1+x+x^2}$$

$$> -\int_0^1 (1-x)\log x dx,$$

because $x^3 \log x/(1+x+x^2)$ is negative in the range of integration considered. Therefore

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} > -\int_0^1 \log x dx + \int_0^1 x \log x dx = 1 - \tfrac{1}{4} = .75$$

while $2\pi^2/27 < .74$.

Mr. H. S. Uhler of Yale University points out that there was published many years ago a monograph entitled

Kongl. Svenska Vetenskaps-Akademiens Handlingar.
Bandet 24 N:o 5.
Examen des Nouvelles Tables d'Intégrales Définies
de
M. Bierens de Haan
Amsterdam 1867.
par
C. F. Lindman.

and in this monograph, on page 61, there appears

"Tab. 113.
1 à 4 sont fautives. Lisez

$$1. \qquad \int_0^1 \frac{\log x dx}{1 + x + x^2} = -\frac{8}{27\sqrt{3}} H(\tfrac{1}{3}) = -0.7813024129."$$

Also solved by D. C. Duncan, Raymond Garver and J. B. Meyer.

E 19. [1933, 51] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Find the radius of a sphere whose surface area and volume are each numerically equal to π times a four-place integer, and show that the solution is unique.

Solution by Raymond Garver, University of California at Los Angeles.

Since $4R^2$ and $4R^3/3$ are each integers, R is obviously rational, with a denominator of 1 or 2, since $4R^2$ is an integer. Since $4R^3/3$ is also an integer, R is an integer divisible by 3. Since $999 < 4R^2$, $15 < R$. Since $4R^3/3 < 10000$, $R < 20$. But the only integer multiple of 3 between 15 and 20 is 18, so $R = 18$. For $R = 18$, the surface area is 1296π and the volume is 7776π, which in each case is a four-place integer times π.

Also solved by W. R. Ransom, Simon Vatriquant, E. E. Whitford and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3612. *Proposed by H. Grossman, New York.*

Prove that the ratio of the sum of the hth, $(h+k)$th, $(h+2k)$th, etc. coefficients to the sum of all the coefficients in the expansion of $(a+b)^n$ converges to the limit $1/k$, as n approaches ∞, where $h = 1, 2, 3, \cdots, k$; and k is a positive integer.

3613. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

Prove that

$$(R_x \cos\theta)^{2/3} + (R_y \cos\phi)^{2/3} + (R_z \cos\psi)^{2/3} = 2R^{2/3},$$

where R is the radius of curvature of a given curve at the given point (x_1, y_1, z_1); R_x, R_y, R_z are the radii of curvature of the projections of this curve on the coordinate planes YOZ, XOZ and XOY at the points (y_1, z_1), (x_1, z_1) and (x_1, y_1),

respectively; and θ, ϕ and ψ are the angles between the binormal to the curve and the coordinate axes.

3614. *Proposed by Leverett Davis, Jr., Seattle, Washington.*

If

$$f(x+y)=f(x)[1-f(y)^2]^{1/2}+f(y)[1-f(x)^2]^{1/2},$$

what conditions must be satisfied so that $f(x)$ will be the ordinary sine function? In other words, under what conditions will the law of addition define the sine?

3615. *Proposed by B. D. Roberts, New Mexico Normal University.*

A dust storm contains particles of two kinds identical except as to color, brown and yellow particles existing in the ratio of 3:2. If five particles of this dust enter my eye at random, determine the probability that two of them are brown and the other three are yellow.

SOLUTIONS

3340 [1928, 378]. *Proposed by the late V. M. Spunar, Chicago, Illinois.*

Solve

$$\begin{vmatrix} x^2dx, & y^2dy, & z^2dz \\ dx, & dy, & dz \\ x, & y, & z \end{vmatrix} = 0$$

consistent with the equation $xyz=1$.

A Note by Otto Dunkel. The differential equation of the problem determines the lines of curvature of the given surface. For at a point P (x, y, z) of the surface the vector (x^{-1}, y^{-1}, z^{-1}) is normal to the surface; and at a neighboring point P' in the direction (dx, dy, dz) the vector $(x^{-1}-x^{-2}dx,\ y^{-1}-y^{-2}dy,\ z^{-1}-z^{-2}dz)$ is also normal to the surface, except for terms of higher order. The condition that the two normals ultimately intersect is the vanishing of the triple scalar product of the above three vectors, or, what is the same thing, the vanishing of the determinant formed from the components of the three vectors. The resulting equation reduces to that of the problem.

This problem is proposed in *A treatise on differential equations*, Forsyth, 2nd. ed. 1888 (Macmillan), p. 265, ex. 4. A solution is given in the translation into German by Maser, *Lehrbuch der Differential-Gleichungen*, 1889 (Vieweg), p. 626. See also *Solutions of examples in a treatise on differential equations*, Forsyth, 1918 (Macmillan), p. 117.

The last reference states that the differential equation is that of the lines of curvature of the surface $xyz=1$; and that the integral to be associated with the given equation of the surface is

$$(x^2+\omega y^2+\omega^2z^2)^{3/2}+(x^2+\omega^2y^2+\omega z^2)^{3/2}=A. \tag{1}$$

This may be verified in an indirect manner as follows: Set the first parenthesis

in (1) equal to U, and the second equal to $\bar{U}$. Since ω and ω^2 are the imaginary cube roots of unity, U and $\bar{U}$ are conjugate imaginaries when x, y, z are real; therefore A is a real constant. Set

$$
\begin{aligned}
X_1 &= U^{1/2} + \bar{U}^{1/2} & X_2 &= U^{1/2} - \bar{U}^{1/2}, \\
Y_1 &= \omega U^{1/2} + \omega^2 \bar{U}^{1/2}, & Y_2 &= \omega U^{1/2} - \omega^2 \bar{U}^{1/2}, \\
Z_1 &= \omega^2 U^{1/2} + \omega \bar{U}^{1/2}, & Z_2 &= \omega^2 U^{1/2} - \omega \bar{U}^{1/2}.
\end{aligned} \tag{2}
$$

Consider the three surfaces

$$
U^{3/2} + \bar{U}^{3/2} \equiv X_1Y_1Z_1 = A, \quad U^{3/2} - \bar{U}^{3/2} \equiv X_2Y_2Z_2 = B, \quad xyz = C, \tag{3}
$$

where A, C are real constants and B is a pure imaginary constant. At a point common to these three surfaces the three vectors with the components

$$
\begin{matrix}
xX_1, & yY_1, & zZ_1, \\
ixX_2, & iyY_2, & izZ_2, \\
yz, & xz, & xy,
\end{matrix} \tag{4}
$$

are normal to the respective surfaces. It may be more convenient to obtain the first two sets from the U form of the equations of the surfaces. These vectors form a mutually orthogonal system. For in the case of the first and third, the scalar product is $xyz(X_1+Y_1+Z_1)$, and this last factor is clearly zero, since $1+\omega+\omega^2=0$ and (2) shows then that the parenthesis is zero. A similar reason applies to the second and third. For the first and second, we have for the scalar product, neglecting the factor i,

$$
\begin{aligned}
&x^2X_1X_2 + y^2Y_1Y_2 + z^2Z_1Z_2 = \\
&\qquad x^2(U - \bar{U}) + y^2(\omega^2 U - \omega\bar{U}) + z^2(\omega U - \omega^2\bar{U}) = \\
&\qquad (\omega - \omega^2)[x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)] = 0.
\end{aligned}
$$

Hence any pair of these surfaces cut the third in its lines of curvature. Thus the first two in conjunction with the third are solutions of the differential equation of the problem, where C is there taken as unity.

3513. [1931, 461]. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$
f(s, p) = \sum_{k=1}^{p}(-1)^{p+k}(2k-1)_sH_{p-k}
$$

where

$$
{}_sH_r \equiv \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad {}_sH_0 \equiv 1.
$$

Prove: (a) If p is odd, then $f(s, p)$ is positive for all values of s;

(b) If p is even, then $f(s, p)$ is positive, equal to zero, or negative according as s is less than, equal to, or greater than 3.

II. *Solution by Frank Ayres, Jr., Dickinson College.*

It will be shown in this solution that the statement of the problem is true for all real values of s. It has already been shown in a former solution [1933, 117] that it is true for all real values of s except in the following intervals in which the sign of $f(s, p)$ was not then determined:

$$(1) \qquad p = 2m + 1, \quad 4 - 2j < s < 5 - 2j, \quad j = 1, 2, \cdots, m,$$

$$(2) \qquad p = 2m, \qquad 3 - 2j < s < 4 - 2j, \quad j = 1, 2, \cdots, m - 1.$$

It will now be shown that $f(s, p)$ is positive in these intervals. In the previous solution it was shown that

$$(3) \qquad f(s, p) = (-1)^{p-1} \sum_{k=1}^{p} (-1)^{k-1} k_{s-1} H_{p-k}.$$

If the terms with positive sign in the summation (except the last for p odd) are transformed by the relation $r\ {}_sH_r = (s+r-1)\,{}_sH_{r-1}$ there results for p odd

$$(4) \qquad f(s, p) = p + (-1)^{p-1} \sum_{k=1}^{q-1} \frac{(2k-1)(s-1) - p}{p - 2k + 1}\,{}_{s-1}H_{p-2k},$$

where q is the greatest integer in $(p+1)/2$. The expression (4) shows that $f(s, p)$ is positive in the intervals of (1) for which $j = 2, 3, \cdots, m$. The case $j = 1$ will be considered later.

For p even the first term p in (4) is dropped and the upper limit $q-1$ is replaced by q; and the resulting expression shows that $f(s, p)$ is positive in all the intervals of (2).

We shall now use the development of the previous solution

$$(5) \qquad f(s, p) = (-1)^{p-1} \sum_{k=1}^{q} k_{s-3} H_{p-2k+1}.$$

Then

$$(6) \qquad f(s, p + 2) = f(s, p) + (-1)^{p-1} \sum_{k=1}^{q+1} {}_{s-3}H_{p-2k+3},$$

as is easily seen by combining like terms on the right in the two developments. It will be shown that the second term on the right is positive for p odd and $2 < s < 3$. Since $f(s, 1) = 1$, it will then follow by repeated use of (6) that $f(s, p)$ is positive for all odd values of p and for $2 < s < 3$. We have

$$(7) \qquad \sum_{k=1}^{q+1} {}_{s-3}H_{p-2k+3} = \sum_{k=1}^{p+2} (-1)^{k+1}\,{}_{s-2}H_{p-k+2},$$

by making use of the relation ${}_{s-3}H_n = {}_{s-2}H_n - {}_{s-2}H_{n-1}$. In the summation on

the right the terms with the minus sign are now transformed by means of $r\ {}_sH_r=(s+r-1)_sH_{r-1}$ and there results the equivalent expression

$$ {}_{s-2}H_{p+1}+(3-s)\sum_{k=1}^{q}\frac{{}_{s-2}H_{p-2k+1}}{p-2k+2}\,. \tag{8}$$

This last expression is easily seen to be positive for $2<s<3$, and the proof is now complete.

A Note by Otto Dunkel. This proof consists of two parts both of which are interesting, but it appears that the second part concerning the difference formula with the unit difference two (6) would suffice. For it is easily seen that (6) is true for p odd or even. Moreover the last term on the right in this formula is positive for the intervals (1) and (2) for integral values of j from 1 to ∞. Hence $\Delta f(s, p)$ is positive in the respective intervals; and, since $f(s, 1)$ and $f(s, 2)$ are positive in the same intervals, it follows that $f(s, p)$ is also positive in the same intervals.

3522. [1932, 45]. *Proposed by A. Galbraith, Ash Grove, Mo.*

To trisect any angle, approximately, with straight edge and compasses only take the vertex of the given angle as a center and with any radius describe the arc AC. Divide the chord AC into six equal parts, the middle point being B and the other two points between B and C being F and G in order B, F, G, C. On AC produced, locate the points H and K such that $CH=\frac{1}{6}AC$ and $HK=\frac{1}{2}CH$. At F erect a perpendicular to AC intersecting the arc AC in E. With G as a center and a radius EG, describe an arc intersecting the chord AC in M. With K as a center and radius EK describe an arc intersecting AC in N. Locate point P, the mid-point of MN. With E as a center and radius PH, describe an arc intersecting AC in L. With L as a center and a radius EL describe an arc intersecting AC in O. Then CO will be the chord of the trisected arc AC.

A Note by the Editors: This is the most accurate method of trisecting any angle approximately by means of a straight edge and compass only that has ever come under our observation.

We are publishing it for the double purpose of acquainting our readers with this rather complex though very accurate method and to have some of them determine for what angles the method is absolutely exact. We suggest that those who are interested, take a ruler and compasses and follow out the construction. An ocular proof is thus seen to be very convincing to the non-mathematical mind.

A Note by Otto Dunkel. The point L is located on the segment CK rather than on AC. The direct computation of the angle subtended by the chord CO is quite laborious, and the ordinary five place logarithm tables may not be sufficiently accurate to determine the error. Also an exact expression for the error ϵ in terms of the trisected angle θ is not easily obtained. An approximate expression for the error may be obtained by series developments which gives a fairly

good idea of the accuracy of the method. The first two non-vanishing terms are given by

$$\epsilon = \frac{8 \sin^5 \frac{1}{2}\theta}{45927 \cos \theta/6}\left[17 - \frac{40955}{3969} \sin^2 \tfrac{1}{2}\theta \cdots \right], \tag{1}$$

where ϵ is the excess of $\theta/3$ over the approximate trisection expressed in radians. For $\theta = 90°$, (1) gives $\epsilon = 1'\,17.9''$, whereas the direct computation gives $1'\,22''$. For $\theta = 45°$, (1) gives $\epsilon = 4.67''$, whereas the direct computation gives $4.8''$. For very small angles, an approximate expression is

$$\epsilon = \frac{\theta^5}{10806.+} \cdot \tag{2}$$

For angles not greater than 45° a simpler and more accurate method is as follows:

If BAC is the angle to be trisected, describe a circle with the vertex A as center and with any convenient radius, AC, cutting AB in the diameter DB. Produce AD to N and M so that $AN = 2AC$ and $AM = DC$. Determine the point P so that it divides the segment MN in the ratio 5:4. Let PC cut the circle again in Q: determine on AN the point P' so that $QP' = QA$. Then the angle $AP'C$ is approximately one third the angle BAC.

The first part of the construction giving the approximate trisection angle APC was given by Wedderburn in problem 2972 [1922, 224], while the rest was described by Kennedy in his paper, *Angle Division*, in this Monthly [1932, 478]. The approximation exceeds the exact trisection, and the excess for very small angles is approximately

$$\epsilon = \frac{\theta^7}{47239.2} \cdot \tag{3}$$

For angles near 90° this method may not be as accurate as that of the problem, since then the operation for determining P' is not so effective in reducing the error of the first part. For $\theta = 90°$ this operation reduces the error of the first part by only $\frac{1}{2}$: thus the error by the first part is $6'\,30.6''$, and the last step reduces the error to $3'\,15.3''$. However, it is not necessary to consider angles between 45° and 90°. For $\theta = 45°$, the error by the complete method is about $0.94''$, and for very small angles (3) shows that it is much more accurate than the problem method. Moreover, this method has the additional advantage that it can be made of any degree of accuracy by repeating the process for finding P'.

A construction of about the same order of accuracy as that of the problem is as follows:

If the angle to be trisected is BAC, take any convenient length AC on that side and produce BA to D and F so that $AD = DF = AC$. Draw DC and take on this segment $DE = DF$. On AF lay off $AM = DC$ and $AN = FE$. On MN take

the point P so that $MP:PN=20:7$; then angle APC is approximately one third of angle BAC.

For $\theta=45°$ the error is about $4''$: for angles not greater than $45°$ the error is given approximately by

$$\epsilon = 67.775 \sin \theta/3 \sin^4 \theta/24. \tag{4}$$

For very small angles

$$\epsilon = \frac{\theta^5}{15996.+}. \tag{5}$$

The approximation is too small. The error is likely to be less than the result given by (4).

Approximate methods for trisecting an angle, or for finding any part of a given angle, may be obtained in a variety of ways. One method is to determine the constants in

$$\tan \delta = \frac{a \sin \theta + b \sin \frac{1}{2}\theta}{c \cos \theta + d \cos \frac{1}{2}\theta + e}, \tag{6}$$

so that for very small angles δ is very nearly $\theta/3$. It will be seen at a glance that we must have $2(c+d+e)=6a+3b$, and this makes the error of the third order in θ. If greater accuracy is desired, other conditions are imposed so that the error is of higher order in θ with as small a coefficient of the principal part of the error as possible. The resulting formula for very accurate constructions may not turn out to be very convenient for construction, as may be observed in the last entry of the table below. In this table the first three rows give very simple constructions, but they are not nearly so accurate as those given by the lower rows.

	a	b	c	d	e
1	1	0	1	0	2
2	1	0	1	2	0
3	1	0	1	1	1
4	9	0	9	8	10
5	1	16	1	16	10
6	7	40	7	44	30

The first four in the table have been mentioned in this MONTHLY. The errors for very small angles for the last two are approximately

$$\epsilon_5 = \frac{\theta^5}{34992}, \quad \epsilon_6 = \frac{\theta^7}{2519424}.$$

The method may be varied by introducing sines in the numerator and cosines in the denominator of $\theta/2^2$, $\theta/2^3$, etc. The construction above to which (4) applies will be found to employ $\theta/4$.

3541. [1932, 175]. *Proposed by J. B. Reynolds, Lehigh University.*

A rod of length $2a$ standing on a rough level plane of coefficient of friction μ falls from a vertical position. At what angle with the vertical will the rod begin to slip at the bottom?

Solution by V. F. Ivanoff, San Francisco, California.

The equations of the motion of the rod before it has begun to slip, are:

$$Mg\cos\theta - R_n = Ma\omega^2, \tag{1}$$

$$Mg\sin\theta - R_t = Ma\alpha, \tag{2}$$

$$Mg\sin\theta\cdot a = I\alpha, \tag{3}$$

where

M is the mass of the rod;

$I=(4/3(Ma^2$, the moment of inertia of the rod about its center of rotation;

α, the angular acceleration;

ω, the angular velocity;

θ, the angle between the rod and vertical;

R_t, the reaction normal to the rod; and

R_n, the reaction directed along the rod.

There is a relation between θ, ω, and α, namely: $d\theta/d\omega=\omega/\alpha$; whence, using (3) we get, $I\omega d\omega=aMg\sin\theta d\theta$, and taking into account that when $\theta=0$, ω also is 0, we get by integration,

$$\omega^2 = 2aMg(1-\cos\theta)/I = 3g(1-\cos\theta)/2a. \tag{4}$$

Now, from (1), (2), (3), and (4) we have

$$R_n = Mg\cos\theta - Ma\omega^2 = \tfrac{1}{2}Mg(5\cos\theta - 3)$$

and

$$R_t = Mg\sin\theta - Ma\alpha = \tfrac{1}{4}Mg\sin\theta.$$

The vector sum of these reactions has the components in the vertical and horizontal directions as follows:

$$R_v = R_t\sin\theta + R_n\cos\theta = \tfrac{1}{4}Mg(9\cos^2\theta - 6\cos\theta + 1)$$

$$R_h = R_t\cos\theta - R_n\sin\theta = \tfrac{3}{4}Mg\sin\theta(2 - 3\cos\theta).$$

The rod begins to slip when the horizontal force R_h becomes equal to μR_v, taken with the same sign as R_h. If $\theta>\cos^{-1}\frac{2}{3}$, R_h is positive and we have

$$\mu(9\cos^2\theta - 6\cos\theta + 1) = 3\sin\theta(2 - 3\cos\theta):$$

whence

$$\mu = \frac{3 \sin\theta(2 - 3\cos\theta)}{-3\cos\theta(2 - 3\cos\theta) - 1};$$

that is, μ is negative, which is inadmissible.

Again, if $\theta < \cos^{-1} \frac{2}{3}$, R_h is negative and we have

$$\mu(9\cos^2\theta - 6\cos\theta + 1) = -3\sin\theta(2 - 3\cos\theta),$$

or

$$3(\mu\cos\theta - \sin\theta)(2 - 3\cos\theta) = \mu.$$

The solution of this equation (quartic in trigonometric functions) gives the desired angle.

Also solved by William Hoover.

A Note by the Proposer. The last equation above is of the form $f(\theta) = \mu$. In order for this equation to be satisfied we must have

$$\tan^{-1}\mu < \theta < \cos^{-1}\tfrac{2}{3},$$

since

$$f(\theta) = 3\cos\theta(\mu - \tan\theta)(2 - 3\cos\theta) = \mu$$

can be satisfied only if both parentheses are negative. This condition will always obtain since $\tan(\cos^{-1}\frac{2}{3}) = 5^{1/2}/2 > 1 > \mu$.

3559. [1932, 359]. *Proposed by G. A. Yanosik, New York University.*

Variable circles are drawn having any point on a central conic and one of its foci as ends of a diameter. Prove that the envelope of these circles is the auxiliary circle.

Solution by E. F. Allen, Stillwater, Oklahoma.

Assume the equation of the conic to be

$$\frac{x^2}{a} + \frac{y^2}{a(1 - e^2)} = 1, \text{ where } a > 0.$$

The equation of the circle upon the line joining the focus, $F(e\sqrt{a}, 0)$ and the point $P(x, y)$ of the conic, as a diameter is

$$\xi^2 + \eta^2 - \xi(x + \sqrt{a}e) + \sqrt{a}ex - \eta y = 0,$$

where x is the parameter and y is a function of x, obtained from the equation of the conic. The envelope of this family of circles, obtained in the usual way, is represented by the equation

$$(\xi^2 + \eta^2 - \xi\sqrt{a}e)^2 - [a(\xi - \sqrt{a}e)^2 + a(1 - e^2)\eta^2] = 0.$$

The left member of this equation being expanded and then factored gives

$$(\xi^2 + \eta^2 - a)(\xi^2 - 2\sqrt{a}e\xi + ae^2 + \eta^2) = 0.$$

Hence, we have

$$\xi^2 + \eta^2 - a = 0 \text{ and } \xi^2 - 2\sqrt{a}e\xi + ae^2 + \eta^2 = 0,$$

the first being the equation of the auxiliary circle and the second the equation of the focus, which was used as the end of the diameter.

Also solved by W. B. Campbell, A. Pelletier, C. A. Rupp, F. Underwood, Paul Wernicke and the proposer.

A Note by Otto Dunkel. If Q and Q' are two neighboring points on the curve (here an ellipse), the two circles with diameters FQ and FQ' intersect in P' so that FP' is perpendicular to the chord QQ' of the curve. Hence the point of contact P of the circle FQ with its envelope is the foot of the perpendicular from F to the tangent at Q to the curve. The envelope is thus the pedal of the curve with respect to F.

Let O and F' be the center and the other focus of the ellipse, and let $F'Q$ produced meet FP produced in G. Since for the ellipse the tangent QP bisects the angle FQG, we have $F'G = F'Q + FQ = 2a$. It now follows that $OP = a$, and hence the envelope is the auxiliary circle of the ellipse. A similar analysis applies to the hyperbola and to the parabola. See the note [1933, 58] to the solution of 3535 for the converse theorems.

3560. [1932, 359]. *Proposed by Frank Morley, Johns Hopkins University.*

In a Euclidean space, perpendiculars from the vertices of a regular tetrahedron to a plane meet the plane at the points represented by the complex numbers x_i, $i = 1, 2, 3, 4$. Show that the four points obey the relation

$$\sum^{6}(x_i - x_j)^2 = 0.$$

Solution by F. Underwood, University College, Nottingham.

In a regular tetrahedron $ABCD$, pairs of opposite edges (such as AB, CD) may be taken as diagonals of opposite faces of a cube which is the enveloping (or surrounding) parallelepiped for the tetrahedron.

If the edge of this cube is $2a$, by taking rectangular axes through its centre (each axis being perpendicular to two faces), the co-ordinates of the vertices of the tetrahedron can be taken as

$$A(a, -a, a); \; B(-a, a, a); \; C(a, a, -a); \; D(-a, -a, -a).$$

If A_1, B_1, C_1, D_1, are the feet of the perpendiculars from A, B, C, D respectively to the plane $lx + my + nz = p$, where l, m, n are actual direction-cosines, we find

$$\begin{aligned} A_1 \text{ is } \{&a + lp - al(l - m + n), \; -a + mp - am(l - m + n), \\ &a + np - an(l - m + n)\}; \end{aligned}$$

B_1 is $\{-a+lp-al(-l+m+n),\ a+mp-am(-l+m+n),$
$a+np-an(-l+m+n)\}$;
C_1 is $\{a+lp-al(l+m-n),\ a+mp-am(l+m-n),$
$-a+np-an(l+m-n)\}$;
D_1 is $\{-a+lp+al(l+m+n),\ -a+mp+am(l+m+n),$
$-a+np+an(l+m+n)\}$.

Taking a new origin O_1 at (lp, mp, np), the foot of the perpendicular from O to the plane, and rectangular axes O_1Y, O_1Z in this plane, we can write

$$Y = \lambda_1(x-lp)+\mu_1(y-mp)+\nu_1(z-np),$$
$$Z = \lambda_2(x-lp)+\mu_2(y-mp)+\nu_2(z-np),$$

where λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 are subject to the restrictions

$$l\lambda_1+m\mu_1+n\nu_1 = l\lambda_2+m\mu_2+n\nu_2 = 0$$
$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2 = 0$$
$$\lambda_1^2+\mu_1^2+\nu_1^2 = \lambda_2^2+\mu_2^2+\nu_2^2 = 1,$$

but are otherwise arbitrary.
Then

$$\frac{l}{\mu_1\nu_2-\mu_2\nu_1} = \frac{m}{\nu_1\lambda_2-\nu_2\lambda_1} = \frac{n}{\lambda_1\mu_2-\lambda_2\mu_1} = 1.$$

To simplify the algebra in the remaining work it is convenient to take $\nu_1=\lambda_2=0$. Then $l=\mu_1\nu_2$, $m=-\lambda_1\nu_2$, $n=\lambda_1\mu_2$. Also $\mu_1\mu_2=0$; $\lambda_1^2+\mu_1^2=\mu_2^2+\nu_2^2=1$. The complex numbers for the points A_1, B_1, C_1, D_1 can be taken as $x_n=Y_n+iZ_n$, and if the suffixes 1, 2, 3, 4 are used in this order, it is easily found that

$$x_1/a = \lambda_1-\mu_1+i(\nu_2-\mu_2).$$
$$x_2/a = \mu_1-\lambda_1+i(\mu_2+\nu_2).$$
$$x_3/a = \lambda_1+\mu_1+i(\mu_2-\nu_2).$$
$$x_4/a = -\lambda_1-\mu_1-i(\mu_2+\nu_2).$$

Hence if the required sum is $S=\sum^6(x_i-x_j)^2$,

$$\begin{aligned} S/(4a^2) &= \{(\lambda_1-\mu_1)-i\mu_2\}^2+\{\mu_1+i(\mu_2-\nu_2)\}^2+(\lambda_1+i\nu_2)^2 \\ &\quad +(-\lambda_1+i\nu_2)^2+\{\mu_1+i(\mu_2+\nu_2)\}^2+\{(\lambda_1+\mu_1)+i\mu_2\}^2 \\ &= 4(\lambda_1^2+\mu_1^2-\mu_2^2-\nu_2^2+i\mu_1\mu_2) = 0. \end{aligned}$$

Also solved by J. M. Feld and the proposer.

A Note by Otto Dunkel. The proof may be put in the following form: Let X', Y', Z' denote the system of coordinate axes as above. The projections of the four points on any one of the coordinate planes, say $X'Y'$, form a square of

side $2a$; and, if its vertices be denoted by $X_j' + Y_j' i$, it is verified by inspection of the sides and diagonals of the square that the theorem is true for these special cases. Hence in this case we have

$$(1)\quad \sum^{6}(X_j' - X_k')^2 = \sum^{6}(Y_j' - Y_k')^2 = 16a^2, \quad \sum^{6}(X_j' - X_k')(Y_j' - Y_k') = 0,$$

and there are two similar sets of equations for the remaining two planes.

Consider now any new system of rectangular axes X, Y, Z with the same origin. Then

$$(2)\qquad \begin{aligned} X &= l_1X' + l_2Y' + l_3Z', \\ Y &= m_1X' + m_2Y' + m_3Z', \end{aligned}$$

$$x = X + Yi = (l_1 + m_1i)X' + (l_2 + m_2i)Y' + (l_3 + m_3i)Z', \; i = \sqrt{-1}.$$

Hence

$$(3)\qquad \begin{aligned} \sum^{6}(x_j - x_k)^2 &= (l_1 + m_1i)^2 \sum^{6}(X_j' - X_k')^2 \\ &\quad + \cdots 2(l_1 + m_1i)(l_2 + m_2i) \sum^{6}(X_j' - X_k')(Y_j' - Y_k') \cdots \\ &= 16a^2\left[\sum l^2 + 2i\sum lm - \sum m^2\right] = 0, \end{aligned}$$

where in the first two lines two sums in each have been indicated by the dots.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of American will hold its summer meeting in Chicago during the week of June 19, 1933 in conjunction with the summer meetings of the American Association for the Advancement of Science and the American Mathematical Society. The American Association will have a program for two weeks, beginning June 19, in connection with the Century of Progress International Exposition, with emphasis on pure science the first week and on applied science the second week. The mathematical meetings will be held for the most part at the University of Chicago. There will be addresses by two foreign mathematicians, Professor Leopold Fejér of the University of Budapest and Professor Tullio Levi-Civita of the University of Rome, and symposia in which foreign and American mathematicians will participate. The full program of the Mathematical Association will be sent to our members in ample season.

Accommodations for mathematicians and their guests will be furnished at

Judson Court, college residence halls for men at the University of Chicago, located at 60th Street and Ellis Avenue. The charge for room will be $2.00 per day per person, and the charge for meals will be $2.00 per day. A weekly rate of $25 for room and board has been set. Suitable arrangements for families can be made at the residence halls. It is anticipated that many of those attending the mathematical meeting will wish to remain for a few days after its conclusion for a more thorough examination of the many interests of the Century of Progress Exposition. For those persons who wish to remain longer, arrangements have been made enabling them to continue residence at Judson Court for a short time. Requests for residence hall or any other accommodations, and for information concerning the Exposition, should be sent to Professor H. S. Everett, University of Chicago, Chicago, Ill.

Plans are being made for the Conference of Engineers at Chicago during the week of June 25–30. On the evening of June 25 the International Union of Pure and Applied Physics will have a joint session with section M, of the American Association for the Advancement of Science. At this meeting Professor R. A. Millikan will speak on the topic "Applications of physics to engineering."

The Danish government reports the approval of a sum of 50,000 kroner for exhibits at the Century of Progress Exposition at Chicago. These exhibits will consist of originals and replicas of the apparatus used by Tycho Brahe, Ole Römer, Oersted, Niels Bohr, and others.

Dr. William Bowie of the U. S. Coast and Geodetic Survey has been awarded the Charles Lagrange prize by the Royal Academy of Belgium in recognition of his having affected the unification of the triangulation systems of the United States, Canada, and Mexico.

Dr. Ernest Brown, professor of mathematics at Yale, gave the second Arthur lecture at the Smithsonian Institution on January 25. His subject was "Gravitation in the solar system."

Professor Edward Kasner of Columbia University gave three public lectures in January and February before the Peoples Institute. First, Numbers and infinity; second, Spaces and dimensionality; third, Geometry and physics.

Professor Warren Weaver, chairman of the department of mathematics of the University of Wisconsin, has resigned his post to accept permanent charge of the Natural Science Division of the Rockefeller Foundation, a post which he assumed temporarily during a year's leave of absence. His resignation has been accepted by the regents and Professor M. H. Ingraham has been named chairman of the department.

Professor Jesse Douglas of the Massachusetts Institute of Technology has been granted leave of absence for the second semester of the present academic year.

The following courses in mathematics are announced for the summer of 1933:

University of Chicago, first term, June 19 to July 21; second term, July 24 to August 25. In addition to Calculus I and Elementary differential equations, the following advanced courses will be offered: By Professor G. A. Bliss: Calculus of variations; Multiple integrals. By Professor A. C. Lunn: Vector analysis; Physical applications of the theory of groups. By Professor M. I. Logsdon: Analytic projective geometry; Algebraic geometry I. By Professor L. M. Graves: Introduction to higher algebra; Integral equations. By Professor A. A. Albert: Elementary theory of equations; Topics in modern algebraic theories. By Professor Walter Bartky: Special perturbations; Modern theories of analytic differential equations I.

Columbia University, July 10 to August 18. In addition to courses in trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor E. Kasner: General concepts of mathematics; Geometric transformations and continuous groups. By Professor W. B. Fite: Differential equations; Introduction to higher algebra. By Professor P. A. Smith: Theory of functions of a real variable.

Cornell University, July 8 to August 18. In addition to the usual elementary work, the following advanced courses will be offered: By Professor Hutchinson: Modern algebra. By Professor Virgil Snyder: Projective geometry. By Professor W. A. Hurwitz: Advanced calculus. By Professor W. B. Carver: Advanced analytic geometry. By Professor D. C. Gillespie: Elementary differential equations. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, and C. F. Craig.

The George Washington University, nine weeks term, June 12 to August 11. In addition to the regular courses in plane analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Professor F. E. Johnston: Advanced algebra. By Professor E. W. Woolard: Theory of potential.

University of Illinois, June 19 to August 12. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor R. D. Carmichael: Analysis. By Professor J. B. Shaw: Algebra; Introduction to higher algebra. By Associate Professor E. B. Lytle: Teacher's course; History of mathematics. By Assistant Professor Harry Levy: Geometry; Introduction to higher geometry. By Dr. Ketchum: Introduction to higher analysis.

University of Iowa, First term, June 12 to July 20. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following

subjects are offered: By Miss Ruth Lane: Subject matter and teaching of mathematics. By Professor Chittenden: Differential equations; Functions of real variables; Seminar in analysis. By Associate Professor Wylie: Mathematics of finance, astronomy, meteors. By Associate Professor Woods: Theory of equations; Projective geometry; Theory of algebraic invariants. By Dr. Craig: Topics in Advanced statistics. By the Staff: Reading and research.

Second term, July 24 to August 24. By Professor Reilly: Elementary finite difference equations; Introduction to Laplace's and Poisson's equations; Seminar in interpolation. By Assistant Professor Ward: Infinite series; the Calculus of variations. By Dr. Conkwright: Differential equations; Group theory. By the Staff: Reading and research.

Johns Hopkins University, June 26 to August 5. By Professor F. D. Murnaghan: College algebra; Differential and integral calculus; Functions of a complex variable.

University of Kansas, June 14 to August 9. In addition to the usual elementary courses, the following advanced courses are offered: By Professor Mitchell: Theory of numbers; History of mathematics; Teachers' course; Seminar. By Professor Smith: Modern synthetic geometry; Seminar. By Professor Wheeler: Differential equations; Mathematical theory of statistics.

University of Kentucky, First term. By Professor H. H. Downing: Theory of equations. By Dean P. P. Boyd: Algebraic plane curves. By Professor F. E. LeStourgeon: Calculus of variations.

Second term. By Professor C. G. Latimer: Theory of numbers; and another course to be chosen. By Professor L. W. Cohen: Theory of functions of a real variable.

University of Maine, July 5 to August 12. In addition to the usual elementary work, the following advanced courses are offered: By Associate Professor Bryan: History of mathematics; Teachers' course. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Differential equations, or other graduate courses by arrangement.

Massachusetts Institute of Technology, First period, June 13 to July 25, Calculus and differential equations, covering the prescribed work of the first two years; Course in theoretical aeronautics; Advanced calculus. July 5 to July 25. Course in aeronautics continued. Second period, July 26 to September 6. Courses in the first period repeated; Vector analysis. August 7 to September 9. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in these subjects.

University of Michigan, June 26 to August 18. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered: By Professor John W. Bradshaw: Topics in calculus; The figures of solid geometry. By Professor V. C.

Poor: Differential equations; Applied mathematics—Engineering problems. By Professor Peter Field: Higher algebra; Vector analysis. By Professor Cecil C. Craig: Theory of probability; Graphical methods; Mathematical theory of statistics. By Professor H. C. Carver: Finite differences. By Professor R. V. Churchill: Solid analytic geometry. By Professor L. J. Rouse: Solid analytic geometry; Advanced calculus. By Professor L. A. Hopkins: Analytic mechanics. By Professor W. L. Ayres: Advanced calculus. By Professor C. J. Coe: Infinite series, with special reference to Fourier series. By Professor L. C. Karpinski: Teaching of geometry; History of mathematics. By Professor G. Y. Rainich: Higher geometry; Differential geometry; Mathematics of relativity. By Professor R. L. Wilder: Introduction to the foundations of mathematics; Studies in the foundations and point sets. By Professor T. H. Hildebrandt: Theory of functions of a real variable; Partial differential equations. By Professors Hildebrandt, Rainich, and others: Seminar in pure mathematics.

University of Minnesota, first term, June 20 to July 29. In addition to the usual elementary work the following courses will be offered: By Professor Dunham Jackson: Vector analysis. By Associate Professor Anthony L. Underhill: Differential equations. By Professors Elizabeth Carlson, Jackson, and Underhill: Reading in advanced mathematics. By Assistant Professor Gladys Gibbens: Famous problems in geometry. Second term, July 31 to September 2. No courses will be offered beyond the sophomore level.

Northwestern University, June 19 to August 12. In addition to courses in trigonometry, college algebra, analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Professor F. E. Wood: Determinants and theory of equations. By Dr. N. E. Rutt: Advanced calculus. By Professor H. S. Wall: Definite integrals.

Ohio State University, Summer Quarter, June 20 to September 1. In addition to elementary courses in college algebra, analytics, and calculus, the following advanced courses are offered: By Professor S. E. Rasor: Functions of a complex variable; Fourier Series. By Professor Tibor Radó: Vector analysis; Differential geometry. By Assistant Professor Grace M. Bareis: Theory of equations; Introduction to higher geometry. By Dr. L. E. Bush: Introduction to higher algebra.

University of Pittsburgh, July 3 to August 11. In addition to the undergraduate courses the following more advanced courses will be offered: By Professor F. A. Foraker: Advanced calculus; Analytical projective geometry. By Professor J. S. Taylor: Vector analysis; Modern algebraic theories. By Professor M. M. Culver: Differential equations.

Stanford University, June 22 to September 2. In addition to the usual courses in calculus and differential equations, the following courses will be offered: By Professor W. A. Manning of Stanford University: Higher geometry; Fuchsian

groups; Theory of functions of a complex variable. By Professor M. H. Stone of Yale University: Some fundamental concepts of mathematics. By Professor G. Polys of Technical School, Zürich, Switzerland: Lectures on singularities of power series.

University of Southern California, first term, June 19 to July 28. In addition to the usual elementary courses, the following advanced courses are offered: By Professor L. D. Ames: History of mathematics; Selected topics—algebra; Theory of probability and statistics. By Associate Professor D. V. Steed: Projective geometry. Second term, July 31 to September 1. By Professor L. E. Gurney: Theory of equations and determinants; Mathematical astronomy; Seminar (subject to be announced during first term).

Syracuse University, July 6 to August 11. In addition to the regular courses in plane and solid geometry, intermediate and advanced algebra, trigonometry, analytic geometry, and differential and integral calculus, the following courses will be offered: By Professor I. S. Carroll: Teaching of mathematics. By Professor F. F. Decker: Elementary theory of numbers or Fundamental concepts in mathematics. By Professor A. D. Campbell: Advanced plane analytic geometry or Elementary solid analytic geometry.

University of Vermont. By Professor Bullard: Differential calculus; Analytical geometry. By Professor Butterfield: Descriptive astronomy; History of mathematics; Plane trigonometry. By Professor Millington: Elementary algebra; College algebra; Solid geometry. By Professor Swift: Integral calculus; The teaching of plane geometry.

University of Wisconsin, six weeks session, June 26 to August 4. By Professor H. P. Evans: Differential and integral calculus; Topics in the theory of probability. By Dr. M. L. Hartung: The content of secondary mathematics; The teaching of mathematics. By Professor M. H. Ingraham: Mathematics of educational statistics; The elementary properties of number systems. By Professor H. W. March: Analytical geometry; Dimensional analysis. By Professor I. S. Sokolnikoff: Advanced calculus; Differential equations; Conformal representation. By Professor E. B. Skinner: Differential geometry; Theory of equations; Theory of numbers. Special nine weeks session for graduates, June 26 to August 25. These courses may be taken for six weeks. By Professor M. H. Ingraham: Advanced analytic theory of equations. By Professor H. W. March: Harmonic analysis.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merges with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

The Chauvenet Prize

IN THE YEAR 1925, the Mathematical Association of America established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the Carus Monographs are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the Chauvenet Prize will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1932, to Professor G. H. Hardy. The next award will be in December, 1935, for the period 1931-1934.

Note that the prize is to be awarded only to a member of the Association—one more of the many good reasons for membership.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-four persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

M. W. AYLOR, M.S. (Virginia) Instr., Univ. of Virginia, University, Va.

FANNIE W. BOYCE, A.M. (Wisconsin) Asst. Prof., Wheaton Coll., Wheaton, Ill.

O. Z. BURDICK, M.S. (Illinois) Engineer with firm of C. W. Hills, Patent Lawyers, Chicago, Ill.

SISTER M. CLAUDETTE, A.M. (Minnesota) Head of Dept., Coll. of St. Benedict, St. Joseph, Minn.

F. F. CRANDELL, A.B. (Wooster) Grad. student, Ohio State Univ., Columbus, Ohio

R. D. DALAL, Stock Broker, Bombay, India

D. C. DEARBORN, A.B. (Hastings Coll.) Asst. Instr., Univ. of Nebraska, Lincoln, Nebr.

H. T. ENGSTROM, Ph.D. (Yale) Asst. Prof., Yale University, New Haven, Conn.

NICHOLÀS FATTU, M.S. (Idaho) Instr., Itasca Jr. Coll., Coleraine, Minn.

J. A. FLEMING, B.S. (Cincinnati) Acting Director, Dept. of Terrestrial Magnetism, Carnegie Institution of Washington, Washington, D. C.

BEATRICE L. HAGEN, Ph.D. (Chicago) Instr., Pennsylvania State Coll., State College, Pa.

I. L. HEBEL, B.S. (Colorado) Asst. Prof., Colorado School of Mines, Golden, Colo.

B. O. KOOPMAN, Ph.D. (Harvard) Asst. Prof., Columbia Univ., New York, N. Y.

ABE LASSER, Student, Armour Inst. of Tech., Chicago, Ill.

F. S. NOWLAN, Ph.D. (Chicago) Prof., Univ. of British Columbia, Vancouver, B. C.

W. T. PUCKETT, JR., M.S. (Virginia) Grad. student, Univ. of Virginia, University, Va.

S. R. TARAZI, A.B. (Amer. Univ. of Cairo, Egypt) Teacher, Friends' Boys' School, Ram Allah, Palestine

R. B. THOMPSON, A.B. (Hastings Coll.) Asst. Instr., Univ. of Nebraska, Lincoln, Nebr.

H. F. TODD, Teacher, Jr. High School, Leewood, W. Va.

JOHN VON NEUMANN, Ph.D. (Budapest), Ing. Chem. (Zurich) Prof., Inst. for Advanced Study, Princeton, N. J. (Until Sept. 1933, Budapest, Hungary)

T. L. WADE, JR., M.S. (Virginia) Asst. Instr., Univ. of Virginia, University, Va.

H. E. WAHLERT, A.M. (Princeton) Instr., Washington Square Coll., New York Univ., New York, N. Y.

H. A. WRIGHT, Ph.D. (New York Univ.) Acting Head of Dept., Transylvania Coll., Lexington, Ky.

J. H. ZANT, A.M. (Columbia) Oklahoma A & M Coll., Stillwater, Okla.

To Institutional Membership

ST. MARY'S COLLEGE, Notre Dame, Ind.

The Trustees have approved the organization and by-laws of the Oklahoma Section.

W. D. CAIRNS, *Secretary*

THE NINETEENTH ANNUAL MEETING OF THE KANSAS SECTION

The nineteenth annual meeting of the Kansas Section of the Mathematical Association of America was held in Topeka, at the High School Building, on Saturday, February 11, 1933. In the morning the session was a joint meeting with the Kansas Association of Mathematics Teachers; Professor O. J. Peterson, Chairman of the Section, presided at both sessions.

There were forty-eight in attendance, including the following twenty-two members of the Association: R. W. Babcock, Wealthy Babcock, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, A. S. Householder, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, U. G. Mitchell, O. J. Peterson, A. W. Philips, P. S. Pretz, B. L. Remick, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler.

At the business meeting, the following officers were chosen for next year: Chairman, Dean R. W. Babcock, Kansas State College; Vice-Chairman, Professor G. W. Smith, University of Kansas; Secretary, Lucy T. Dougherty, Junior College, Kansas City.

The morning session opened with a tribute to Homer S. Myers, former head of the Department of Mathematics and Registrar of Southwestern College, Winfield, who died suddenly on October 23, 1932, of a heart attack. Professor Garrett said in part: "The death of Professor Homer S. Myers brought a shock not only to his immediate associates on the faculty, but to all the members of the Kansas Section. We had grown to appreciate his qualities of mind and heart, and his genial comradeship. His passing leaves an unfilled gap in our ranks."

There were two papers at the morning session. Professor W. H. Garrett of Baker University gave a short resumé of Sir Arthur Eddington's "Nature of the Physical Universe," and by graph and illustration explained his theory of the expanding universe. Professor Garrett also showed some interesting views which he had taken of the 1932 eclipse. Professor U. G. Mitchell of University of Kansas told of some of the unusual features of his trip to Europe, which included attendance at the Congress of Mathematicians at Zürich. In London he was fortunate to meet the Director of the British Museum, who is author of a book on "jetons," a subject in which Professor Mitchell has been much interested. He also found that the Director of the Museum of the City of London was a specialist in "stocks," another of Professor Mitchell's hobbies.

After lunching together, the two associations met in separate session, and the following papers were presented before the Section:

1. "Some properties of orthogonal functions" by Dean R. W. Babcock, Kansas State College.
2. "Certain loci related to a variable triangle" by N. F. Shell, University of Kansas, by invitation.
3. "Generalized coordinates and the Lagrangian equation of motion" by Professor R. G. Smith, Kansas State Teachers College, Pittsburg, by invitation.
4. "The early life of Gauss" by G. W. Dunnington, Horner Junior College, Kansas City, Missouri, by invitation.

Abstracts of these papers follow:

1. Two functions $f_i(x)$ and; $f_j(x)$ are orthogonal over the interval ab if

$$\int_a^b f_i(x)f_j(x)dx = 0.$$

Included with the function in the integrand may be found a weight function $w(x)$. Various choices for the interval and the weight function permit developments of otherwise diverse functions under this single definition of orthogonality, such as the sine cosine Fourier series, the Legendre polynomials or surface zonal harmonics, Bessel functions, the polynomials of Tschebyscheff, Jacobi, Hermite and Laguerre. On the surface of a unit sphere, the tesseral harmonic functions, and the surface spherical harmonic functions of the same degree constitute orthogonal systems. Orthogonal functions provide a convenient method for the development of series as an approximation to an arbitrary function, after which conditions for convergence must be studied, as for example the Dirichlet conditions in Fourier's series. Approximations to a given function by a convergent series formed from orthogonal functions are valid over the entire interval, and as more terms are taken, the closeness of approximation is obtained simultaneously over the entire interval. This differs from Taylor's series, in which case the closeness of approximation is increased in the neighborhood of a point as more terms of the series are used.

2. In his "Recreation in Mathematics" Lick suggests the proposition: "Given any triangle. On its sides as bases construct isosceles triangles with equal thirty degree angles. The triangle formed by joining the vertices of these isosceles triangles is equilateral." Two such triangles are formed which are covariant with the given triangle. This paper discusses the properties of these covariant triangles as the given triangle is allowed to vary according to some fixed law.

3. A brief outline of the derivation of the Lagrangian equations of motion, of points moving under a conservative system of forces, is followed by a few applications: some of which demonstrate the method for determining the tensions and pressures due to certain geometrical or physical constraints.

4. Mr. Dunnington traced the development of the "prince of mathematicians" up to the attainment of his doctorate in 1799. Special attention was called to the textbooks by lesser known authors, and also to works of Euler, Newton, Lagrange and others, which he studied. His own influence on later mathematicians was pointed out and most of his later achievements were shown to have their origin in this period of his life. The more intimate and human side of Gauss during these early years was described; his relations with close friends, especially Wolfgang Bolyai, were discussed. Gauss rose to world fame from the very humblest origin. His father never had the faintest idea of his importance, but his mother lived to see her only child the doyen of German science. In 1792 he was considering the foundations of geometry; in 1794, discovery of the method of least squares; then for several years the theory of numbers, elliptic functions, series, the theory of functions, proof of the fundamental theorem of algebra, construction of the regular polygon of seventeen sides, and allied topics were in the foreground. The speaker has examined all the original sources on Gauss—in Göttingen, Brunswick, and elsewhere. Through friendship with the Gauss family he has access to a large mass of unpublished material and for several years has been at work on a full biography of him. An exhibit of

Gaussiana, including many of his original letters and rare pictures, was shown after the address.

LUCY T. DOUGHERTY, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirteenth regular meeting of the Southern California Section was held at Pomona College, Claremont, on Saturday, March 4, 1933. Professor L. D. Ames presided.

The attendance was fifty including the following thirty-one members of the Association: E. E. Allen, L. D. Ames, Harry Bateman, Clifford Bell, Grace E. Berry, Jessie R. Campbell, Myrtie Collier, P. H. Daus, Iva B. Ernsberger, Raymond Garver, H. H. Gaver, Harriet Glazier, J. M. Gleason, E. R. Hedrick, G. H. Hunt, C. G. Jaeger, G. A. Linhart, G. R. Livingston, W. E. Mason, A. D. Michal, G. F. McEwen, Lena E. Reynolds, G. E. F. Sherwood, Marcus Skarstedt, F. C. Touton, S. E. Urner, H. C. Van Buskirk, L. E. Wear, Mabel G. Whiting, H. C. Willett, E. R. Worthington.

The meeting began with a luncheon at the Claremont Inn, after which it adjourned to Mason Hall for a short business meeting and the program. The following officers were elected for the year 1933–34: Chairman, O. W. Albert, University of Redlands; Vice-Chairman, C. G. Jaeger, Pomona College; Program Committee, G. A. Linhart, Riverside Junior College and L. E. Wear, California Institute of Technology. The next meeting was tentatively scheduled for March 3, 1934 at Riverside Junior College.

A resolution of sympathy for the death of President Victor L. Duke of the University of Redlands was adopted and later transmitted to the proper persons.

The following program was presented:

1. "Time series: their analysis by successive smoothings" by Professor Lewis A. Maverick, Department of Economics, University of California at Los Angeles, by invitation.

2. "Linear functional differential equations" by Professor Harry Bateman, California Institute of Technology.

3. "The transformation $y=f'(x)$" by Professor Raymond Garver, University of California at Los Angeles.

4. "Estimation of vertical circulation and turbulence in the ocean from a solution of the equation of heat conduction" by Professor George F. McEwen, The Scripps Institute of Oceanography of the University of California.

5. "A formula for the profile of a stream" by Professor Edward Taylor, Pomona College, by invitation.

Abstracts of these papers follow, the numbers corresponding to the numbers of the titles.

1. It has been suggested by Ragnar Frisch that a trend or smoothing line may be drawn to pass through the points of inflection in the original curve, and that, if the smoothing line be itself examined, it will in turn show fluctuations

which will necessarily be marked by points of inflection. These points may be connected to give a trend or smoothing line of the second order, and it may be submitted to the same process. The first smoothing line is freed from the fluctuation of shortest period, but contains all fluctuations of longer period; the second smoothing line is in turn freed from the lowest order fluctuation remaining in the smoothing line of first order.

In the present paper the concept is accepted of a succession of smoothing lines which set apart the fluctuations into homogeneous orders. The criterion of points of inflection is accepted as but one of several criteria, aiding in the location of the successive smoothing lines; the short-coming lies in the indeterminateness of the location of the points of inflection. Added criteria employed are moving averages, minimum radius of curvature, and *bounds*, which are two curves that touch, respectively, the successive peaks and the successive troughs.

Each order fluctuation may be studied for period and amplitude and a standard pattern may be determined.

In forecasting, the highest order trend may be continued forward and those of lower orders superimposed in the form of their standard patterns. In correlation, the study may be carried on separately for each order fluctuation. In constructing index numbers, the selection and weighting of the component series may be undertaken separately for each order fluctuation, according to the significance of the series for the fluctuation of the particular order.

This paper will be published in full in the forthcoming issue of *Econometrica*.

2. A study of equations which are linear in a function of x, the same function of $-x$ and the derivatives of these functions. An operational calculus is developed with the operation of differentiation and the operation changing x into $-x$ as basic operations. Associated variational principles are considered.

3. The application of the transformation $y=f'(x)$ to the equation $f(x)=x^n+c_1x^{n-1}+\cdots+c_n=0$ leads to a transformed equation in y whose constant term, except possibly for sign, is the discriminant of $f(x)=0$. In the case of the cubic, quartic, and some special higher degree equations, this gives a convenient method of evaluating the discriminant. In fact, applied to the general nth degree equation, it leads to a determinant form for the discriminant which has certain advantages over the usual determinant forms.

4. The classical Fourier differential equation of heat conduction in which a term is included to take account of the circulation of the water is solved with special reference to conditions in the ocean. The diffusivity coefficient under these conditions is not a physical constant of the fluid but is a measure of the turbulence. The equation is solved so as to correspond to the observed initial vertical distribution of temperature below a level near the surface and the observed relation of temperature to time at that level. Tables have been prepared to facilitate the computation, and estimates are presented of turbulence and vertical flow based upon records of ocean temperatures in the San Diego region.

5. The formula for the long profile of a stream has been sought since 1875. Sternberg developed an exponential curve. Unwin developed a parabola. They

were both based upon such restriction and exception as to make their applications to the entire length of a stream impossible. The present analysis is based upon the hydraulic formulae of Chezy and Bazin. It shows that no single formula can represent all streams, nor all of a single stream; and that an analytic formula of the form y equals a single term in x is not possible.

P. H. DAUS, *Secretary*

THE TENTH ANNUAL MEETING OF THE MICHIGAN SECTION

The tenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan on Saturday, March 18, 1933, in conjunction with the Michigan Academy of Science. The chairman of the Section, Professor R. W. Clack, presided.

The attendance was over a hundred, including the following forty-four members of the Association: N. H. Anning, W. L. Ayres, W. D. Baten, Harold Blair, W. M. Borgman, J. W. Bradshaw, J. B. Brandeberry, R. W. Clack, C. J. Coe, J. J. Corliss, C. C. Craig, S. E. Crowe, Wayne Dancer, J. D. Elder, L. C. Emmons, J. P. Everett, Peter Field, W. B. Ford, B. C. Getchell, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, H. S. Kaltenborn, L. C. Karpinski, Theodore Lindquist, C. E. Love, A. L. Nelson, J. A. Nyswander, H. L. Olson, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, R. C. Shellenbarger, E. R. Sleight, A. G. Swanson, Dorothy Van Deusen, T. O. Walton, R. L. Wilder, J. B. Winslow.

A luncheon was held at noon at the Michigan Union. A short business session was held at this time and the following officers were chosen for the year 1933–34: Chairman, Professor T. O. Walton, Kalamazoo College; Secretary-Treasurer, Professor W. L. Ayres, University of Michigan; Third member of Executive Committee, Professor R. W. Clack, Alma College.

The following papers were read during the morning and afternoon sessions:

1. "The ellipsoidal viscosity distribution function for molecular velocities" by Professor W. S. Kimball, Michigan State College.
2. "Laplace transforms of conjugate nets" by W. M. Borgman, Jr., College of the City of Detroit.
3. "Differentiation with respect to a function" by Dr. Ben Dushnik, University of Michigan.
4. "The remainder theorem as a test for divisibility" by R. E. Gaskell, student, Albion College.
5. "Highlights of the Zürich Mathematical Congress" by Professor G. Y. Rainich, University of Michigan.
6. "Rare old books in mathematics" by Professor Harold Blair, Western State Teachers College.
7. "Double divisibility in a division algebra" by Professor H. L. Olson, Michigan State College.

8. "Geometrical studies suggested by a proof of the fundamental theorem of algebra" by Professor T. O. Walton, Kalamazoo College.

9. "An experiment in second semester mathematics for freshmen" by E. E. Ingalls, Albion College.

10. "Maya mathematics" by Director Carl Guthe, Museum of Anthropology, University of Michigan, by invitation.

11. "Practical harmonic analysis" by Professor N. H. Anning, University of Michigan.

12. "The introduction of the calculus into high school mathematics" by Professor R. F. McDaid, Michigan State Normal College.

Abstracts of the papers follow:

1. The paper discusses the change in the Maxwell spherical distribution function produced by a shearing force. It is shown that the distribution then becomes elliptical and that it satisfies Bolzmann's equation.

2. Nets of curves $u=c$ and $v=k$ on surfaces such that the tangents to the u-curves on one surface are tangent to the v-curves on the next surface are discussed. It is shown that if the tangents form a linear family then the surfaces are quadratic.

3. The derivative of a function $f(x)$ with respect to a function $\alpha(x, y)$ is defined as

$$f_\alpha'(x) = \lim_{y\to 0} \frac{f\{\alpha(x, y)\} - f\{\alpha(x, 0)\}}{\alpha(0, y)}$$

Dr. Dushnik proves that this operation has most of the elementary properties of ordinary derivatives except for some variation in the formulas for the derivative of a product and quotient. An illustrative example indicates a possible application of this operation to the solution of a certain type of functional operation.

4. The remainder theorem is used to develop the well-known tests for divisibility of numbers by 3, 9, and 11. Divisibility tests in number systems with a base other than 10 are also discussed and the results generalized.

5. The nature of Professor Rainich's paper is clear from the title.

6. Professor Blair exhibited and discussed some rare scientific books from the collection of the late Mr. A. M. Todd of Kalamazoo. These books are now in the library of the Western State Teachers College.

7. In this paper is developed a necessary and sufficient condition on an integral element v of a division algebra in order that, for every integral element x, there shall exist an integral element y such that $xv=vy$.

8. The paper studies some relations between the roots of the complex function $f(z)=u(x, y)+iv(x, y)$ and the curves $u=0$ and $v=0$. If $f(z)$ has a multiple root of order n at a point, then $u(x, y)=0$ and $v(x, y)=0$ will each have a multiple point of order n at this point. Further the branches of $u=0$ (and $v=0$) intersect at this point at angles of π/n, and the branches of $u=0$ bisect the angles between adjacent $v=0$ branches.

9. Three years ago the experiment of replacing trigonometry by special topics in charting and graphing, statistics, and compound interest was tried on a group of 40 students not planning to take further work in mathematics. Recently a questionnaire was sent to these students asking their opinions on the desirability of the course and whether they had actually used the knowledge gained from it. The paper discussed the content of the course and the favorable response to the questionnaire.

10. Dr. Guthe discussed the mathematics of the Mayas, "The Greeks of the New World." He gave a full account of their number system of base 20, showing their number symbols and their methods of combining them to represent large numbers. He described their calendar and pointed out that it was as accurate as and simpler than our Gregorian calendar. As there are no fractions in the Maya number system, he showed how, using integers only, they adjusted their man-made calendar to fit observed astronomical phenomena.

11. The paper was a summary of current methods—graphical, mechanical, and numerical. A Henrici-Coradi analyser, the property of the University of Michigan, was exhibited. This and the Mader machine were described in some detail. Professor Anning pointed out the advantages for rapid analysis of Runge's numerical method as modified by Zipperer and, more recently, by Terebesi.

12. The paper was an account of some lessons on maxima and minima given in the University High School, Ann Arbor, in October 1932.

W. L. AYRES, *Secretary*

ON THE NUMERICAL INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

By W. E. MILNE, Oregon State Agricultural College

Differential equations of the second order of the type

$$\frac{d^2y_i}{dx^2} = u_i(y_1, y_2, \cdots, y_n, x) \qquad (i = 1, 2, \cdots, n) \tag{1}$$

in which the first derivatives do not appear are of such frequent occurrence in physics, dynamics, and astronomy, that special methods for their numerical integration are of considerable interest and importance.*

The method here described is particularly well adapted for use with a calculating machine, provides a check on each step, and requires a minimum of written calculation. Trial computations using several methods indicate that when a calculating machine is available the method given below is the most rapid of all, though it is probable that some of the methods using differences are more satisfactory when all of the work has to be done by hand.

* A survey of various special methods of solving equations of type (1) numerically is given by Nyström, Acta Societatis Scientiarium Fennicae, vol. 50, page 14, #4A, page 23, #5A, pages 35–37.

1. For the sake of simplicity the treatment will be limited to a single differential equation with a single dependent variable, which may be written

$$d^2y/dx^2 = u(x, y). \tag{1}$$

The problem is to determine approximately the particular solution of this differential equation for which $y=y_0$, $dy/dx=y_0'$, when $x=x_0$. Let us assume that we have already obtained the values of y and of u corresponding to a set of uniformly spaced values $x_0, x_1, x_2, \cdots, x_n$, of the independent variable x, and that these quantities have been tabulated as follows:

x_0	y_0	u_0
x_1	y_1	u_1
x_2	y_2	u_2
. .	. .	. .
x_n	y_n	u_n
x_{n+1}		

The next step is to determine the values of y_{n+1} and u_{n+1} which are to be inserted opposite x_{n+1}. After these are found the next set of values for x_{n+2} will be secured by a precisely similar operation and thus the computation will proceed step by step.

For the calculation of y_{n+1} in each step we employ formulas of a special type which give y_{n+1} in terms of the tabulated y's and u's, and which do not contain the first derivative y'. The derivation of these special formulas is taken up in paragraph 2.

2. When the solution $y(x)$ is substituted for y in the expression $u(x, y)$ of equation (1) the function $u(x, y)$ becomes a function of x alone. We shall assume that this function u can be represented with a degree of accuracy[1] sufficient for our purpose by a few terms of Newton's interpolation formula, so that we may write

$$d^2y/dx^2 = u = u_n + \Delta u_n(x - x_n)/h + \Delta^2 u_n(x - x_n)(x - x_{n-1})/2h^2 + \cdots, \tag{2}$$

in which

$$\Delta u_n = u_n - u_{n-1}, \ \Delta^2 u_n = \Delta u_n - \Delta u_{n-1}, \text{ etc.}, \ h = x_n - x_{n-1}.$$

Integrating equation (2) twice and determining the constants so that $dy/dx = y_n'$, $y=y_n$, when $x=x_n$, we have

$$y(x) = y_n + y_n'(x - x_n) + h^2[u_n A_0(x) + \Delta u_n A_1(x) + \Delta^2 u_n A_2(x) + \cdots], \tag{3}$$

in which for the sake of brevity we have set

$$h^2 A_0(x) = (x - x_n)^2/2,$$

$$h^2 A_k(x) = \int_{x_n}^{x} ds \int_{x_n}^{s} (t - x_n)(t - x_{n-1}) \cdots (t - x_{n-k+1})dt/k!h^k, \ (k = 1, 2, \cdots).$$

[1] Cf., e.g., Steffensen, *Interpolation*, pp. 22–26.

The values of the first eight A's have been calculated for six different values of x and are given in the following table:

x	$A_0(x)$	$A_1(x)$	$A_2(x)$	$A_3(x)$	$A_4(x)$	$A_5(x)$	$A_6(x)$	$A_7(x)$
x_{n+1}	1	+1	3	+19	45	+863	9625	+67906
x_n	0	0	0	0	0	0	0	0
x_{n-1}	1	−1	−1	−4	−7	−107	−995	−6031
x_{n-2}	4	−8	0	−8	−16	−256	−2432	−14912
x_{n-3}	9	−27	27	−27	−27	−405	−3807	−23328
x_{n-4}	16	−64	128	−256	0	−512	−5120	−31744
x_{n-5}	25	−125	375	−1250	625	−1375	−6875	−40625
Common Denominator	2	6	24	180	480	10080	120960	907200

Equation (3) contains the first derivative y_n' and therefore is not directly applicable to our problem, but from equation (3) can be obtained a general formula of the desired type. Let m and k be any two integers, and substitute in (3) successively the following four values of x; $x=x_{m+1}$, $x=x_m$, $x=x_{k+1}$, $x=x_k$. Then we have four equations from which the term y_n' can be eliminated by subtracting the sum of the second and third from the sum of the first and fourth, so that we have the general equation

$$(4)\quad y_{m+1} - y_m - y_{k+1} + y_k = h^2 \sum_i \Delta^i u_n [A_i(x_{m+1}) - A_i(x_m) - A_i(x_{k+1}) + A_i(x_k)].$$

By giving m and k integral values in (4) we obtain a variety of particular formulas, all of which lack the term y_n'. Thus if $m=n$, $k=n-1$, we get, after transposition of the second, third, and fourth terms,

$$y_{n+1} = 2y_n - y_{n-1} + h^2[u_n + (1/12)\Delta^2 u_n + (1/12)\Delta^3 u_n + (19/240)\Delta^4 u_n + \cdots].$$

This equation is not new but was employed by Störmer.[1] We shall not use this particular case but shall endeavor to construct similar formulas in which the coefficient of the third difference shall vanish. It will be found that if $m=n$, $k=n-3$, we get

$$(5)\quad y_{n+1} = y_n + y_{n-2} - y_{n-3} + h^2[3u_n - 3\Delta u_n + (5/4)\Delta^2 u_n + (17/240)\Delta^4 u_n + \cdots].$$

while if $m=n-1$, $k=n-2$, we have

$$(6)\quad y_n = 2y_{n-1} - y_{n-2} + h^2[u_n - \Delta u_n + (1/12)\Delta^2 u_n - (1/240)\Delta^4 u_n + \cdots].$$

Now in (5) and (6) we substitute the identities

[1] Archives des Sciences physiques et naturelles, Genève, juillet–octobre 1907, p. 63ff.

$$\Delta u_n = u_n - u_{n-1}, \ \Delta^2 u_n = u_n - 2u_{n-1} + u_{n-2},$$

neglect all terms containing differences of the fourth and higher orders, and obtain finally the two formulas that we actually use in calculation:

$$y_{n+1} = y_n + y_{n-2} - y_{n-3} + (h^2/4)(5u_n + 2u_{n-1} + 5u_{n-2}), \tag{7}$$

and

$$y_n = 2y_{n-1} - y_{n-2} + (h^2/12)(u_n + 10u_{n-1} + u_{n-2}). \tag{8}$$

It will be noted that if u is a polynomial of the third degree in x the fourth and higher differences vanish, so that the formulas (7) and (8) are exact. The exceptional simplicity of these two formulas is apparent when we observe that ordinarily a formula of this type which gives exact results for a polynomial of the third degree will contain the quantity u_{n-3} in addition to u_n, u_{n-1}, and u_{n-2}. Moreover we see from the table of A's that when the third difference does not cancel out the numerical coefficients will not be as simple as they are in (7) and (8). In their class these formulas are as noteworthy as Simpson's Rule is among quadrature formulas.

Equation (7) is of particular value for the integration of differential equations since it expresses y_{n+1} in terms of y_n, y_{n-1}, u_n, u_{n-1}, etc. thus permitting the evaluation of y_{n+1} by means of quantities already obtained. Equation (8) is used for recalculation, and in general is more accurate than (7).

Rigorous upper bounds for the magnitude of the error may be calculated in the usual manner, but as they involve the higher derivatives of $u(x, y)$ they are not usually convenient for practical use.

3. The actual procedure in integrating a differential equation is best shown by an example, for instance

$$\frac{d^2y}{dx^2} = -y^3,$$

with the initial values $y=0$, $dy/dx=1$, when $x=0$. Let $h=0.1$. In order to get started we need to have three values of y besides the initial value. These may be obtained from Taylor's series for y, a few terms of which are found to be

$$y = x - \frac{1}{20}x^5 + \frac{1}{480}x^9 + \cdots,$$

whence we obtain the following values

x	y	$u(x, y)$
−0.1	−0.0999995	0.0010000
0.0	0.0000000	0.0000000
0.1	0.0999995	−0.0010000
0.2	0.1999840	−0.0079981
0.3	. . . . .	. . . . .
0.4	. . . . .	. . . . .

To continue the computation we use (7) for $n=2$ to predict the value of y_3, obtaining $y_3=0.2998785$. From y_3 we find u_3 and now use (8) for $n=3$ to recalculate y_3, getting in this case precisely the same value as before. We conclude that y_3 is now correct, and proceed to predict y_4 by (7) and to recalculate y_4 by (8). The correction δy is $+1$ in the last figure, so y_4 and u_4 are corrected and we go on to y_5. The computation up to $x=0.7$ appears as follows when the corrected values are given. The column headed δy gives the correction.

x	y	u	δy
0.2	0.1999840	−0.007998	
0.3	0.2998785	−0.026967	
0.4	0.3994885	−0.063755	1
0.5	0.4984415	−0.123835	2
0.6	0.5961329	−0.211850	7
0.7	0.6916799	−0.330914	12

4. When fifth differences are negligible fourth differences are practically constant and we see by (6) that the error of (8) is $E_8=-h^2\Delta^4u_m/240$ and by (5) that the error of (7) is $E_7=17h^2\Delta^4u_n/240$ so that $\delta y=18h^2\Delta^4u_n/240$ and we have

$$E_8 = \frac{\delta y}{18}, \tag{9}$$

Consequently from the tabulated values of δy we can form an estimate[1] of the error that is liable to occur in using formula (8). When the error given by (9) is large enough to affect the last digit retained it is advisable either to shorten the interval or to use more accurate formulas, such as those derived in the next section.

5. When in formula (4) we set $m=n$ and $k=n-5$, and afterwards replace the first four differences on the right by their values in terms of the u's we obtain a formula analogous to (7), which reads

$$y_{n+1} = y_n + y_{n-4} - y_{n-5} + \frac{h^2}{48}[67u_n - 8u_{n-1} + 122u_{n-2} - 8u_{n-3} + 67u_{n-4}]. \tag{10}$$

Again setting $m=n-1$, $k=n-4$ we obtain similarly a formula analogous to (8)

$$y_n = y_{n-1} + y_{n-3} - y_{n-4} + \frac{h^2}{240}[17u_n + 232u_{n-1} + 222u_{n-2} + 232u_{n-3} + 17u_{n-4}]. \tag{11}$$

These formulas are both exact if sixth differences of u vanish, and hence are in general considerably more accurate than (7) and (8).

[1] Of course (9) is strictly true only when fifth differences of u vanish, but it is nevertheless a very useful guide in actual practice.

6. Returning to our example we note that δy is increasing at such a rate as to make the error in the next calculation significant. We therefore abandon (7) and (8) and use instead (10) to predict, and (11) to correct, the values of y. The next five lines of the calculation appear as follows:

x	y	u	δy
0.8	0.7838909	−0.481689	+2
0.9	0.8712609	−0.661370	+1
1.0	0.9519989	−0.862799	−3
1.1	1.0241004	−1.074058	−5
1.2	1.0854666	−1.278938	−6

When seventh differences are negligible the error of (11) is

$$E_{11} = \frac{-318h^2}{120960}\Delta^6 u_n$$

and that of (10) is

$$E_{10} = \frac{7870h^2}{120960}\Delta^6 u_n$$

so that

$$\delta y = \frac{8188h^2}{120960}\Delta^6 u_n$$

and

$$E_{11} = -\frac{159}{4094}\delta y = -\frac{\delta y}{26} \text{ (approx.)} \tag{12}$$

When the error E_{11} given by (12) is significant it is advisable to choose a smaller value of h.

NEGATIVE-RECIPROCAL EQUATIONS

By H. S. UHLER, Yale University

In a certain investigation in geometrical optics I found it necessary to transform a rational algebraic equation in x to a new one in y by the relation $x=2y(1-y^2)^{-1}$. [$x\equiv\tan 2\theta$, $y\equiv\tan\theta$]. The resulting equation in y was very closely related to the reciprocal equations discussed in the elementary treatises on the theory of equations, for example, in those by Todhunter, and by Burnside and Panton. The following examination of the chief properties of the less familiar type of equation was suggested by the original problem. The results may be of interest to others since they supplement and extend the discussions

given in the elementary text books.[1] The manner of presentation will parallel that of the texts both in style and in making no claim to formal rigor.

Definition: The epithet *negative-reciprocal* will be applied to a rational algebraic equation which (after multiplication by simple factors) is reproduced identically when its variable x is replaced by $-x^{-1}$.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0. \tag{1}$$

Substitute $-x^{-1}$ for x, and then multiply by x^n/p_n to obtain

$$x^n - \frac{p_{n-1}}{p_n}x^{n-1} + \frac{p_{n-2}}{p_n}x^{n-2} - \cdots \pm \frac{p_2}{p_n}x^2 \mp \frac{p_1}{p_n}x \pm \frac{1}{p_n} = 0, \tag{2}$$

where the upper or lower signs are to be taken according as n is even, or odd, respectively.

In order that equation (2) may be identical with equation (1) the following conditions must be fulfilled

$$p_1 = -\frac{p_{n-1}}{p_n}, \quad p_2 = \frac{p_{n-2}}{p_n}, \quad p_3 = -\frac{p_{n-3}}{p_n}, \cdots, p_{n-2} = \pm\frac{p_2}{p_n},$$

$$p_{n-1} = \mp\frac{p_1}{p_n}, \quad p_n = \pm\frac{1}{p_n};$$

the double signs conforming to the earlier qualification. The last equation gives $p_n^2 = \pm 1$ hence $p_n = \pm 1$ when n is even, and $p_n = \pm(-1)^{1/2} \equiv \pm i$ when n is odd. Accordingly, a negative-reciprocal equation of odd degree with exclusively real coefficients can not exist.

Case 1(a). *n even, $p_n = +1$*. Then

$$p_1 = -p_{n-1}, \; p_2 = p_{n-2}, \; p_3 = -p_{n-3}, \cdots, p_{n-3} = -p_3,$$

$$p_{n-2} = p_2, \; p_{n-1} = -p_1$$

or, more compactly,

$$p_j = (-1)^j p_{n-j}; \; j = 1, 2, 3, \cdots, n-1.$$

Thus an equation is a negative-reciprocal equation when the coefficient of x^n and the constant term are both $+1$, and the coefficients of the terms at equal odd distances from the first and last are equal in magnitude and opposite in sign, while the coefficients of terms at equal even distances from the beginning and end agree both in magnitude and in sign. Furthermore, when $n/2$ is odd the middle term must vanish. [$n/2 = 2m-1 = j \;\therefore\; p_{2m-1} = (-1)^{2m-1}p_{2m-1} = -p_{2m-1}$.] Then the self negative-reciprocal numbers $+i$ and $-i$ are roots of

[1] The broad features of the solution of the more general problem in which x and h/x are roots may be found in: J. Carnoy's *Cours d'Algèbre Supérieure*, pp. 256 to 262; and B. Niewenglowski's *Cours d'Algèbre*, vol. 2, pp. 344 to 348.

the equation, hence x^2+1 is a factor of the left member so that the degree of the equation can be reduced by two. The equation obtained by removing the factor in question is also a negative-reciprocal equation, and it likewise falls under the present case 1(a). Evidently $n/2$ is now even and hence further division by x^2+1 is, in general, no longer possible.

The total number of different but complete negative-reciprocal equations of degree n (n even, $p_n=+1$) in which the coefficients of the corresponding terms are equal in magnitude but have their signs permuted in every possible way consistent with the pairing required by the equations of condition, and the middle term (when present) is also alternately given the plus and the minus sign, is expressed by 2^e where e equals $n/2$ or $n/2-1$ according as $n/2$ is even or odd, respectively.

Case 1(b). *n even*, $p_n=-1$. Now

$$p_1 = p_{n-1},\ p_2 = -p_{n-2},\ p_3 = p_{n-3}, \cdots , p_{n-3} = p_3,$$
$$p_{n-2} = -p_2,\ p_{n-1} = p_1$$

or

$$p_j = (-1)^{j+1}p_{n-j};\ j = 1, 2, 3, \cdots , n-1.$$

Hence, an equation is a negative-reciprocal equation when the coefficient of x^n and the constant term are $+1$ and -1 respectively, and the coefficients of the terms at equal even distances from the terms x^n and -1 are equal in absolute value but of contrary signs, while the coefficients of terms at equal odd distances from the extreme terms coincide both in magnitude and in sign. Moreover, when $n/2$ is even the term involving $x^{n/2}$ must be absent $[n/2=2m=j\ \therefore\ p_{2m}=(-1)^{2m+1}p_{2m}=-p_{2m}]$, and x^2+1 is a factor of the left member of the equation. Under these circumstances the degree of the polynomial can be lowered by two. The resulting equation is again a negative-reciprocal equation, and it too falls under the present case 1(b). Obviously $n/2$ is now odd so that further division by x^2+1 can not, in general, be performed.

The total number of distinct but complete negative-reciprocal equations of degree n (n even, $p_n=-1$) in which the coefficients of the corresponding terms are equal in absolute value but have their signs permuted in every possible arrangement compatible with the pairing necessitated by the conditional equations, while the middle term (when present) is also successively assigned the plus and the minus sign, is represented by 2^e where e equals $n/2-1$ or $n/2$ according as $n/2$ is even or odd, respectively.

Case 2(a). *n odd*, $p_n=+i$. Here

$$p_1 = ip_{n-1},\ p_2 = -ip_{n-2},\ p_3 = ip_{n-3}, \cdots , p_{n-3} = -ip_3,$$
$$p_{n-2} = ip_2,\ p_{n-1} = -ip_1$$

or

$$p_j = (-1)^{j+1}ip_{n-j};\ j = 1, 2, 3, \cdots , n-1.$$

Accordingly, an equation is a negative-reciprocal equation when it has the following properties: the coefficient of x^n and the constant term are $+1$ and $+i$, respectively; the quotient obtained by dividing the coefficient of a term occupying an odd position after x^n by the coefficient of the term occupying the corresponding odd position before the constant term is $+i$; the quotient found by dividing the coefficient of a term occupying an even position following x^n by the coefficient of the term occupying the corresponding even position preceding the constant term is $-i$.

In the present case the equation has $+i$ or $-i$ as a root according as $(n+1)/2$ is even or odd, respectively. Correspondingly, the degree of the equation can always be depressed by unity to the next lower and even degree by removing either the factor $x-i$ or the factor $x+i$. The resulting equation is also always a negative-reciprocal equation having, in general, complex coefficients. For illustration

$$\begin{aligned} x^9 &+ a_1x^8 + a_2x^7 + a_3x^6 + a_4x^5 + ia_4x^4 - ia_3x^3 + ia_2x^2 - ia_1x + i \\ &\equiv (x+i)\{x^8 + (a_1 - i)x^7 - [(1-a_2) + ia_1]x^6 \\ &\quad - [(a_1 - a_3) - (1-a_2)i]x^5 + [(1 - a_2 + a_4) + (a_1 - a_3)i]x^4 \\ &\quad + [(a_1 - a_3) - (1-a_2)i]x^3 - [(1-a_2) + ia_1]x^2 - (a_1 - i)x + 1\}. \end{aligned}$$

The special values $a_1=b_1+i$, $a_2=b_2+ib_1$, $a_3=b_3+ib_2$, and $a_4=b_4+ib_3$ are required to make all of the preceding coefficients real.

Case 2(b). *n odd, $p_n=-i$.* Then

$$p_1 = -ip_{n-1},\ p_2 = ip_{n-2},\ p_3 = -ip_{n-3},\ \cdots,\ p_{n-3} = ip_3,$$
$$p_{n-2} = -ip_2,\ p_{n-1} = ip_1$$

or

$$p_j = (-1)^j ip_{n-j};\ j = 1, 2, 3, \cdots, n-1.$$

Consequently, an equation is a negative-reciprocal equation when it possesses the following characteristics: the coefficient of x^n and the constant term are $+1$ and $-i$ respectively; the quotient obtained by dividing the coefficient of a term occupying an odd position after x^n by the coefficient of the term occupying the corresponding odd position before the constant term is $-i$; the quotient found by dividing the coefficient of a term occupying an even position following x^n by the coefficient of the term occupying the corresponding even position preceding the constant term is $+i$.

In the present case the equation has $+i$ or $-i$ as a root according as $(n+1)/2$ is odd or even, respectively. Correspondingly, the degree of the equation can always be decreased by one to the next lower and even degree by removing either the factor $x-i$ or the factor $x+i$. The resulting equation is also always a negative-reciprocal equation having, in general, complex coefficients. A typical example may be obtained by changing the signs before all the i's in the illustration with which case 2(a) closes.

The total number of different but complete negative-reciprocal equations of degree n (n odd, $p_n=+i$) in which the coefficients of the corresponding terms have equal moduli while the real units ± 1 and the pure imaginary i are independently permuted in every possible way consistent with the pairing required by the $(n-1)/2$ mutually independent equations of condition, is given by 2^{n-1}. Obviously this result also applies to the case in which $p_n=-i$ (n odd).

It is interesting to observe that the relation $x=\pm iy$ will transform any negative-reciprocal equation into an ordinary reciprocal equation (of the same degree), and vice versa. Let x_1 and x_2 be a pair of roots of a negative-reciprocal equation, then $x_1x_2=-1$. Also define y_1 and y_2 by the relations $x_1=\pm iy_1$ and $x_2=\pm iy_2$, in which the signs at the same level must be taken together. By multiplication $x_1x_2=i^2y_1y_2$, hence $y_1y_2=+1$. That is to say, y_1 and y_2 constitute a pair of roots of an ordinary reciprocal equation. Obviously, if all the coefficients of the equation to be transformed are real and non-zero, certain ones of the coefficients of the transformed equation will contain the pure imaginary as an explicit factor. For illustration, the complete negative-reciprocal equation

$$x^8+a_1x^7+a_2x^6+a_3x^5+a_4x^4-a_3x^3+a_2x^2-a_1x+1=0$$

transforms into the two following ordinary reciprocal equations

$$y^8\mp ia_1y^7-a_2y^6\pm ia_3y^5+a_4y^4\pm ia_3y^3-a_2y^2\mp ia_1y+1=0$$

in which it may be supposed that none of the a's involves i as a factor.

Any negative-reciprocal equation of degree n can be reduced to the form

$$\begin{aligned}(3)\quad x^{2m}+c_1x^{2m-1}+c_2x^{2m-2}+\cdots+c_mx^m+\cdots\\ +(-1)^{m-2}c_2x^2+(-1)^{m-1}c_1x+(-1)^m=0\end{aligned}$$

by operating upon it as indicated by the fifth column of the following table.

n	p_n	$n/2$	$(n+1)/2$	Factor removed	m	Constant term
even	$+1$	even			$n/2$, even	$+1$
even	-1	odd			$n/2$, odd	-1
even	$+1$	odd		x^2+1	$(n/2)-1$, even	$+1$
even	-1	even		x^2+1	$(n/2)-1$, odd	-1
odd	$+i$		even	$x-i$	$(n-1)/2$, odd	-1
odd	$+i$		odd	$x+i$	$(n-1)/2$, even	$+1$
odd	$-i$		even	$x+i$	$(n-1)/2$, odd	-1
odd	$-i$		odd	$x-i$	$(n-1)/2$, even	$+1$

The coefficients (c) may be real or complex, and they are linear functions of the coefficients of the original, unreduced negative-reciprocal equation.

It will now be shown that the degree of equation (3) can always be depressed by one-half, that is, to m. By dividing this equation by x^m and pairing both the extreme terms and the terms which occupy positions at equal distances from x^{2m} and the constant term, it is found that

$$\begin{aligned}&\left[x^m+\left(-\frac{1}{x}\right)^m\right]+c_1\left[x^{m-1}+\left(-\frac{1}{x}\right)^{m-1}\right]\\ (4)\qquad &+c_2\left[x^{m-2}+\left(-\frac{1}{x}\right)^{m-2}\right]+\cdots\\ &+c_{m-2}\left[x^2+\left(-\frac{1}{x}\right)^2\right]+c_{m-1}\left[x+\left(-\frac{1}{x}\right)\right]+c_m=0.\end{aligned}$$

Let

$$(5)\qquad u\equiv x+(-1)(x^{-1}),$$

and

$$(6)\qquad U_k\equiv x^k+(-1)^k(x^{-1})^k.$$

Then

$$\begin{aligned}U_k=u^k+ku^{k-2}&+\frac{k(k-3)}{2!}u^{k-4}+\frac{k(k-4)(k-5)}{3!}u^{k-6}\\ (7)\qquad &+\frac{k(k-5)(k-6)(k-7)}{4!}u^{k-8}+\cdots\\ &+\frac{k(k-r-1)(k-r-2)\cdots(k-2r+1)}{r!}u^{k-2r}+\cdots,\end{aligned}$$

and

$$(8)\qquad U_k=uU_{k-1}+U_{k-2}.$$

[See I. Todhunter's *Theory of Equations*, Art. 261 (1895).] Hence, definition (6) enables equation (4) to be written

$$(9)\qquad U_m+c_1U_{m-1}+c_2U_{m-2}+\cdots+c_{m-2}U_2+c_{m-1}U_1+c_m=0.$$

Formula (7) shows at once that equation (9) is of degree m in u, and this is tantamount to saying that the degree of equation (3) has been decreased to one-half its original value.

Let u_0 be a root of equation (9). Then definition (5) gives the quadratic

$$(10)\qquad x^2-u_0x-1=0.$$

The constant term of this equation shows that the product of its two roots equals -1, that is to say, either root is the negative-reciprocal of the other. Thus equation (10) pairs the related roots of equation (3).

Each of the roots obtained by equating to zero the factors given in the last six rows of the fifth column of the preceding table, namely $+i$ and $-i$, is the negative reciprocal of itself.

The following list of values of U_k will facilitate the reduction of negative-reciprocal equations of degree at most as high as sixteen.

$$\begin{aligned} U_1 &= u, & U_5 &= u^5 + 5u^3 + 5u, \\ U_2 &= u^2 + 2, & U_6 &= u^6 + 6u^4 + 9u^2 + 2, \\ U_3 &= u^3 + 3u, & U_7 &= u^7 + 7u^5 + 14u^3 + 7u, \\ U_4 &= u^4 + 4u^2 + 2, & U_8 &= u^8 + 8u^6 + 20u^4 + 16u^2 + 2. \end{aligned}$$

In conclusion attention may be called to the fact that an unlimited number of equations of transformation which will convert any rational algebraic equation into a negative-reciprocal equation of higher degree can be constructed in the following obvious manner. First write an irreducible negative-reciprocal equation having literal coefficients and y as the variable. Next replace all of the symbols for these coefficients by the single letter x, (or by $1/x$), and then solve for x. For illustration, put $a_1 = a_2 = x$ in

$$y^4 + a_1y^3 + a_2y^2 - a_1y + 1 = 0$$

to obtain

$$x = (1 + y^4)/[y(1 - y - y^2)].$$

ON THE CLASSIFICATION OF COLLINEATIONS IN THE PLANE

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1. *Introduction.* A collineation in the plane is a projective transformation of a plane of points into itself in such a way that points which are on a line are transformed into points which are on a line. Analytically, a collineation is defined by equations of the type

$$\rho x_i' = \sum a_{ij}x_j \qquad (i = 1, 2, 3), \tag{1}$$

where the a's are any numbers, real or complex, subject to the single restriction that the determinant, $\Delta = | a_{ij} |$, shall be different from zero. Here (x_1, x_2, x_3) are the homogeneous coordinates of a point $P(x)$ and (x_1', x_2', x_3') are the coordinates of the transformed point $P'(x')$.

If the a's are complex and if we interpret any triple of numbers, $(x_1, x_2, x_3) \neq (0, 0, 0)$, real or complex, as the coordinates of a point in a plane, there are, exclusive of the identity, five distinct types of collineations in the plane.[1] If, however, the collineation is *real*, i.e., the a's are real numbers, and if we confine our attention to *real* points (x_1, x_2, x_3), one of these types subdivides further, so that, exclusive of the identity, there are six types of real collineations in the plane.

[1] Cf. for example, the classifications in Veblen and Young, *Projective Geometry*, Vol. I, p. 106, p. 273; Winger, *Projective Geometry*, pp. 298–306; Woods, *Higher Geometry*, pp. 83–86.

This classification can be made from either the geometric or the algebraic standpoint.

Most of the American texts on analytic projective geometry adopt the geometric viewpoint and make the classification through the orientation of the fixed elements. Those, however, that do make an algebraic classification employ the theory of elementary divisors,[1] an elegant and powerful theory and one which is practically indispensable in case four or more variables are involved. But that theory is much more elaborate and complicated than is necessary for the problem under consideration. Moreover, the average student of projective geometry is totally unacquainted with it at that stage in his mathematical career. It is probably for the purpose of avoiding this extensive algebraic theory that many writers adopt the geometric viewpoint rather than the algebraic, the latter being, in the opinion of the writer, much the more satisfactory of the two.

In this paper we are interested primarily in an algebraic classification. However, no use is made of the theory of elementary divisors. On the contrary, the discussion presupposes a knowledge of only the simplest facts from the theory of equations with which every student of projective geometry should be familiar.

It is true that any instructor in projective geometry might work up for his class just such a classification as is made here, and doubtless many of them do so. Still, there should be some convenient place in the literature to which the instructor might refer his students before taking up the subject in class.

2. *The induced line transformation.* Suppose that the point x' runs along the line u' so that $\sum u_i' x_i' = 0$. We then have by (1)

$$\sum a_{ij} u_i' x_j = \rho \sum u_i' x_i' = 0$$

so that the corresponding point x runs along the line u, where

$$\sigma u_j = \sum a_{ij} u_i' \qquad (j = 1, 2, 3). \tag{2}$$

Hence, the point transformation (1) induces the line transformation (2).

3. *Fixed elements under a collineation.* A point x is transformed into itself by (1), or is a fixed point of the collineation, provided there exists a number $\rho \neq 0$, such that

$$\sum a_{ij} x_j = \rho x_i \qquad (i = 1, 2, 3),$$

or

$$\begin{aligned} (a_{11} - \rho)x_1 + a_{12}x_2 + a_{13}x_3 &= 0, \\ a_{21}x_1 + (a_{22} - \rho)x_2 + a_{23}x_3 &= 0, \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \rho)x_3 &= 0. \end{aligned} \tag{3}$$

[1] For example, both Veblen and Young, and Woods refer to Bôcher, *Introduction to Higher Algebra*, Chaps. XX and XXI, wherein is given an exposition of the theory of elementary divisors as a basis for an algebraic classification.

This system of equations will be satisfied by a set of values $(x_1, x_2, x_3) \neq (0, 0, 0)$ if, and only if, ρ be so chosen that the determinant

$$(4) \qquad D(\rho) = \begin{vmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{21} & a_{22} - \rho & a_{23} \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix}$$

vanishes.

The matrix of the coefficients of the system of equations (3) is called the *characteristic matrix* of the collineation (1) and is denoted by $A - \rho I$. The determinant $D(\rho)$ of this matrix is the *characteristic determinant* and the equation $D(\rho) = 0$ is the *characteristic equation* of the collineation. When expanded, this equation is

$$(5) \qquad D(\rho) \equiv -\rho^3 + \rho^2 \sum a_{ii} - \rho \sum A_{ii} + \Delta = 0,$$

where A_{ii} is the cofactor of a_{ii} in $\Delta = |a_{ij}|$.

Corresponding to each root ρ of (5) there exists at least one set of solutions of the system (3); i.e., at least one fixed point of the collineation.

Similarly, a line u will be a fixed line under the collineation provided there exists a number $\rho \neq 0$, such that

$$(6) \qquad \sum a_{ji} u_j = \rho u_i \qquad (i = 1, 2, 3).$$

These equations will be satisfied by a set of u's, not all zero, if, and only if, ρ be so chosen that the determinant of the coefficients in (6) vanishes. This is seen to be precisely the condition $D(\rho) = 0$.

THEOREM I. *Corresponding to each root of the characteristic equation $D(\rho) = 0$, of the collineation* (1), *there is at least one fixed point and one fixed line of the collineation.*

It is clear that the matrix of the system (6) is precisely the transpose of the matrix of the system (3). Hence, the ranks of the two matrices are the same. Since the number of linearly independent solutions of a system of homogeneous linear equations in n unknowns depends entirely on the rank of the matrix of the system,[1] we have the theorem:

THEOREM II. *Corresponding to any root ρ of the equation $D(\rho) = 0$, there are just as many linearly independent fixed lines as fixed points of the collineation.*

THEOREM III. *The rank of the matrix $A - \rho I$ is never less than* 2 *unless ρ is a multiple root of $D(\rho) = 0$.*

For if the rank of this matrix is less than 2, all the second order principal minors vanish, so that in addition to $D(\rho) = 0$, we have also

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[1] Bôcher, *loc. cit.*, Theorem 1, p. 50.

$$\rho^2 - (a_{11} + a_{22})\rho + A_{33} = 0,$$
$$\rho^2 - (a_{11} + a_{33})\rho + A_{22} = 0,$$
$$\rho^2 - (a_{22} + a_{33})\rho + A_{11} = 0.$$

On adding we get

$$3\rho^2 - 2\rho \sum a_{ii} + \sum A_{ii} = 0,$$

or

$$D'(\rho) = 0.$$

Hence, ρ is a multiple root of $D(\rho)=0$.

The rank of $A-\rho I$ will be zero, i.e., every element of the matrix will be zero, if and only if $a_{ij}=0$, $i\neq j$, and $a_{11}-\rho=a_{22}-\rho=a_{33}-\rho=0$, so that $a_{11}=a_{22}=a_{33}$. The transformation is then the identity:

$$\tau x_i' = kx_i \qquad (i = 1, 2, 3).$$

THEOREM IV. *The rank of the matrix $A-\rho I$ is never zero unless the collineation* (1) *reduces to the identity.*

Since the characteristic equation of the identical transformation has a triple root, we have the corollary:

COROLLARY. *The rank of $A-\rho I$ is never less than* 1 *unless the equation $D(\rho)=0$ has a triple root.*

4. *Incidence of fixed points and fixed lines.*

THEOREM V. *The same fixed point* (*line*) *cannot arise from two distinct roots of $D(\rho)=0$.*

For suppose that the point x arose from the two distinct roots ρ and σ, so that we would have as in (3):

$$\sum a_{ij}x_j = \rho x_i, \quad \sum a_{ij}x_j = \sigma x_i \qquad (i = 1, 2, 3).$$

On subtracting these two equations, member for member, we have

$$(\rho - \sigma)x_i = 0 \qquad (i = 1, 2, 3),$$

whence, since $\rho\neq\sigma$, $(x_1, x_2, x_3)=(0, 0, 0)$ contrary to supposition.

THEOREM VI. *A fixed line u arising from one root ρ of $D(\rho)=0$ always passes through a fixed point x arising from a root $\sigma\neq\rho$.*

For the coordinates of the fixed line satisfy the equations

$$\sum a_{ji}u_j = \rho u_i, \tag{6'}$$

and the coordinates of the fixed point satisfy the equations

$$\sum a_{ij}x_j = \sigma x_i. \tag{7}$$

Multiply (6′) through by x_i and sum as to i; similarly, multiply (7) through by u_i and sum as to i. We then obtain:

$$\sum a_{ji}u_jx_i = \rho \sum u_ix_i,$$
$$\sum a_{ij}u_ix_j = \sigma \sum u_ix_i.$$

Since the left hand members of these last two equations are identical, we have on subtracting

$$(\rho - \sigma) \sum u_ix_i = 0,$$

whence, since $\rho \neq \sigma$, $\sum u_ix_i = 0$.

THEOREM VII.[1] *If the matrix $A - \rho_1 I$ is of rank 2, the fixed point x and the fixed line u arising from the root ρ_1 will be incident if, and only if, ρ_1 is a multiple root of $D(\rho) = 0$.*

Let us choose the fixed point x arising from the root ρ_1 as the vertex (1, 0, 0) of our triangle or reference. It will then follow that

$$a_{11} = \rho_1,\ a_{21} = a_{31} = 0,$$

so that the matrix A is of the form

$$A = \begin{pmatrix} \rho_1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

The three characteristic roots of A are obviously ρ_1 and the two roots of

$$F(\rho) = \begin{vmatrix} a_{22} - \rho & a_{23} \\ a_{32} & a_{33} - \rho \end{vmatrix} = 0.$$

The coordinates of the fixed line u arising from the root ρ_1 may be taken as

$$u_1 : u_2 : u_3 = \begin{vmatrix} a_{22} - \rho_1 & a_{23} \\ a_{32} & a_{33} - \rho_1 \end{vmatrix} : \begin{vmatrix} a_{32} & a_{33} - \rho_1 \\ a_{12} & a_{13} \end{vmatrix} : \begin{vmatrix} a_{12} & a_{13} \\ a_{22} - \rho_1 & a_{23} \end{vmatrix}.$$

This line will pass through the point x (1, 0, 0) if, and only if, $F(\rho_1) = 0$, i.e., ρ_1 is a multiple root of $D(\rho) = 0$.

5. *The classification of collineations.* We are now in a position to classify collineations.

Type I. $D(\rho) = 0$ *has three distinct roots*, ρ_1, ρ_2, ρ_3.

By Theorem III, each matrix $A - \rho_i I$ is of rank 2, so that corresponding to each root ρ_i there is a single fixed point P_i and a single fixed line p_i of the collineation. Moreover, by Theorem VI, p_1 passes through P_2 and P_3, but, by

[1] My thanks are due to Professor J. W. Lasley, Jr. for a suggestion as to the proof of this theorem.

Theorem VII, does not pass through P_1; etc. The three fixed points are therefore the vertices of a proper triangle where p_i is the side opposite the vertex P_i. If we choose this triangle as the triangle of reference so that $x_1=0$, $x_2=0$ and $x_3=0$ are fixed lines, and (1, 0, 0), (0, 1, 0) and (0, 0, 1) are fixed points, it is easily shown that the equations of the collineation are[1]

$$\tau x_i' = \rho_i x_i \qquad (i = 1, 2, 3).$$

Next, suppose that the characteristic equation $D(\rho)=0$ has a double root ρ_1 and a simple root ρ_3. The matrix $A-\rho_3 I$ is of rank 2 by Theorem III. There is a separation of cases according as $A-\rho_1 I$ is of rank 2 or rank 1. We consider the latter type first.

Type II. *$D(\rho)=0$ has roots ρ_1, ρ_1, ρ_3; $A-\rho_1 I$ of rank* 1. Corresponding to the root ρ_1, there is an entire line p of fixed points and an entire pencil of fixed lines with vertex at P, say. Corresponding to the root ρ_3, there is a single fixed point Q and a single fixed line q, where, by Theorem VII, Q does not lie on q. Since by Theorem VI, q passes through every point arising from the root ρ_1, obviously $q\equiv p$. Since Q lies on every line arising from the root ρ_1, Q is the vertex of the pencil of these lines and $Q\equiv P$. The fixed elements therefore consist of a line p of fixed points and a pencil P of fixed lines, where P is not on p[2]. If we choose as the vertices of our triangle of reference the points $P(0, 0, 1)$ and two distinct points $R(1, 0, 0)$ and $S(0, 1, 0)$ on p, it is easily shown that the equations of the collineation are in the canonical form:[3]

$$\begin{aligned} \tau x_1' &= \rho_1 x_1 \\ \tau x_2' &= \rho_1 x_2 \\ \tau x_3' &= \rho_3 x_3. \end{aligned}$$

This is the so-called *homology*. If $\rho_3=-\rho_1$, we have the harmonic homology or reflexion.

Type III. *$D(\rho)=0$ has roots ρ_1, ρ_1, ρ_3; $A-\rho_1 I$ of rank* 2. Corresponding to the root ρ_1 there is a single fixed point P and a single fixed line p, where, by Theorem VII, P lies on p. Corresponding to the root ρ_3 there is a single fixed point Q and a single fixed line q, where Q does not lie on q. But by Theorem VI, P lies on q and Q lies on p. Hence P is the point of intersection of p and q.[4]

If we choose as vertices of our triangle of reference the points Q (0, 0, 1), P (1, 0, 0) and R (0, 1, 0), some point on q distinct from P, the equations of the collineation are easily shown to be:

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[1] Winger, *loc. cit.*, p. 300.

[2] That $P=Q$ is not a point of p follows also from Theorem V.

[3] Winger, *loc. cit.*, pp. 301–2.

[4] The point of intersection of two fixed lines of the collineation is obviously a fixed point of the collineation. Hence, p and q intersect in either P or Q; but we cannot tell which of these points lies at the intersection of p and q, unless we go back to Type I and invoke the principle of continuity, or else use some theorem such as VII.

$$\begin{aligned} \tau x_1' &= \rho_1 x_1 + k x_2 \\ \tau x_2' &= \qquad \rho_1 x_2 \\ \tau x_3' &= \qquad\qquad \rho_3 x_3. \end{aligned} \qquad (k \neq 0).$$

We consider finally the cases where the characteristic equation $D(\rho)=0$ has a triple root ρ, ρ, ρ. There is a separation of cases depending on the rank of $A-\rho I$, which may be 2, 1 or 0.

Type IV. *$D(\rho)=0$ has a triple root ρ; $A-\rho I$ of rank* 2. In this case there is a single fixed point P and a single fixed line p, where, by Theorem VII, P lies on p. If we choose P as the vertex (0, 0, 1) of our triangle of reference and p as the side $x_1=0$, the equations of the collineation are easily shown to be of the form:

$$\begin{aligned} \tau x_1' &= \rho x_1 \\ \tau x_2' &= a_{21} x_1 + \rho x_2 \\ \tau x_3' &= a_{31} x_1 + a_{32} x_2 + \rho x_3. \end{aligned} \qquad (a_{21} a_{32} \neq 0).$$

Type V. *$D(\rho)=0$ has a triple root ρ; $A-\rho I$ of rank* 1. Corresponding to the root ρ there is an entire line p of fixed points and an entire pencil P of fixed lines. Obviously, the vertex P of the pencil of fixed lines is a fixed point, so that P must be one of the points of p. This is the so-called *elation*. If we choose P as the vertex (0, 0, 1) and p as the side $x_1=0$ of our triangle of reference, the equations of the collineation reduce to the canonical form:

$$\begin{aligned} \tau x_1' &= \rho x_1 \\ \tau x_2' &= \qquad \rho x_2 \\ \tau x_3' &= k x_1 + \rho x_3. \end{aligned} \qquad (k \neq 0).$$

Type VI. *$D(\rho)=0$ has a triple root ρ; $A-\rho I$ of rank* 0. By Theorem IV, this type of collineation is the identity.

6. *Real collineations.* The above classification has been made on the assumption that any triple of numbers, $(x_1, x_2, x_3) \neq (0, 0, 0)$, real or complex, are to be interpreted as the coordinates of a point in the plane. In case we confine our attention to the geometry of reals, where the a's in (1) are real and where triples of real numbers only are to be interpreted as the coordinates of a point, the classification is somewhat different.

By reference to (3) and (6) it is obvious that when the a's are real, a fixed point x (line u) will be real if, and only if, ρ is real. Hence, since a real cubic has a multiple root only when all its roots are real, Types II, $\cdots$, VI arise in the case of real collineations just as before, with the understanding that *fixed* point (line) means *real* fixed point (line). Type I, however, subdivides into two types depending on whether $D(\rho)=0$ has *one* or *three* real roots.

Type I. *$D(\rho)=0$ has three distinct real roots.* In this case we have Type I as listed above where fixed point (line) is understood to mean real fixed point (line).

Type I′. $D(\rho)=0$ *has one real root and two complex roots.* Corresponding to the real root ρ_1 there is a single real fixed point P_1 and a single real fixed line p_1, where, by Theorem VII, p_1 does not pass through P_1. If now we choose P_1 as the vertex (1, 0, 0) and the line p_1 as the side $x_1=0$ of our triangle of reference, the collineation assumes the canonical form

$$\begin{aligned}\tau x_1' &= \rho_1 x_1\\ \tau x_2' &= \quad a_{22}x_2 + a_{23}x_3\\ \tau x_3' &= \quad a_{32}x_2 + a_{33}x_3,\end{aligned}$$

where the quadratic

$$F(\rho) = \rho^2 - (a_{22}+a_{33})\rho + a_{22}a_{33} - a_{23}a_{32} = 0,$$

has imaginary roots, i.e.,

$$(a_{22}-a_{33})^2 + 4a_{23}a_{32} < 0.$$

A Correction

Professor Morgan Ward has kindly called my attention to an unintentional misstatement of Theorem I of my note "*On a Certain Transformation of Infinite Series*" in the April number of this Monthly (vol. 40, p. 226). It should read as follows:

If $\lim_{n\to\infty} nu_n = l$ *exists, then the two series* (U) *and* (V) *both diverge, if* $l\neq 0$. *If* $l=0$, *the convergence of one implies that of the other, and the two series have the same sum.*

J. A. Shohat

QUESTIONS, DISCUSSIONS, AND NOTES

Edited by R. E. Gilman, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A Certain Class of Trigonometric Integrals

By Morgan Ward, California Institute of Technology

1. In the December issue of the Monthly[1] Professor Uhler has raised some questions about the functions defined by the indefinite integrals

$$\int \begin{matrix}\cos\\ \sin\end{matrix}(\cot\theta)\begin{matrix}\cos\\ \sin\end{matrix}\theta d\theta$$

[1] American Mathematical Monthly, vol. 39 (1932), p. 589.

which I propose to answer here. Let us define four functions of the real variable θ:

$$K_1(\theta) = \int_0^\theta \cos(\tan\phi)\cos\phi d\phi, \quad K_2(\theta) = \int_0^\theta \cos(\tan\phi)\sin\phi d\phi,$$

$$K_3(\theta) = \int_0^\theta \sin(\tan\phi)\cos\phi d\phi, \quad K_4(\theta) = \int_0^\theta \sin(\tan\phi)\sin\phi d\phi.$$

The integrals under discussion are immediately expressible in terms of these functions; for example,

$$\int \cos(\cot\theta)\sin\theta d\theta = \text{const.} - K_1\left(\frac{\pi}{2} - \theta\right).$$

We shall show that the functions $K(\theta)$ are not expressible in finite terms by any simple known functions. However, in the range $-\pi/2 < \theta < \pi/2$, they are representable by convergent series of which

$$\begin{aligned}(1.1)\quad K_1(\theta) = {} & \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{1}{6}\cos\frac{3\theta}{2}\left(2\sin\frac{\theta}{2}\right)^3 - \frac{1}{8}\sin 2\theta\left(2\sin\frac{\theta}{2}\right)^4 \\ & + \frac{1}{30}\cos\frac{5\theta}{2}\left(2\sin\frac{\theta}{2}\right)^5 - \frac{1}{36}\sin 3\theta\left(2\sin\frac{\theta}{2}\right)^6 \\ & + \frac{2}{45}\cos\frac{7\theta}{2}\left(2\sin\frac{\theta}{2}\right)^7 + \frac{1}{30}\sin 4\theta\left(2\sin\frac{\theta}{2}\right)^8 + \cdots\end{aligned}$$

may be quoted as typical. I shall give recursion formulas by which the numerical coefficients in these series may be calculated, and an estimate of the error terms. It turns out that the convergence is fairly good in the range $-\pi/4 \leqq \theta \leqq \pi/4$.

There also exist asymptotic expansions giving the behavior of the functions near $\pi/2$ and $-\pi/2$ which limitations of space forbid my developing here.

2. We begin by observing that from our defining relations, it follows that[1]

$$K_1(\theta+\pi) = -K_1(\theta); \quad K_2(\theta+\pi) = 2K_2\left(\frac{\pi}{2}\right) - K_2(\theta);$$

$$K_3(\theta+\pi) = 2K_3\left(\frac{\pi}{2}\right) - K_3(\theta); \quad K_4(\theta+\pi) = -K_4(\theta).$$

We may therefore assume that $-\pi/2 \leqq \theta \leqq \pi/2$.

If we let

$$P(\theta) = K_1(\theta) + iK_2(\theta), \quad Q(\theta) = K_3(\theta) + iK_4(\theta),$$

we obtain immediately from (1.1) the integral formulas

[1] The constants $K_2(\pi/2)$, $K_3(\pi/2)$ may be expressed as infinite integrals by writing $\tan\phi = x$, and these integrals may be evaluated in terms of Bessel functions, and related expressions.

$$P(\theta) + iQ(\theta) = \int_0^\theta \cos(\tan\phi)e^{i\phi}d\phi + i\int_0^\theta \sin(\tan\phi)e^{i\phi}d\phi = \int_0^\theta e^{i(\tan\phi+\phi)}d\phi,$$

$$P(\theta) - iQ(\theta) = \int_0^\theta \cos(\tan\phi)e^{i\phi}d\phi - i\int_0^\theta \sin(\tan\phi)e^{i\phi}d\phi = \int_0^\theta e^{-i(\tan\phi-\phi)}d\phi.$$

We shall now introduce two functions of a complex variable in terms of which these last integrals are easily expressible.

3. Let Z denote a complex variable. Consider the two functions

$$(3.1)\qquad F(Z) = \int_1^Z \exp\left(\frac{1-z^2}{1+z^2}\right)dz,\quad G(Z) = \int_1^Z \exp\left(\frac{z^2-1}{z^2+1}\right)dz.$$

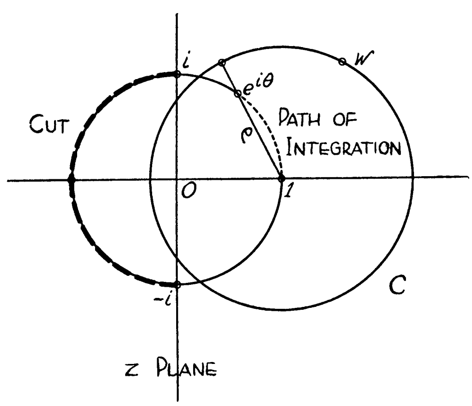
Then if we join the points $z=i$ and $z=-i$ by a cut which it is convenient to take along the left half of the unit circle (see figure), the reader may verify that

(a) The functions $F(Z)$ and $G(Z)$ are one-valued and analytic at all points Z of the cut z-plane save the point at infinity where each has a pole of order one, and their value at any point Z is independent of the path of integration joining 1 and Z.

(*b*) In particular, both functions are one-valued and analytic in the interior of a circle of radius $\sqrt{2}$ about the point $Z=1$.

(c) Both functions have essential singularities and logarithmic branch points at $z=i$ and $z=-i$, but remain finite as we approach these points along the right half of the unit circle.

(*d*) Upon a circle C with centre 1 and radius $\rho<\sqrt{2}$,



$$|F(Z)| \leqq \rho \exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)},\qquad |G(Z)| \leqq \rho \exp\frac{\sqrt{\rho^4+4}-\rho^2}{2(2-\rho^2)}.$$

4. Now let $Z=e^{i\theta}$, $-\pi/2<\theta<\pi/2$, be a point on the right half of the unit circle. Then if in the integrals (3.1) we choose for our path of integration the arc of the unit circle from 0 to θ, we obtain on writing $e^{i\phi}$ for z and reducing,

$$F(e^{i\theta}) = i\int_0^\theta e^{-i(\tan\phi-\phi)}d\phi, \quad G(e^{i\theta}) = i\int_0^\theta e^{i(\tan\phi+\phi)}d\phi.$$

On combining these results with the formulas of sections 1 and 2, we find that

$$\begin{aligned} K_1(\theta) &= \text{Real part of } \frac{F(e^{i\theta}) + G(e^{i\theta})}{2i}, \\ K_2(\theta) &= \text{Imaginary part of } \frac{F(e^{i\theta}) + G(e^{i\theta})}{2i}, \\ K_3(\theta) &= \text{Real part of } \frac{F(e^{i\theta}) - G(e^{i\theta})}{2}, \\ K_4(\theta) &= \text{Imaginary part of } \frac{F(e^{i\theta}) - G(e^{i\theta})}{2}. \end{aligned} \tag{4.1}$$

It is clear from these formulas that the function-theoretic nature of the $K(\theta)$ is determined by that of $F(Z)$ and $G(Z)$.

5. The nature of the functions $F(Z)$ and $G(Z)$ may be best seen from the differential equations which they satisfy. From formula (3.1)[1]

$$\frac{dF}{dz} = \exp\frac{1-z^2}{1+z^2}; \qquad \frac{dG}{dz} = \exp\frac{z^2-1}{z^2+1}.$$

Hence differentiating logarithmically, we see that

$$(z^2+1)^2\frac{d^2F}{dz^2} + 4z\frac{dF}{dz} = 0; \ (z^2+1)^2\frac{d^2G}{dz^2} - 4z\frac{dG}{dz} = 0. \tag{5.1}$$

Consider the differential equation for $F(z)$. If we make the substitution $w = z^2/(z^2+1)$, this differential equation becomes

$$w(1-w)\frac{d^2F}{dw^2} - (2w^2 - w - \tfrac{1}{2})\frac{dF}{dw} = 0. \tag{5.3}$$

The equation has 0 and 1 for regular points and ∞ for an irregular point. Now drawing upon the results of the theory of second order linear differential equations,[2] we see that if (5.3) is regarded as obtained by confluence from a differential equation with only regular points, the initial equation must have had five or more regular points. By actual trial, we find it is impossible to derive (5.3) from a differential with exactly five regular points. But all of the elementary functions of mathematical physics may be derived as solutions of confluent

[1] For convenience in printing, we hereafter write z for Z.

[2] See for example Whittaker and Watson, *Modern Analysis* Chap. X; or Ince, *Ordinary Differential Equations*, Chap. XX.

forms of such an equation. Hence $F(z)$ cannot be expressed in finite form by means of such functions.

Neither can $F(z)$ be expressed by means of elliptic integrals of the first or second kinds, for such integrals have no essential singularity in any part of the plane. A precisely similar argument holds for $G(z)$.

6. We are thus driven to seek series representations of $F(z)$ and $G(z)$ in the range in which we are interested. One such series is immediately obvious; namely an expansion about the point $z=1$ in ascending powers of $z-1$.

We have by Taylor's theorem

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!}(z-1)^n, \quad G(z) = \sum_{n=0}^{\infty} \frac{G^{(n)}(1)}{n!}(z-1)^n,$$

the radius of convergence of both series being $\sqrt{2}$ in accordance with section 3, (b), (c). On writing $z=e^{i\theta}$, we find that $z-1=2e^{(\pi+\theta)i/2}\sin(\theta/2)$. Hence

$$(6.1)\qquad \begin{aligned} F(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!} e^{ni/2(\pi+\theta)}\left(2\sin\frac{\theta}{2}\right)^n, \\ G(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{G^{(n)}(1)}{n!} \alpha^{ni/2(\pi+\theta)}\left(2\sin\frac{\theta}{2}\right)^n, \end{aligned} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Since $F(z)$ and $G(z)$ are real when z is real and greater than -1, the constants $F^{(n)}(1)$ and $G^{(n)}(1)$ in (6.1) are all real. Hence on combining these formulas with (4.1), we find that

$$(6.2)\quad K_1(\theta) = \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{F^{(n)}(1)+G^{(n)}(1)}{n!}\sin\frac{n}{2}(\pi+\theta)\left(2\sin\frac{\theta}{2}\right)^n, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

with similar formulas for $K_2(\theta)$, $K_3(\theta)$ and $K_4(\theta)$.

To calculate the constants $F^{(n)}(1)$ and $G^{(n)}(1)$, we differentiate the equations (5.1) $n+2$ times and set $z=1$. We thus obtain the recursion formulas

$$(6.3)\qquad \begin{aligned} F^{(n+4)}(1) = &-(2n+5)F^{(n+3)}(1)-(n+2)(2n+3)F^{(n+2)}(1) \\ &-(n+2)(n+1)nF^{(n+1)}(1)-\frac{(n+2)(n+1)n(n-1)}{4}F^{(n)}(1); \\ G^{(n+4)}(1) = &-(2n+3)G^{(n+3)}(1)-(n+2)(2n+1)G^{(n+2)}(1) \\ &-(n+2)(n+1)nG^{(n+1)}(1)-\frac{(n+2)(n+1)n(n-1)}{4}G^{(n)}(1). \end{aligned}$$

On setting $n=-2, -1, 0, 1, \cdots$ in these formulas, we find from (5.2) that[1]

[1] These coefficients have been checked from (3.1) by expanding $\exp\pm(z^2-1)/(z^2+1)$ in ascending powers of $z-1$ up to the terms of order $(z-1)^7$, and integrating term by term.

$F^{(0)}(1) = 0$; $F^{(1)}(1) = 1$; $F^{(2)}(1) = -1$; $F^{(3)}(1) = 2$; $F^{(4)}(1) = -4$; $F^{(5)}(1) = 4$; $F^{(6)}(1) = 34$; $F^{(7)}(1) = -374$; $F^{(8)}(1) = 2498$.

$G^{(0)}(1) = 0$; $G^{(1)}(1) = 1$; $G^{(2)}(1) = 1$; $G^{(3)}(1) = 0$; $G^{(4)}(1) = -2$; $G^{(5)}(1) = 4$; $G^{(6)}(1) = 6$; $G^{(7)}(1) = -74$; $G^{(8)}(1) = 190$.

On substituting these values in the first nine terms of (6.2), we obtain the series for $K_1(\theta)$ given in section 1.

7. Let $z = e^{i\theta}$ be a fixed point on the unit circle to the right of the y axis, and C a circle of radius $\rho < \sqrt{2}$ about the point $z = 1$ including the point $e^{i\theta}$. Then

$$F(e^{i\theta}) = \frac{1}{2\pi i}\int_C \frac{F(w)dw}{w - z}, \tag{7.1}$$

where w denotes a complex current co-ordinate upon the circle C.

From the identity

$$\frac{1}{w - z} = \frac{1}{w - 1} + \frac{z - 1}{(w - 1)^2} + \cdots + \frac{(z - 1)^n}{(w - 1)^{n+1}} + \frac{(z - 1)^{n+1}}{(w - 1)^{n+1}(w - z)},$$

we obtain

$$F(z) = c_0 + c_1(z - 1) + \cdots + c_n(z - 1)^n + \Re_n$$

where $c_k = F^{(k)}(1)/k!$ $(k = 0, \cdots, n)$ and

$$\Re_n = \frac{(z - 1)^{n+1}}{2\pi i}\int_C \frac{F(w)dw}{(w - 1)^{n+1}(w - z)}. \tag{7.2}$$

Now

$$|z - 1| = 2\sin\frac{|\theta|}{2}, \quad |w - 1| = \rho, \quad |w - z| \geqq \rho - |z - 1| = \rho - 2\sin\frac{|\theta|}{2}$$

and by 3 (d),

$$|F(w)| \leqq \rho \exp\frac{\sqrt{\rho^4 + 4} + \rho^2}{2(2 - \rho^2)}.$$

Hence from (7.2),

$$|\Re_n| \leqq \left(\frac{2\sin\frac{|\theta|}{2}}{\rho}\right)^{n+1} \frac{\rho^2}{\rho - 2\sin\frac{|\theta|}{2}} \exp\frac{\sqrt{\rho^4 + 4 + \rho^2}}{2(2 - \rho^2)}.$$

The inequality for the remainder in the series for $G(z)$ is precisely the same, save that the numerator of the exponential is replaced by $\sqrt{\rho^2 + 4} - \rho^4$.

A somewhat better inequality when n is large may be obtained by integrating the right side of (7.2) by parts before obtaining the dominant. It gives

$$|\mathfrak{R}_n| \leqq \frac{1}{n}\left(\frac{2\sin\frac{|\theta|}{2}}{\rho}\right)^{n+1} \frac{2\rho^2\left(\rho+\sin\frac{|\theta|}{2}\right)}{\left(\rho-2\sin\frac{|\theta|}{2}\right)^2} \exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)}.$$

If we take $\rho^2=3/2$, $\theta=\pi/4$ in the first inequality, we obtain

$$|\mathfrak{R}_n| < (\tfrac{5}{8})^{n+1}\times 51.08 = .0042 \text{ for } n = 19.$$

The second inequality gives

$$|\mathfrak{R}_n| < \frac{1}{n}\left(\frac{5}{8}\right)^{n+1} 438 = .0019 \text{ for } n = 19.$$

If θ is quite small, we may take $\rho=1$, obtaining

$$|\mathfrak{R}_n| < \left(2\sin\frac{|\theta|}{2}\right)^{n+1}\frac{e^{(\sqrt{5}+1)/2}}{1-2\sin\frac{|\theta|}{2}} \text{ for } F(z)$$

and

$$|\mathfrak{R}_n| < \left(2\sin\frac{|\theta|}{2}\right)^{n+1}\frac{e^{(\sqrt{5}-1)/2}}{1-2\sin\frac{|\theta|}{2}} \text{ for } G(z).$$

Homogeneous Linear Differential Equations of the Second Order

By T. C. Benton, Pennsylvania State College

1. *Introduction.* After teaching the subject of linear differential equations to sophomore students a number of times, it has seemed to the author that the customary methods of assuming the correct answers and then verifying them are highly unsatisfactory from a pedagogical viewpoint. The student always asks how the form of solution used was obtained in the first place. Also the better student is left with the feeling, that except for a lucky guess, there is no way to obtain the solution of similar problems. It is the purpose of this development of the subject to present a method in which every step is forced—a method in which there is no guesswork at all. The actual work is all of well known character but the fact that the general methods of the higher theory of differential equations work out in such a simple way for the elementary cases seems worthy of attention

2. *General Theory* of the solution of:

$$(A)\qquad \frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0,$$

P, Q being functions of x or constants.

Definition—If $y=u(x)$, $y=v(x)$, $y=w(x)$, etc. are solutions of (A) such that $c_1u+c_2v+c_3w+\cdots\neq 0$ for all values of x unless $c_1=c_2=c_3=\cdots=0$, we will say that $u(x)$, $v(x)$, $w(x)$, etc. are independent solutions. By this definition, if u and v are independent solutions neither u nor v is identically zero.

Theorem 1.—If u, v, w, $\cdots$ are solutions of (A) the function $y=c_1u+c_2v+c_3w+\cdots$ is also a solution.

This follows by direct substitution in the equation.

Theorem 2.—If u and v are two independent solutions of (A), and w is any solution not identically zero, there must be two constants k_1 and k_2 not both zero, such that $w\equiv k_1u+k_2v$.

Proof—Let u and v be two independent solutions of (A). Then,

$$u''+Pu'+Qu=0 \tag{1}$$

$$v''+Pv'+Qv=0, \tag{2}$$

where the primes denote differentiation with respect to x. Multiplying (1) by v and (2) by u and subtracting we obtain:

$$u''v-uv''+P(u'v-uv')=0. \tag{3}$$

Now by direct differentiation $d\,(u'v-uv')/dx\ =u''v-uv''$, so (3) becomes $d\,(u'v-uv')/dx=-P(u'v-uv')$. Then since, by the independence of u and v, $uv'-u'v\not\equiv 0$, we have

$$\frac{d(u'v-uv')}{(u'v-uv')}=-Pdx.$$

Solving this we obtain:

$$u'v-uv'=Ce^{-\int Pdx},\quad C\neq 0, \tag{4}$$

which gives a relation between any two independent solutions and their derivatives.

By a similar process, if we assume that $u'w-uw'\not\equiv 0$, we obtain:

$$u'w-uw'=De^{-\int Pdx},\quad D\neq 0; \tag{$\bar{4}$}$$

and if we admit the case $D=0$, $(\bar{4})$ holds for any pair of solutions. Hence multiplying (4) by D and $(\bar{4})$ by C and subtracting it follows that

$$(Dv-Cw)u'-(Dv'-Cw')u=0.$$

If $Dv-Cw\equiv 0$, the conclusion of the theorem is satisfied with $k_1=0$, and $k_2=D/C$. If $Dv-Cw\not\equiv 0$, we can divide by $(Dv-Cw)u$, obtaining

$$\frac{u'}{u}=\frac{Dv'-Cw'}{Dv-Cw}\,.$$

Hence $\log u+\log k=\log(Dv-Cw)$; but then, since $C\neq 0$, $w=(-k/C)u+(D/C)v$, which proves the theorem for this case.

3. *Constant Coefficients*. Let the constants p, q replace the functions P, Q, giving the equation:

$$u''+pu'+qu=0. \tag{B}$$

In this equation let $z=u'/u$. Then $u'=zu$, $u''=z'u+zu'=z'u+z^2u$, so (B) becomes

$$(z'+z^2+pz+q)u=0. \tag{5}$$

If u is not identically zero, we must have $z'=-(z^2+pz+q)$. This first order equation might be solved by separating the variables; but as the different cases of integrating dz/z^2+pz+q are somewhat complicated, a much simpler attack is to notice that a constant value for z means $z'=0$ and therefore $z^2+pz+q=0$. Now this equation has two roots if we admit complex numbers, so there are always two constants $z=k_1$, $z=k_2$, which satisfy (5). But for these values of z, we obtain

$$u=C_1e^{k_1x} \text{ or } u=C_2e^{k_2x}.$$

Case 1: Suppose $k_1\neq k_2$. We will now show that the two solutions obtained for u are independent. For if not, let C and D be constants not both zero such that $Ce^{k_1x}+De^{k_2x}=0$ for all values of x. We have for $x=0$, $C+D=0$; and for $x=1$, $Ce^{k_1}+De^{k_2}=0$. These two equations imply that $C=0$ and $D=0$. As this contradicts our premise, the two solutions must be independent.

It then follows that the general solution of (B) is by theorem 1,

$$u=C_1e^{k_1x}+C_2e^{k_2x}.$$

Case 2: Suppose $k_1=k_2$. Then we have $p^2=+q$ and $k_1=k_2=-p/2$. By equation (4) we have, for independent solutions u and v,

$$u'v-uv'=e^{-\int p\,dx}.$$

Using $u=Ce^{-(p/2)x}$, we have

$$-(p/2)Ce^{-(p/2)x}\cdot v-Ce^{-(p/2)x}\cdot v'=e^{-px+h}.$$

Since h is arbitrary we can choose it so that $e^h=C$. Then dividing by $-e^{-(p/2)x}$, the result is $v'+(p/2)v=-e^{-(p/2)x}$. Applying the usual formula to solve this linear equation of the first order we obtain $v=Ke^{-(p/2)x}-xe^{-(p/2)x}$. Since we wish only the simplest solution which is independent of $e^{-(p/2)x}$ we can take $K=0$ without any loss of generality and use $v=-xe^{-(p/2)x}$ as the other solution, proving the independence by much the same method as in case 1.

Hence in this case the general solution is:

$$u=e^{-(p/2)x}(C_1+C_2x).$$

4. *Comments*. The equation (4) of theorem 2 part 1 is the relation which is

usually called Abel's Identity. Its use in the case of equal roots seemed to make students feel that there was much more reason for the form of the second particular solution.

There is, of course, no theoretical objection on a logical ground to the usual presentation but it seems to the author that each step in the proofs here given is almost forced upon one, and that this is a big factor in giving the student assurance that there is nothing amiss with his work.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

Vector Analysis. By H. B. Phillips. New York, John Wiley & Sons, 1933. viii +236 pages. $2.50.

This book gives an account of three dimensional vector analysis with many applications to electromagnetic theory and hydrodynamics. The treatment is sound and careful, and each of the first eight chapters carries a score or more of exercises and problems. Chapter 9 is devoted to the theory of the retarded potential and chapter 10 to a rather outmoded account of linear vector functions (dyadics, etc.) There is no particular claim to novelty either in subject matter or in treatment, and the book will undoubtedly be useful to students.

However, in writing this evaluation, the reviewer cannot forget a review by Clifford of a book of Booth's in which he referred to the author as serenely indifferent to the new-fangled methods which were tending to disturb the complacency of the older British School. So many interesting things have happened in the field of vector analysis since, say 1913, when the present book might well have been written, that it seems a shame not to let the student know about them. But whilst expressing this opinion, we recognize that we have been asked to tell about a book which *has* been written, and not about one which *might* be written.

The present actual book may be recommended as one of the best accounts of the vector analysis of Gibbs, and of its applications, to problems of mathematical physics.

F. D. MURNAGHAN

The Theory of Ruled Surfaces. By W. L. Edge. Cambridge University Press, 1931. x+324 pp. $7.00.

This work on the theory of algebraic ruled surfaces in Euclidean space of three dimensions, while not introducing any methods not previously used, has made use of two of the most important of these methods in conjunction to classify the quartic, quintic, and sextic ruled surfaces. No such classification so far published has been shown to be complete, and this enumeration is no excep-

tion, in spite of the author's claim. In fact, surfaces other than those listed have been mentioned by other writers; in particular, for sextic surfaces, compare the series of papers of Virgil Snyder in the American Journal of Mathematics, vol. 25 (1903) and vol. 27 (1905). This fact does not mean, however, that the work fails to be a real contribution to mathematical literature; for, to quote from the author's preface, "there exists at present no work, easily accessible to English readers, which tests the application of the general ideas here employed in anything like the same detail. One might mention especially the use of higher space and the principle of correspondence, and these two ideas are vital and fundamental in all modern algebraic geometry."

In the first chapter, the author recalls to the reader the salient points of space curve and surface theory, and discusses the theory of correspondence of Chasles-Cayley-Brill, which later is of such great value. He also outlines the two methods which he intends to use in classifying ruled surfaces. The first of these employs the Klein representation of lines in space as points of a quadric four-dimensional hypersurface Ω in five-space. Thus a ruled surface f is represented by a curve C on Ω, and the properties of f are related to properties of C. The second method, one which works equally well for ruled surfaces in n-space, is to consider the given surface as the projection of a normal ruled surface in a space of higher dimensionality. This latter method is particularly valuable for the study of rational surfaces, and often aids in setting up the given surfaces as the totality of lines joining corresponding points of two directrix curves. Here the principle of correspondence comes into play. In the last analysis, the classification is made according to the nature of the double curve and the bitangent developable.

The quartic, quintic, and sextic varieties are treated in detail in separate chapters, and one chapter is devoted to the developables. The book proper closes with a classification table summarizing the author's results, which is followed by a note concerning the intersections of two curves on a ruled surface.

It seems to the reviewer that an index would have added to the value of the book as a work of reference, preferably a general index, but at least a page reference list in connection with the tables at the close of the work.

The book is well written and well printed, and has the decided advantage of being one of the first compilations in one volume of the outstanding known facts in this interesting field.

ROBIN ROBINSON

Numerology. By E. T. Bell. New York, The Century Co., 1933. vii+187 pages. $2.00.

In order that the meaning of the term *numerology*, or number-mysticism, may be clear at the outset, the author in Chapter I gives three illustrations: *Hollywood in the Middle Ages*, concerning (as he says) Hollywood and theology; *Vibrations, Bulls and Bears*, featuring a stock broker who changed his name to harmonize with that "inner and insensible number" of which his "vibrations"

Solid Geometry. By W. H. Macaulay. Cambridge University Press, 1930. xii+303 pages. $4.75.

The book is a text on solid analytic geometry, using rectangular coordinates, considering only real loci, and making no use of the notion of points at infinity. Besides the usual topics through quadric surfaces, there are chapters on twisted curves and developables, the general theory of surfaces, spherical trigonometry; and then four chapters for the purpose of connecting the theory of solid analytics with some of its applications, namely moments of inertia, strain, stress, and vector distributions. There is also an appendix containing a few miscellaneous notes and examples.

The author uses his own ideas in places. When classifying the quadric surfaces, for instance, he first forms all quadrics of revolution by rotating the various conics about their axes, then subjects them to a homogeneous strain, then to a limiting process which carries a system of parallel elliptic or hyperbolic sections into parabolas. This process gives us a conception of how the surfaces look. He also takes the general equation of second degree and simplifies it by rotation and translation. Again, in representing a vector distribution, instead of picturing an arrow of determined length and direction emanating from each point of space, he pictures space as filled with curved lines whose direction gives the direction of the vector, and so packed that the number per unit area cutting a small piece of surface equals the component of the vector perpendicular to the surface. He has coined a convenient word, "conformable," to denote the relation between two triads of rectangular axes which can be made to coincide by a continuous motion, without a reversal of direction. We had been saying, "both right-handed, or both left-handed." A knowledge of calculus through partial differentiation and Taylor's series is presupposed.

The style of the book is decidedly English. The exposition is very lucid and scholarly, and the text free from errors. There are few figures, and no special devices for catching the eye and calling attention to the main facts and formulas. The equations are not even numbered for later reference. I should say that the book would not be so good as a reference work, but would be excellent for cultural purposes.

D. F. BARROW

Linguistic Analysis of Mathematics. By A. F. Bentley. Bloomington, Indiana, Principia Press, Inc., 1932. xii+315 pages. $3.00.

The language of mathematics here treated includes not merely mathematical symbols but also "those immediately surrounding forms of expression and assertion through which the symbols are developed, communicated and interpreted." Language, with all of its imperfections, affords the only means of communicating a critical analysis of language itself (p. viii). "Development must therefore be precarious: it must be in part impressionistic . . . . Only as an island in this sea of linguistic confusion may a small region of precision be established."

The author insists that the foundations of mathematics must be sought in mathematics itself and that no reliance should be placed on any external source, such as logic for instance (pp. 166, 308), since (p. 8) "mathematics is the safest and most certain knowledge the world possesses or thus far has had prospect of possessing." The author realizes (p. 49) that "the reader whose habits of work require him to commence with firm definition and classification will find herein nothing but disappointment." His point of view is strictly empirical. He does not try to give satisfaction to the mathematician (see p. 60); he seeks rather to disturb him in the hope of uncovering "possible seeds of fertility."

There is too much lack of precision in the book for the reviewer to be able to extract from it any very useful definitive results. The author himself speaks of it (p. 311) as "a record of exploration, not an achieved formulation." The most clearly written part is Chapter XI (pp. 181–211) on the denumerability of decimals; but the principal part of this chapter is demonstrably invalid, in particular that part in which the author undertakes to show that the totality of decimals is denumerable. A certain satisfaction with vagueness (see p. 248 for instance) characterizes a considerable part of the exposition. The reviewer believes that the exploration should have been carried to a stage of more definite achievement before publication. The work is too tentative in character to call for much attention from other thinkers. Unless the author himself can more fully justify his method and procedure by results achieved he is not entitled to hope that it will have any very useful effect upon mathematical thinking.

R. D. CARMICHAEL

Economic Control of Quality of Manufactured Product. By W. A. Shewhart. New York, D. Van Nostrand Company, Inc., 1932. viii+501 pages. $6.50.

Upon reading the title of this book the natural question is, "Why review it in a mathematical journal?" It is true that the book is not written primarily for mathematicians nor for students of mathematics but for the manufacturer and the engineer. Nevertheless, it is interesting to the mathematician be he pure or applied. No matter how often we may tell the story of the mathematician who took great pleasure in asserting that his own special interest in mathematics had no possible applications, we take more or less pride in saying that sooner or later any branch of mathematics will find an application.

What is this problem of economic control? To answer the question from the book, let us consider a fountain pen point. A maker of such pen points turns out thousands per year. He has in his mind a point which he calls standard. Then he tries to make pens which conform to this standard. This seems a simple problem, but we all know the variability of pens of the same make, price and style as well as the variability of the hands of the persons using them. Unknown or chance causes have their effect. If we cannot make all pens to the exact pattern of the standard pen, the next best thing is to know how a given shipment may vary from this standard. The manufacture of an article is said to be controlled

when we can predict the variations from the standard which is set up—that is, when we can state the probability that the measure of quality will fall within given limits. Put more specifically, each maker of fountain pen points will have associated with his pen certain data, which put into geometrical form is the ordinary frequency histogram.

The structure of the book has mathematical form, in that three postulates underlie its theory of control. In shortened form these postulates are

1. All chance systems of causes are not alike.
2. Constant systems of chance causes exist.
3. Assignable causes of variation may be found and eliminated.

By weeding out assignable causes of variation the product is considered as controlled and the advantages to the manufacturer are

1. Reduction in the cost of inspection.
2. Reduction in the cost of rejection.
3. Attainment of maximum benefits from quantity production.
4. Attainment of uniform quality even though the inspection test is destructive.
5. Reduction in tolerance limits where quality measurement is indirect.

The book is made up of seven "Parts" and appendices. The above is a short summary of Part I which is introductory. In Part II the word "quality" used in the introduction is discussed and defined. The concept of variability used in the definition of quality leads naturally to the main features of this Part, which are those of a good text book on the fundamentals of statistics, taking up the basic notions of grouping, measures of central tendency, dispersion, correlation, skewness, and in general, the methods of reducing large numbers of observations of quality to a few simple statistics which contain the essential information.

In Part III the relations with the theory of probability are taken up. Statistical laws are stated to be the frequency distributions arising from the general law of large numbers. Various systems of frequency distributions are discussed. An interesting idea is that of "Maximum Control," which is defined as the condition reached when the chance fluctuations in a phenomenon are produced by a constant complex of a large number of chance causes in which no cause produces a predominating effect. This notion brings to mind the Galton board and the next thought is on normality, but the author takes pains to show that normality of distribution is not a necessary condition for maximum control. "Most distributions exhibiting control have been found to be sufficiently near normal to be fitted by the first two terms of the Gram-Charlier series." This point is interesting to the reviewer, for at first glance it would appear that as a prescribed standard approached perfection the distribution would develop considerable skewness.

Part IV is a readable chapter on sampling written from the standpoint of the statistician who must look upon the distribution function of a statistic as a working tool. The chapters discuss the usual statistics including the correlation coefficient. Much of the very recent advancement which has been made in

sampling theory is summarized. The establishment of tolerance ranges and the setting up of standard of quality are shown in Part V to be problems based in general on the three statistics, arithmetic mean, standard deviation and relative frequency of defectives.

The determination of lack of control, that is, the establishment of an efficient method for detecting the presence of a cause of variability other than chance is the subject matter of Part VI. The first of two main problems to be considered is that of establishing an efficient method for determining when an observed sample is such that it is unlikely that it came from a complex of causes characterized by the prescribed standard distribution. The second and more complex problem is that of finding out whether the observed data arise from some not specified constant system of causes. Five criteria which have been found to work successfully are given and discussed. Part VII on quality control in practice gives a summary of the fundamental principles underlying the theory of control and an outline of the method of attaining control of quality from the raw material to the finished product. There are two appendices of "Resultant effects of constant cause systems" and on "Experimental results." Appendix III is an extensive classified bibliography.

This brief outline of the main points of Dr. Shewhart's work will give but a sketchy idea of its contents. There is great wealth of illustration by means of problems that have actually arisen in practical work and many statistical processes are illustrated by problems that will be valuable supplements to the ordinary text book discussions. Dr. Shewhart's book will tend to increase the growing use of the term, "Bell Telephone School," in connection with statistics.

A. R. CRATHORNE

Analytic Geometry. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1933. xii+296 pages. $2.25.

This book contains a very thorough treatment of the subject matter of Plane Analytic Geometry plus a chapter on Determinants. It is rigorous, but I believe certain portions could be treated in a more simple fashion without losing rigor.

Orthogonal projections are introduced in the early part of the first chapter and, as the author points out in the preface, are used extensively throughout. There is no confusion regarding the definition of the inclination of a line. The slope of a line is treated in the usual way and in addition there is a non-trigonometric treatment. Polar coordinates are introduced in the first chapter.

Functional notation is introduced in the first of the two chapters on loci and several theorems are developed regarding changes in the form of the equation of a locus and possible changes resulting in the loci themselves.

In the chapter on the straight line no mention is made of Hesse's normal form and no use is made of it in deriving the formula for the perpendicular distance from a line to a point. I regret this. Parametric equations of the straight line are introduced in this chapter.

The circle is handled very nicely.

The general conic is defined and the equations of the three types developed from the equation of the general conic. For this purpose the author introduces translation of axes and derives his equations by means of projections. The coordinates of a focus and the equation of a directrix are derived directly for the three types of conics from the equation of the general conic and its consequences, but the details seem a bit formidable for one approaching the subject for the first time.

In the next two chapters, one on tangents, normals and chords of contact, and the other on diameters, poles and polars, extensive use is made of the equations of the straight line in parametric form.

There follow two short chapters, one devoted to properties of conics and one to rotation of axes. The equations for rotation are derived by the use of polar coordinates.

We have then a much longer chapter in which there is a detailed study of the general equation of the second degree with a discussion of invariants in which use is made of determinants.

There is a short chapter on Higher Plane Curves, followed by a good treatment of the elements of determinants.

The book is well written and is suitable for use in a very complete course in Plane Analytic Geometry but would not, I believe, be very practical for a freshman course of which the latter part is devoted to an introduction to the Calculus.

A. H. Sprague

MATHEMATICS CLUBS

Edited by F. M. Weida, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide.

Club Activities

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Oklahoma

Our chapter reports the following officers for the year 1931–1932: C. E. Springer, Director, Dora McFarland, Vice Director, Dorothy Huff, Secretary, Carl Ransbarger, Treasurer, Professor J. O. Hassler, Librarian. They were elected May 31, 1931, by ballot.

Of the thirty-seven active members, fifteen are members of the faculty, six are graduate students, and sixteen are undergraduates. An election of members was held on March 10, 1932 and fifteen new members were admitted. These fifteen were formally initiated at a banquet given April 8, 1932.

Regular meetings were held on the second and fourth Thursdays of each month. During the year the following papers and topics were discussed:

October 22, 1931: "Fourier series" by Mr. Carl Ransbarger.

November 5, 1931: "Matrices and determinants" by Mr. Woodward and Miss Huff.

November 19, 1931: "Fundamental properties of determinants" by Miss Wynne and Mr. Humphrey.

December 10, 1931: "Mathematical aspects of philosophy" by Dr. N. A. Court.

January 14, 1932: "Integral equations" by Mr. Stephen Brixey.

February 25, 1932: "The expanding universe" by Mr. Whitney; "Numbers which are equal to the sum of their aliquot parts diminished by the number of aliquot parts" by Mr. John Brixey.

April 21, 1932: "Gamma functions" by Miss McGregor and Miss Gassett; "The trisection of an angle" by Miss Dorsett.

May 12, 1932: "Dynamic aspects of the theory of moving axes" by Mr. Dorsett.

The business meetings of the year were:

March 10, 1932: Election of new members.

May 19, 1932: The election of officers for the year 1932–1933.

The social meetings of the year were as follows:

October 11, 1931: A tea at the Faculty Club entertaining the mathematics majors.

October 29, 1931: A picnic at the Norman Country Club.

April 19, 1932: A luncheon given at the Faculty Club in honor of Dr. W. D. MacMillan of the University of Chicago.

DOROTHY HUFF, *Secretary*

Pi Mu Epsilon of Syracuse University

This chapter had a successful year under the guidance of Paul D. Jose, the Director. The chapter has expanded and now has 61 active members and 28 faculty members. The affairs of the fraternity were handled by the following officers: Paul D. Jose, Director; Jane Armstrong, Vice Director; John E. Backman, Treasurer; Mazie Chapman Lloyd, Secretary.

These officers were elected by ballot May 13, 1931. Upon the resignation of Mazie Lloyd, the position of Secretary was filled by Helen Heineman who was elected by acclamation February 13, 1932.

On December 15, 1931, seventeen new members were welcomed into the chapter. After the initiation we had a social hour. At our Spring banquet on May 11, 1932, six more persons were initiated.

At the first meeting of the year, October 28, 1931, Dr. William P. Graham spoke on "Einstein's theory of relativity." This interesting talk was followed on November 17, 1931 by a meeting under the management of the Engineers. At this meeting John Norton spoke on "The relation of mathematics to chemical engineering"; Martin Hogan showed the relation between mathematics and electrical engineering; Edward Backman told how mathematics is involved in the building of bridges; and Joseph Carroll connected mathematics and mechanical engineering. Three Liberal Arts students had charge of the next program meeting of February 16, 1932. Helen Heineman spoke on "Egyptian mathematics"; Francis Persons on "The life of Newton"; and Helen Noble on "A few of the relations between mathematics and physics." Several other meetings were planned which unfortunately had to be postponed when our campus was buried in snow. A meeting was held on April 21, 1932 but the program had to be omitted because of the length of the business meeting.

In the Fall, Pi Mu Epsilon had a picnic at one of the near-by State Parks, to which were in-

vited juniors and seniors who were majoring in mathematics. A similar picnic was held on May 18, 1932.

We have had a busy and interesting year and hope to continue to expand as we have this year.

HELEN HEINEMAN, *Secretary*

Pi Mu Epsilon of The State College of Washington

The year 1931–1932, the first year for our chapter, was a successful one, particularly from the standpoint of the number of interesting papers that were presented. All of the meetings except the business meetings were held jointly with the Newtonian Society which includes freshmen and others who are not members of Pi Mu Epsilon. The starred names are members of Pi Mu Epsilon. The papers were given in the following order:

October 28, 1931: "Nomography" by *Professor C. A. Isaacs.

November 12, 1931: "Descriptive properties of point sets" by *Miss Jane Secrest; "Characteristic sets of points" by Mr. Don Jenne.

December 2, 1931: "The base of our number system" by *Miss Thelma Peterson; "The measure of sets of points" by *Mr. Ward Crowley.

January 6, 1932: "Two geometric fallacies" by Miss Claire Lasater; "Massie's method of the trisection of an angle" by Miss Laura Colpitts; "Algebraic fallacies" by Miss Marianne Hawley; Mr. R. Summers exhibited some algebraic and other fallacies; Miss Doris Lee showed a linkage for the trisection of an angle.

January 21, 1932: "Telescope making" by *Professor H. H. Irwin.

January 27, 1932: "Mars" by Miss Emma Pell; "Astronomical distances" by *Professor E. C. Colpitts.

February 18, 1932: "Flatland" by *Miss Ruth Peterson; "Projective geometry" by *Mr. J. R. Vatnsdal.

March 3, 1932: "Impossibility of angle trisection" by *Mr. L. G. Butler; "Magic squares" by Mr. Sam Rausch.

March 17, 1932: "Invariants" by *Mr. J. Biggerstaff; "Meteorology" by Mr. Don Jenne.

March 31, 1932: "Continued fractions" by *Miss Mildred Hunt; "What mathematics can do for you" by Mr. Stephen Cristopher.

A banquet on April 15, 1932 was held jointly with the Newtonian Society.

During the first semester of 1931–1932, five new members were initiated and during the second semester five more were given membership.

The officers for the year were: Miss Jane Secrest, Director; Miss Grace Leyde, Secretary; Mr. C. A. Isaacs, Treasurer.

GRACE LEYDE, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of The George Washington University

The officers for 1931–1932 were: Dr. F. E. Johnston, President; Mr. Albert Wertheimer, Secretary.

The meetings and programs were as follows:

October 12, 1931: "On the Galois theory of equations" by Dr. F. E. Johnston.

October 28, 1931: "On certain properties of the Beta function" by Daniel B. Fisher.

November 9, 1931: "A generalized error function" by Albert Wertheimer.

November 23, 1931: "Ovals of constant breadth" by Dr. P. J. Federico.

December 7, 1931: "On a problem in permutations and combinations" by A. Sinkov.

January 11, 1932: "On the geometry of paths" by Dr. James H. Taylor.

February 23, 1932: "Integration" by Professor E. R. Hedrick of the University of California at Los Angeles.

March 7, 1932: "A problem in the theory of transformations" by Dr. Tobias Dantzig of the University of Maryland.

March 21, 1932: "Integral equations" by Michael Goldberg.
April 11, 1932: "Product series" by Dr. Florence M. Mears.
April 22, 1932: "The calculus of variations and quantum theory" by Professor G. A. Bliss of the University of Chicago.
May 9, 1932: "The problem of mathematical consistency *im kleinem* and *im grossen*" by Captain E. E. Hagler.

Albert Wertheimer, *Secretary.*

Club Topics

1933 as a Centennial Year in the History of Mathematics

By W. C. Eells, Stanford University, California

In continuation of previously published lists (See this Monthly, vol. 39 (1932), pp. 298–299, for a list of 1932 centennial events, and for references to previous volumes for corresponding lists from 1925 to 1931) of centennial dates in the history of mathematics, the following group of important 1933 centennial dates is presented.

A.D. 833. Death of the Caliph, al-Mamun, Arabian astronomer and patron of learning who supervised two geodetic surveys of Mesopotamia and under whose direction was completed the translation of Euclid's *Elements* into Arabic.

A.D. 1533. Posthumous publication of "Regiomontanus" (Johann Müller) *De triangulis omnimodis libri V*, the first work that may be said to have been devoted solely to trigonometry, written about 1464.

A.D. 1533. Appearance in Europe of the Greek text of Euclid's *Elements*, a publication of the *Editio Princeps* by Grynaeus.

A.D. 1533. Death of Köbel, German author of the popular arithmetic *Rechenbiechlin* (1514) which passed through at least twenty-two editions.

A.D. 1633. Birth of Mei Wen-ting, important Chinese historian of mathematics.

A.D. 1633. Birth of Hudde, Dutch mathematician, author of an ingenious rule for finding equal roots.

A.D. 1633. Posthumous publication of Briggs's *Trigonometria Britannica.*

A.D. 1733. Daniel Bernoulli leaves his professorship of mathematics at St. Petersburg for a professorship at Basel, Switzerland.

A.D. 1733. Euler succeeds Bernoulli as professor of mathematics at St. Petersburg.

A.D. 1733. Death of Saccheri, Jesuit father of Milan, who in the year of his death wrote *Euclides ab omni naevo vindicatus*, important study of the foundations of geometry and precursor of non-Euclidean geometry.

A.D. 1733. Death of Jacob Hermann, early Swiss writer on the differential calculus.

A.D. 1833. Birth of Clebsch, Prussian algebraist and geometer.

A.D. 1833. Birth of L. Fuchs, German mathematician who made contributions to the study of linear differential equations.

A.D. 1833. Organization of the Statistical Section of the British Association for the Advancement of Science, due largely to the visit and influence of the Belgian statistician, Quetelet.

A.D. 1833. Death of the outstanding French analyst, Legendre, who greatly enriched mathematics by important contributions especially on elliptic integrals, theory of numbers, attraction of ellipsoids, and least squares.

A.D. 1833. Benjamin Peirce became professor of mathematics and natural philosophy at Harvard University, marking an epoch in the history of mathematics teaching at Harvard and in the United States. (See *School and Society*, 35: 533, April 16, 1932 and this MONTHLY, January 1925.)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 42. *Proposed by C. A. Rupp, Pennsylvania State College.*

If $abcde$ and $baced$ are squares, and $c+d$ and $b+e$ are successive primes, determine the five distinct digits a, b, c, d and e, and show that the solution is unique.

E 43. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

In the following simple multiplication the x's represent unknown digits to be determined. Show that there are just two solutions.

$$\begin{array}{ccccc} & & x & x & x \\ & & & x & x \\ \hline & & x & x & x \\ & x & x & 4 & \\ \hline x & x & x & 1 & 7 \end{array}$$

E 44. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

ABC is an isosceles triangle with $AB=AC$. ADB is a right triangle with D,

the vertex of the right angle, on the opposite side of AB from C. Angle DAB is equal to angle BAC, and DF and CE are perpendicular to AB and AD at F and E respectively. Prove that AF and FB differ by AE.

E 45. *Proposed by W. R. Ransom, Tufts College.*

The ellipse of minimum area which can be circumscribed about a pair of equal, tangent circles, passes through the centers of its largest circles of curvature, and these centers and the two foci are the vertices of a square.

E 46. *Proposed by B. H. Brown, Dartmouth College.*

Show that $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+1/N$ is never an integer for any N.

E 47. *Proposed by D. C. Duncan, University of California at Berkeley.*

By methods of elementary plane geometry construct an equilateral triangle having a vertex upon each of three general lines in a plane, given the position of one vertex. Consider the case when the lines are parallel, and also the case in which the three lines are replaced by three concentric circumferences. What determines the number of solutions in the last case?

SOLUTIONS

E 18. [1933, 51] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Of all the right triangles whose areas exceed a million square units and whose three sides are integers without common factor, find that one whose perimeter is a minimum.

Solution by W. R. Ransom, Tufts College

When $a^2+b^2=c^2$, we have $a=2qr$, $b=r^2-q^2$, $c=r^2+q^2$, area $A=qr(r^2-q^2)$, and perimeter $P=2rq+2r^2$. Now $10^6<A$, and P is to be a minimum.

Since $2A=(rq-q^2)P$, $(rq-q^2)P<2{,}000{,}000$; and we shall try to get P a minimum by making $(rq-q^2)$ a maximum. This requires that $r=2q$, which gives $A=6q^2=10^6$ as an approximation, whence $q=20$, $r=40$, $P=4800$, and $A=960{,}000$ as values close to the desired solution.

We must now determine Δq and Δr so as to increase the area by at least 40,000, but increase the perimeter as little as possible. When $q=20$ and $r=40$ we find that

$$\partial A/\partial q = r^3 - 3q^2r = 16{,}000, \qquad \partial A/\partial r = 3r^2q - q^3 = 88{,}000,$$
$$\partial P/\partial q = 2r = 80, \qquad \partial P/\partial r = 2q + 4r = 200.$$

From these facts we set up the equations,

$$\Delta A = 16{,}000\Delta q + 88{,}000\Delta r = 40{,}000$$
$$\Delta P = 80\Delta q + 200\Delta r = 0$$

which give $\Delta q=-2.08$, $\Delta r=+.83$. Consequently we take $\Delta q=-2$, $\Delta r=1$, getting our new q and r as 18 and 41 respectively. As we change q by one either way from 18, or r by one either way from 41, or both, we get either a larger

perimeter or else too small an area. Hence $q=18$ and $r=41$ give the desired solution. Then the sides of the triangle are $a=1476=4\cdot 9\cdot 41$, $b=1356=23\cdot 59$, $c=2005=5\cdot 401$, and the perimeter is 4838, and the area is 1,001,466.

Solved also by J. Rosenbaum, E. E. Whitford, and the proposer.

E 20. [1933, 51]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The following letters represent the digits of a simple multiplication:

$$\begin{array}{ccccc} & & A & B & C \\ & & & B & D \\ \hline & C & E & E & B \\ F & B & C & D & \\ \hline F & G & C & G & B \end{array}$$

Solve and show that the solution is unique.

Solution by W. E. Buker, Leetsdale, Pa.

Since 1] $D\times C=$ a number whose right hand digit is B,
2] $B\times C=$ a number whose right hand digit is D,
3] neither B, C nor D could be 1,
4] $B+C<10$ (as no number is carried to give the F in the product).
Therefore the only possible values satisfying conditions [1] to [4] are $B=2$, $C=4$, $D=8$, as may readily be verified by actual trial.

From these facts it follows that $A=6$, $E=9$, $F=1$, $G=7$, and the actual multiplication was

$$\begin{array}{ccccc} & & 6 & 2 & 4 \\ & & & 2 & 8 \\ \hline & 4 & 9 & 9 & 2 \\ 1 & 2 & 4 & 8 & \\ \hline 1 & 7 & 4 & 7 & 2 \end{array}$$

Since all possibilities have been investigated, the solution is unique.

Solved also by Mannis Charosh, M. L. Constable, W. R. Ransom, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 21. [1933, 110]. *Proposed by V. F. Ivanoff, San Francisco.*

Prove that the differences of squares of two consecutive numbers in the third diagonal of the Pascal Triangle:

1, 3, 6, 10, 15, 21, 28, 36, 45, etc.

is always a perfect cube.

Solution by L. S. Johnston, University of Detroit

The rth term of the third diagonal is ${}_{r+1}C_2$, and since ${}_{r+1}C_2=\frac{1}{2}r(r+1)$, we have ${}_{r+1}C_2=1+2+3+\cdots+r$.

Now it is well known that the sum of the cubes of the first r integers is equal to the square of the sum of those integers.

Consequently,

$$({}_{r+1}C_2)^2 - ({}_rC_2)^2 = (1+2+3+\cdots+r)^2 - (1+2+3+\cdots+[r-1])^2 = r^3.$$

Solved also by Max Astrachan, S. F. Bibb, W. E. Buker, F. J. Feinler, D. W. Hall, C. A. Rupp, Simon Vatriquant, R. N. Walter and T. R. C.Wilson.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3616. *Proposed by J. B. Reynolds, Lehigh University.*

What is the surface of buoyancy for a homogeneous cube of edge a, if it floats with one face horizontal?

3617. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the center and the principal axes of an ellipse, and a line A anywhere in the plane of the ellipse, it is required by a ruler-compass construction and without drawing the ellipse to construct those normals to the ellipse that are parallel to the line A.

3618. *Proposed by N. A. Court, University of Oklahoma.*

The sum of the medians of a tetrahedron (i.e. the lines joining the vertices to the centroids of the opposite faces) is smaller than two thirds and greater than four ninths of the sum of the edges of the tetrahedron.

3619. *Proposed by V. F. Ivanoff, Berkeley, California.*

A plane cuts the edges OA, OB, OC of a parallelopiped in points A', B', C', respectively, and the diagonal OP in a point P'. Prove that

$$\frac{OA}{OA'} + \frac{OB}{OB'} + \frac{OC}{OC'} = \frac{OP}{OP'} \cdot$$

3620. *Proposed by S. A. Corey, Des Moines, Iowa.*

Find solutions of the functional equation

$$a^4u^2 + u^2 + 2a^2u^2 + a^4 + 2a^2 = v^2$$

in which u and v are to be determined as rational functions of a.

3621. *Proposed by A. S. Levens, University of Minnesota.*

Extend the graphical method for the solution of real roots of a quadratic, as given in Dickson's *First Course in the theory of equations*, p. 29, to permit the reading of complex roots.

3622. *Proposed by Otto Dunkel, Washington University.*

A set of circles pass through a fixed point A and each circle of the set is tangent to a fixed circle I which does not contain A. Each circle of the set cuts the two tangents from A to I in a pair of corresponding points. Prove by synthetic geometry that the envelope of the straight lines joining the pairs of corresponding points is a circle.

This is the converse of the theorem in 3416 [1930, 157] an analytic proof of which is given 1930, 559.

3623. *Proposed by H. Grossman, New York City.*

Let A and B be two fixed points in a plane and P be a third point. Rotate AP from A as center through the fixed angle θ and then stretch it in the fixed ratio $k:1$ to position AP_A. Rotate and stretch BP similarly to position BP_B. The rotations may be both clockwise, both counterclockwise, or either one clockwise and the other one counterclockwise. Let P_0 be the midpoint of P_AP_B. If P describes any curve C in the plane, then the locus of P_0 is a curve C_0 similar to C. If the rotations are both in the same sense, the linear scale of C_0 to C is $k:1$; if the rotations are in opposite senses, the linear scale of C_0 to C is $k\cos\theta:1$.

This is a generalization of 3455 [1930, 477].

SOLUTIONS

3517. [1931, 539]. *Proposed by Morgan Ward, California Institute of Technology.*

Let $\Delta_N(a, \omega)$ denote the determinant

$$\begin{vmatrix} \binom{1}{0} & 1-\omega & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \binom{2}{0} & \binom{2}{1} & 1-a\omega & \cdot & \cdot & \cdot & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1-a^{N-3}\omega & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \binom{N-1}{N-2} & 1-a^{N-2}\omega \\ \binom{N}{0} & \binom{N}{1} & \binom{N}{2} & \cdot & \cdot & \cdot & \binom{N}{N-2} & \binom{N}{N-1} \end{vmatrix}$$

Establish the formula

$$a^{N(N-1)/2}\Delta_N(a,\ a^{-N+1}) = \Delta_N(a,\ a)$$
$$= (1+a)(1+a+a^2)\cdots(1+a+a^2+\cdots+a^{N-1}).$$

Solution by the Proposer

Consider the system of N equations in the unknowns $x_1, \cdots, x_N$

$$(1)\qquad \binom{r}{0}+\binom{r}{1}x_1+\cdots+\binom{r}{r-1}x_{r-1}+(1-a^r)x_r = 0, \qquad (r=1,\cdots,N).$$

On solving for x_N by determinants, we find that

$$x_N = \frac{(-1)^N\Delta_N(a,\ a)}{(1-a)(1-a^2)\cdots(1-a^N)}\cdot$$

Hence

$$(2)\qquad \Delta_N(a,\ a) = (-1)^N(1-a)(1-a^2)\cdots(1-a^N)x_N.$$

Now by actual calculation, we find on solving the first equation of (1) for x_1, and the first two for x_2, that

$$x_1 = \frac{-1}{1-a}, \qquad x_2 = \frac{1}{(1-a)^2}\cdot$$

If we assume that $x_k=(-1)^k(1-a)^{-k}$, $k=1,2,\cdots,r-1$, we have from the equations (1)

$$\binom{r}{0}-\binom{r}{1}\frac{1}{1-a}+\binom{r}{2}\frac{1}{(1-a)^2}-\cdots + (-1)^{r-1}\frac{1}{(1-a)^{r-1}}+(1-a^r)x_r = 0.$$

Hence

$$x_r = \frac{-1}{1-a^r}\left\{\left(1-\frac{1}{1-a}\right)^r-\frac{(-1)^r}{(1-a)^r}\right\} = \frac{(-1)^r}{(1-a)^r}\cdot$$

Thus by induction, $x_N=(-1)^N(1-a)^{-N}$. On substituting this value of x_N in (2), we obtain the second part of our formula; namely

$$(3)\qquad \Delta_N(a,\ a) = (1+a)(1+a+a^2)\cdots(1+a+a^2+\cdots+a^{N-1}).$$

It does not seem possible to establish the first part of the formula in any such simple manner. Since both $a^{N(N-1)/2}\Delta_N(a,\ a^{-N+1})$ and

$$F_N(a) = (1+a)(1+a+a^2)\cdots(1+a+a^2+\cdots+a^{N-1})$$

are polynomials in a of degree $\frac{1}{2}N(N-1)$ and leading coefficients unity we shall have proved the first part of our formula if we can show that the determinant is

divisible by $F_N(a)$. To prove the latter result, it is sufficient to show that *every root of* $F_N(a)=0$ *is a root of* $a^{N(N-1)/2}\Delta_N(a, a^{-N+1}))=0$ *of equal or greater multiplicity.*

Consider first the roots of $F_N(a)=0$. Since $a=1$ is not a root, every root is a primitive root of unity of degree $k\geqq 2$ and $\leqq N$. Let $a=\rho$ be such a root. Since $F_N(a)$ may be written $(1-a)^{-N+1}(1-a^2)(1-a^3)\cdots(1-a^N)$, and $\rho^n-1=0$ when and only when n is divisible by k, ρ is a root of $F_N(a)=0$ of multiplicity $r=[N/k]$, where $[N/k]$ denotes as usual the greatest integer in N/k.

There are in all $\phi(k)[N/k]$ such primitive roots of degree k, $\phi(k)$ being the totient of k. Since

$$\sum_{k=2}^{N}\phi(k)\left[\frac{N}{k}\right] = \tfrac{1}{2}N(N-1),$$

all the roots of $F_N(a)=0$ are thus accounted for. In particular, every primitive Nth root of unity is a root of $F_N(a)=0$ of multiplicity unity.

It only remains to show that $a=\rho$ *is a root of* $a^{N(N-1)/2}\Delta_N(a, a^{-N+1})=0$ *of multiplicity* $\geqq[N/k]$. For this step we need the following two lemmas which we shall establish later.

Lemma 1. *The determinant*

$$\begin{vmatrix} \binom{M}{M-1} & 1-a^{M-1}\omega & 0 & \cdot & \cdot & 0 \\ \binom{M+1}{M-1} & \binom{M+1}{M} & 1-a^{M}\omega & \cdot & \cdot & \cdot \\ \binom{M+2}{M-1} & \binom{M+2}{M} & \binom{M+2}{M+1} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1-a^{M+N-3}a\omega \\ \binom{M+N-1}{M-1} & \binom{M+N-1}{M} & \binom{M+N-1}{M+1} & \cdot & \cdot & \binom{M+N-1}{M+N-2} \end{vmatrix}$$

has the value

$$\binom{M+N-1}{M-1}\Delta_N(a, a^{M-1}\omega).$$

Lemma 2. *If* ρ *is a primitive kth root of unity, then*

$$\Delta_N(\rho, \rho^{-N+1}) = C\{\Delta_k(\rho, \rho)\}^r\Delta_{N-rk}(\rho, \rho^{-N+1}),$$

where C is a rational integer, and r is the greatest integer in N/k.

Our result now follows immediately from lemma 2. We have seen that $a=\rho$ is a root of multiplicity $[N/k]$ of $F_N(a, a)=0$. On setting $N=k$ in formula (3), we see that it is a root of multiplicity one of $\Delta_k(a, a)=0$. Hence by lemma 2, it is a root of multiplicity at least $[N/k]$ of $a^{N(N-1)/2}\Delta_N(a, a^{-N+1})=0$.

The proofs of the lemmas will be sufficiently illustrated by special cases. Take $M=3$, $N=4$ and let D be the resulting determinant in lemma 1. Then

$$D = \begin{vmatrix} \frac{3!}{2!1!} & 1-a^2\omega & 0 & 0 \\ \frac{4!}{2!2!} & \frac{4!}{3!1!} & 1-a^3\omega & 0 \\ \frac{5!}{2!3!} & \frac{5!}{3!2!} & \frac{5!}{4!1!} & 1-a^4\omega \\ \frac{6!}{2!4!} & \frac{6!}{3!3!} & \frac{6!}{4!2!} & \frac{6!}{5!1!} \end{vmatrix},$$

$$= \frac{6!}{2!}\begin{vmatrix} \frac{1}{1!} & 1-a^2\omega & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1-a^3\omega & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1-a^4\omega \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

on performing upon the first determinant the operations

$$\frac{2!}{6!}\text{ col } 1,\ \frac{3!}{6!}\text{ col } 2,\ \frac{4!}{6!}\text{ col } 3,\ \frac{5!}{6!}\text{ col } 4;\ \frac{6!}{3!}\text{ row } 1,\ \frac{6!}{4!}\text{ row } 2,\ \frac{6!}{5!}\text{ row } 3,\ \frac{6!}{6!}\text{ row } 4.$$

Finally, performing upon the second determinant the operations

$$\frac{1}{1!}\text{ col } 2,\ \frac{1}{2!}\text{ col } 3,\ \frac{1}{3!}\text{ col } 4;\ 2!\text{ row } 2,\ 3!\text{ row } 3,\ 4!\text{ row } 4,$$

we find that

$$D = \binom{6}{2}\Delta_4(a, a^2\omega).$$

For lemma 2, take $N=8$, and let $a=\rho$ be a primitive cube root of unity, so that $k=3$, $r=[N/k]=2$, $N-rk=2$. Then on replacing ρ^3 by 1 whenever it appears, we see that

$$\Delta_8(\rho, \rho^{-7}) = \Delta_8(\rho, \rho^2) =$$

$$
\begin{vmatrix}
\binom{1}{0} & 1-\rho^2 & 0 & & & & & \\
\binom{2}{0} & \binom{2}{1} & 0 & 0 & 0 & & & \\
 & \binom{3}{1} & \binom{3}{2} & 1-\rho & 0 & 0 & & \\
 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & 1-\rho^2 & 0 & & \\
 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 0 & & \\
 & & & & \binom{6}{4} & \binom{6}{5} & 1-\rho & 0 \\
 & & & & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1-\rho^2 \\
\binom{8}{0} & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7}
\end{vmatrix}
$$

It is evident that this last determinant is the product of the three determinants set off by dotted lines. On applying lemma 1 to the last two of these, we see that

$$\Delta_8(\rho, \rho^{-7}) = \binom{8}{5}\binom{5}{2}\{\Delta_3(\rho, \rho)\}^2\Delta_2(\rho, \rho^{-7}).$$

A Note by Otto Dunkel. The determinant $\Delta_N(a, \omega)$ of order N is of the type in which the elements are of the form

$$
\begin{aligned}
(i, j) &= \binom{i}{j-1}, && 1 \leqq i \leqq N,\ 1 \leqq j \leqq i, \\
(i, i+1) &= a_{i,i+1}, && 1 \leqq i \leqq N-1, \\
(i, j) &= 0, && 1 \leqq i \leqq N-2,\ i+2 \leqq j \leqq N,
\end{aligned}
$$

where i, j denote the row and column, respectively, of the element, and where the elements $a_{i,i+1}$ are arbitrary. These latter elements lie in a line parallel to the principal diagonal and just above it; call this line the a parallel for brevity. Such a determinant has the property that its value is unaltered if the elements in the a parallel are written in the reverse order; such a reversal may be obtained by rotating the a parallel through 180° about the secondary diagonal as an axis while each other element remains fixed. This may be proved as follows: Multiply the elements in the j column by $(N-j+1)!(j-1)!$, and divide the elements in the i row by $(N-i)!i!$. When this is done for all the columns and

rows, it will be easily seen that the elements $a_{i,i+1}$ are unaltered, whereas any other element (i, j) not zero is replaced by the element which was at $(N-j+1, N-i+1)$. In other words the elements, except those in the a parallel, have been interchanged by a rotation through 180° about the secondary diagonal. The value of the determinant has not been changed, since the multiplications and divisions cancel in the end. Now rotate all the elements of the resulting determinant through 180° about the secondary diagonal. This operation does not alter the value of the determinant, and the result is that each element not in the a parallel has been restored to its original position, but the succession of the elements in the a parallel has been reversed in order. This completes the proof of the proposition.

Now consider a determinant D of this kind in which $a_{i,i+1}=1-x^i$. To each element of the second column add the product of $x-1$ times the corresponding element of the first column; to the elements of the third column add the products of $x-1$ times the resulting elements of the second column; and continue in this same manner. In the end we have for the element (i, i) in the ith row

$$(i, i) = \sum_{k=1}^{i} \binom{i}{i-k}(x-1)^{k-1},$$
$$= \frac{1}{x-1}\left\{ \sum_{k=0}^{i} \binom{i}{k}(x-1)^k - 1\right\} = \frac{x^i-1}{x-1}\cdot$$

In the same row we have zero as the result for $(i, i+1)$; and hence zeros for all following elements. Thus

$$D = \prod_{i=1}^{N}(1 + x + x^2 + \cdots + x^{i-1}).$$

Next consider D', where the element $a_{i,i+1}=1-y^{N-i}$, $y=x^{-1}$. As shown above D' is unaltered if each $a_{i,i+1}$ is replaced by $a_{N-i,N-i+1}$, or if $1-y^{N-i}$ is replaced by $1-y^i$. When this is done D' is evaluated by replacing x in D by y. Thus we have finally the desired result

$$x^{N(N-1)/2}D' = D = \prod_{i=1}^{N}(1 + x + x^2 + \cdots + x^{i-1}).$$

3561. [1932, 359]. *Proposed by Emmanuel Wad, Baltimore, Md.*

The number 12345678 is not divisible by 11, but by placing the eight figures in different orders we can form other numbers which are divisible by 11. Determine how many such numbers can be formed.

Solution by D. B. Perry, Stanford University and J. D. Hill, Brown University.

We first note that

$$N = a_0 10^n + a_1 10^{n-1} + \cdots + a_n \equiv 0 \bmod 11,$$

if and only if

$$D = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n \equiv 0 \bmod 11.$$

This, of course, follows merely from the fact that $N+D\equiv 0 \bmod 11$, if n is odd, and that $N-D\equiv 0 \bmod 11$, if n is even. In the present example, $n=7$, and the a's form a permutation of 1, 2, $\cdots$, 8. Then if we place $a_0+a_2+a_4+a_6=d$, we have $a_1+a_3+a_5+a_7=36-d$. The condition that N be a multiple of 11 is

$$D = 2(d-18) \equiv 0 \bmod 11,$$

which requires that $d=7$, 18, or 29. Of these values, however, 7 and 29 are evidently impossible; hence $d=36-d=18$.

It is not difficult to verify that the given digits admit eight distinct sets of four numbers each whose sum is 18. These are $(1, 2, 7, 8)_1$, $(1, 3, 6, 8)_2$, $(1, 4, 5, 8)_3$, $(1, 4, 6, 7)_4$ $(2, 3, 5, 8)_4$, $(2, 3, 6, 7)_3$, $(2, 4, 5, 7)_2$, $(3, 4, 5, 6)_1$, where complementary sets have the same subscript. We may then choose the terms of d in 8 ways, and corresponding to each such choice the terms of $36-d$ are uniquely determined. But each set of four digits may be permuted independently in 24 ways. Hence, there may be formed with the given digits $8\cdot 24\cdot 24=4608$ different multiples of 11. The smallest and largest of these are, respectively, 12346587 and 87653412.

Solved also by H. T. R. Aude, M. G. Boyce, W. E. Buker, H. A. Campbell, Leverett Davis, Jr., G. H. Graves, H. Grossman, David Katz, Theodore Lindquist, L. C. Mathewson, Ruth G. Mason, A. Pelletier, B. D. Roberts, and F. Underwood.

3562. [1932, 429]. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

If $\phi(x, y, z)=0, \psi(x, y, z)=0$, prove that

$$\begin{vmatrix} d^2x & d^2y & d^2z \\ \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = 0,$$

under suitable conditions.

Solution by J. M. Feld, Brooklyn College.

Let the surfaces $\phi=0$ and $\psi=0$ determine a curve C and let s be the distance measured on C from a fixed point on the curve. Then dx/ds, dy/ds and dz/ds are the direction cosines of a tangent to C, and

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

Differentiating with respect to s we obtain

$$\frac{d^2x}{ds^2}\frac{dx}{ds} + \frac{d^2y}{ds^2}\frac{dy}{ds} + \frac{d^2z}{ds^2}\frac{dz}{ds} = 0. \tag{1}$$

In addition we have

$$\phi_x \frac{dx}{ds} + \phi_y \frac{dy}{ds} + \phi_z \frac{dz}{ds} = 0 \tag{2}$$

$$\psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} + \psi_z \frac{dz}{ds} = 0. \tag{3}$$

Eliminating dx/ds, dy/ds, and dz/ds and multiplying by ds^2, we obtain the stated relationship.

Solved also by J. D. Leith.

A Note by Otto Dunkel. Consider as the independent parameter the time t at which a point is at x, y, z of the curve C and let s denote the length of arc to this point as in the solution above. Denote by A, B, C the cofactors of the first row of the determinant in the problem, and set $R=(A^2+B^2+C^2)^{1/2}\neq 0$, $s''=d^2s/dt^2$. Then the given determinant may be written

$$Ad^2x + Bd^2y + Cd^2z = Rs''(dt)^2.$$

This easily follows by resolving the components of the acceleration, x'', y'', z'', along the path s, which has direction components A/R, B/R, C/R; the sum of the resolved parts must be s''.

The determinant is zero if, and only if, $s''=0$; or if $s=at+b$, where a and b are constants, $a\neq 0$. Hence the suitable conditions are that we must choose our parameter t as a linear integral function of s.

3564. [1932, 429]. *Proposed by H. T. R. Aude, Colgate University.*

A determinant of order $n+1$ has for the elements of its first and second rows, respectively, the successive powers of α and x with exponents from 0 to n inclusive. The third row is the derivative of the second row, and the elements of any row after the second are given by the relation

$$\alpha_{i+1,j} = \frac{1}{i-1}\,\frac{d\alpha_{i,j}}{dx}, \quad i = 2, 3, \cdots, n.$$

Prove that this determinant has the value $(x-\alpha)^n$.

Solution by R. E. Moritz, University of Washington

This problem is an easy corollary to the following general theorem: Let

$$F(x) = \begin{vmatrix} f_0(x), & f_1(x), & \cdots, f_n(x) \\ f_0(\alpha), & f_1(\alpha), & \cdots, f_n(\alpha) \\ f_0'(\alpha), & f_1'(\alpha), & \cdots, f_n'(\alpha) \\ f_0''(\alpha), & f_1''(\alpha), & \cdots, f_n''(\alpha) \\ \cdots & \cdots & \cdots \\ f_0^{(n-1)}(\alpha), & f_1^{(n-1)}(\alpha), & \cdots, f_n^{(n-1)}(\alpha) \end{vmatrix},$$

where $f_i(x)$, $i=0, 1, \cdots, n$, are $n+1$ distinct polynomials in x and $f_i^{(k)}(x)$ is the kth derivative of $f_i(x)$. If m, the highest degree of any one of these polynomials, is not less than n, then $F(x)$ contains $(x-\alpha)^n$ as a factor.

Proof. It is evident that $F(x)$ is a polynomial of degree m in x whose kth derivative is

$$F^{(k)}(x) = \begin{vmatrix} f_0^{(k)}(x), & f_1^{(k)}(x), & \cdots, f_n^{(k)}(x) \\ f_0(\alpha), & f_1(\alpha) & , \cdots, f_n(\alpha) \\ f_0'(\alpha), & f_1'(\alpha), & \cdots, f_n'(\alpha) \\ f_0''(\alpha), & f_1''(\alpha), & \cdots, f_n''(\alpha) \\ \cdots & \cdots & \cdots \\ f_0^{(n-1)}(\alpha), & f_1^{(n-1)}(\alpha), & \cdots, f_n^{(n-1)}(\alpha) \end{vmatrix}.$$

Now obviously both $F(x)$ and $F^{(k)}(x)$ vanish when $x=\alpha$, provided k is less than n. It therefore follows from the well-known theorem regarding multiple roots that $F(x)$ contains $(x-\alpha)^n$ as a factor.

Cor. 1. If $m=n$, $F(x)=c(x-\alpha)^n$, where c is a constant which may be determined by comparing coefficients of like powers of x of the two members of the equation.

Cor. 2. If $f_i(x)=x^i$, then $c=1!2!3!4! \cdots (n-1)!(-1)^n$.

Cor. 3. If $f_i(x)=x^i$ and the coefficients of the rows after the second are those specified in the problem, then $c=(-1)^n$.

Cor. 4. If in Cor. 3, we interchange x and α, we have the determinant, as specified in the problem as originally stated, equal to $(-1)^n(\alpha-x)^n=(x-\alpha)^n$.

Solved also by Frank Ayres, Jr., G. A. Baker, Joseph Lev, W. V. Parker, S. S. Shu, F. W. Sparks, H. S. Thurston, F. Underwood, and the proposer.

3565. [1932, 430]. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

Find the ellipse of least area circumscribing a given triangle.

Solution by F. Underwood, University College, Nottingham, England

Let PQR be the given triangle in a plane α. We can project orthogonally upon another plane α' in such a way that the corresponding points $P'Q'R'$ will be the vertices of an equilateral triangle. (See solution of problem 2895 [1933, 274]; or Müller's *Darstellende Geometrie*, vol. I, p. 97). Let C' be the circle circumscribed to $P'Q'R'$, and let E be the corresponding ellipse circumscribed to PQR in the plane α. The ratio of the area of the circle C' to the inscribed equilateral triangle $P'Q'R'$ is $r=4\pi/3\sqrt{3}$; and, the equilateral triangle being the maximum triangle that can be inscribed in a given circle, the ratio of the area of any circle to that of any triangle inscribed in it can never be less than r. Then since orthogonal projection leaves ratios of areas unchanged, and since any ellipse can be projected orthogonally into a circle, it follows that the ratio of the area of *any* ellipse to that of *any* triangle inscribed in it can never be less

than r. But it also follows that the ratio of the area of the ellipse E to that of the triangle PQR is r, and hence E must be the required ellipse of minimum area.

To construct E in the plane α, it is only necessary to observe that an orthogonal projection preserves parallelism, and hence E must be the ellipse circumscribing the triangle PQR and tangent at each vertex to a line parallel to the opposite side. This ellipse can be constructed by well-known methods, such as are given in Minchin and Dale's *Mathematical Drawing*.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Dr. Henry Eyring, research associate in chemistry at Princeton University, has been awarded the ninth annual $1,000 prize of the American Association for the Advancement of Science, for his paper entitled *Quantum mechanics and chemistry with particular reference to reactions involving conjugate double bonds*, presented at the Atlantic City meeting.

Dr. Charles Roos formerly assistant professor of mathematics at Cornell University and later permanent secretary for the American Association for the Advancement of Science has been awarded a Guggenheim fellowship. He plans to write a book on dynamical economics in consultation with European scholars.

The Alfred Noble prize for the best technical paper of the year on a topic in engineering has been awarded to Frank M. Starr, of the General Electric Company, for his paper entitled *Equivalent Circuits, II.*

Professor H. A. Simmons, of Northwestern University, on sabbatical leave this semester, spent two months in study at Princeton and will be abroad until fall in France, Germany and Italy.

On February 3, 1933, at Tulsa, the teachers of mathematics in the colleges of Oklahoma organized an Oklahoma Section of the Mathematical Association of America, electing the following officers: Chairman, Professor N. A. Court, University of Oklahoma; Vice Chairman, Professor A. M. Wallace, East Central Teachers College; Secretary, Professor E. F. Allen, Oklahoma A. and M. College. By-laws were adopted and submitted to the Trustees of the Association.

The following seventy-eight doctorates with mathematics or mathematical physics as major subject were conferred during 1932 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

J. R. Abernethy, Michigan, On the application of divided differences to approximation.

Alice L. Ambrose, Wisconsin, June, major in philosophy, In defense of an extensional logic.

Sister Mary Nicholas Arnoldy, Catholic, October, minors in physics and education, The reality of the double tangents of the rational symmetric quartic curve.

N. H. Ball, Massachusetts Institute of Technology, June, minor in physics, Projective geometry of element-manifolds.

R. A. Beaver, Cornell, September, minor in physics, Finite plane euclidean geometry.

A. H. Black, Cornell, June, minor in physics, Types of involutorial space transformations associated with certain rational curves.

J. J. Brady, California (Berkeley), May, major in physics, A quantitative study of the photo-electric properties of thin alkali metal films.

G. S. Bruton, Wisconsin, June, Certain aspects of the theory of equations for a pair of matrices.

R. C. Bullock, Chicago, June, Non-conjugate osculating quadrics of a curve on a surface.

J. H. Butchart, Illinois, June, minor in physics, Helices in euclidean n-spaces.

R. H. Cameron, Cornell, September, minor in physics, Almost periodic transformations.

J. F. Carlson, California (Berkeley), May, major in physics, The energy losses of fast particles.

J. M. Clarkson, Cornell, June, minor in physics, Some involutorial line transformations interpreted as points of V of S.

H. H. Conwell, Wisconsin, June, Linear associative algebras of infinite rank whose elements satisfy finite algebraic equations.

Frances T. Cope, Radcliffe, June, Formal solutions of irregular linear differential equations.

A. H. Diamond, California (Berkeley), December, The complete existential theory of the Whitehead-Huntington set of postulates for the algebra of logic.

J. L. Doob, Harvard, June, The boundary values of analytic functions.

C. H. Fischer, Iowa, June, minor in economics, On correlation surfaces of sums with a certain number of random elements in common.

D. G. Fulton, Michigan, Generalizations of the Cauchy integral formula.

C. E. M. Gewertz, Massachusetts Institute of Technology, June, major in electrical engineering, Synthesis of a finite four-terminal network whose driving point and transfer functions are prescribed.

G. D. Gore, Chicago, August, Surfaces associated with a congruence.

H. E. H. Greenleaf, Indiana, October, minor in physics, Curve approximation by means of functions analogous to the Hermite polynomials.

Harvey Hall, California (Berkeley), May, major in physics, Relativistic theory of the photo-electric effect.

C. H. Harry, Johns Hopkins, June, Concerning spaces without local cut points and a geometry of acyclic spaces.

M. C. Hartley, Illinois, June, minor in astronomy, Properties of algebraic, plane quintics which are invariant under finite collineation groups.

G. G. Havens, Wisconsin, June, major in physics, The magnetic susceptibility of some common gases.

M. R. Hestenes, Chicago, June, Sufficient conditions for the general problem of Mayer with variable end-points.

E. H. C. Hildebrandt, Michigan, Systems of polynomials connected with the Charlier expansions and the Pearson differential and difference equations.

Banesh Hoffmann, Princeton, April, On the spherically symmetric field in relativity.

I. O. Horsfall, Cornell, June, minor in physics, Transformations associated with the lines of a cubic, quadratic, or linear complex.

Kuen-Sen Hu, Chicago, June, The problem of Bolza and its accessory boundary value problem.

Ralph Hull, Chicago, August, The numbers of solutions of congruences involving only kth powers.

R. E. Huston, Chicago, August, Asymptotic generalizations of Waring's theorem.

R. D. James, Chicago, June, Analytical investigations in Waring's theorem.

M. W. Keller, Indiana, June, minor in physics, A study of the angular velocity about a point between the foci in Keplerian elliptic motion.

S. H. Kimball, Harvard, June, On rigid motions in four dimensions, with applications to the Laguerre geometry of three dimensions.

R. W. P. King, Wisconsin, June, major in electrodynamics, Characteristics of vacuum tube circuits having distributed constants at ultra-radio frequencies.

E. C. Klipple, Texas, June, Spaces in which there exist contiguous points.

H. L. Krall, Brown, June, An asymptotic expression of the characteristic values of the elliptic partial differential equations.

J. H. Kusner, Pennsylvania, June, On continuous curves with cyclic connection of higher order.

A. J. Lewis, Colorado, August, Solution of algebraic equations with one unknown by infinite series.

D. C. Lewis, Harvard, June, Infinite systems of ordinary differential equations with applications to certain second order non-linear partial differential equations of hyperbolic type.

C. F. Luther, Stanford, June, minor in physics, Concerning primitive groups of class U.

M. H. Martin, Johns Hopkins, June, On infinite orthogonal matrices.

R. S. Martin, California Institute of Technology, June, minor in physics, Contribution to the theory of functionals.

Ruth G. Mason, Chicago, March, Studies in the Waring problem.

David Moskovitz, Brown, June, Certain irregular non-homogeneous linear difference equations.

A. F. Moursund, Brown, October, On a method of summation of Fourier series.

S. B. Myers, Harvard, June, Sufficient conditions in the problem of the calculus of variations in n-space in parametric form and under general end conditions.

Leo Nedelsky, California (Berkeley), May, major in physics, Radiation from slow electrons.

Emma J. Olson, Chicago, August, Conjugate systems characterized by special properties of their ray congruences.

G. B. Price, Harvard, June, On the double pendulum and similar dynamical systems.

Irene Price, Indiana, October, minor in astronomy, On a certain type of polynomials.

E. J. Purcell, Cornell, June, Involutorial space Cremona transformations determined by non-linear null reciprocities.

A. W. Raab, Chicago, August, Jacobi's condition for multiple integral problems of the calculus of variations.

Francis Regan, Michigan, The application of the theory of admissible numbers to time series.

C. E. Rhodes, Cincinnati, June, Concerning the double Poisson integral and its derivatives.

Nathan Rosen, Massachusetts Institute of Technology, June, major in physics, Calculation of energies of diatomic molecules.

Helen G. Russell, Radcliffe, June, On the degree of convergence and overconvergence of polynomials of best simultaneous approximation to several functions analytic in distinct regions.

Jacob Sherman, Pennsylvania, June, On the numerators of the convergents of the Stieltjes continued fractions.

D. T. Sigley, Illinois, June, minor in physics, Groups involving a small number of complete sets of conjugates.

A. Sisk, Cornell, June, The plane symmetric quintic Cremona involutions.

A. H. Smith, Brown, October, On the summability of derived series of the Fourier-Lebesgue type.

A. E. Staniland, Pittsburgh, June, On the Segre curved four-space representation of the plane of two complex variables.

Mildred M. Sullivan, Radcliffe, June, On the derivatives of Newtonian and logarithmic potentials near the acting masses.

J. F. Thomson, Michigan, Motion of the electrons of the helium atom.

Sister Mary Domitilla Thuener, Catholic, June, minors in physics and education, On the number and reality of the self-symmetric quadrilaterals in- and circumscribed to the triangular-symmetric rational quartic.

E. W. Titt, Princeton, June, Systems of partial differential equations and their characteristic surfaces.

A. W. Tucker, Princeton, June, An abstract approach to manifolds.

C. H. Vehse, Brown, June, Acceleration stresses in a heavy wire rope.

C. W. Vickery, Texas, June, minor in philosophy, Spaces in which there exist uncountable convergent sequences of points.

J. P. Vinti, Massachusetts Institute of Technology, June, major in physics, The variational calculation of atomic wave functions.

E. H. Wagner, Michigan, A treatment of systems of linear difference equations having faculty series as coefficients.

A. M. Wahl, Pittsburgh, June, minor in mechanics. Bending of semi-circular plates with and without radially slotted peripheral portions.

A. E. Whitford, Wisconsin, June, major in physics, Zeeman effect of the K II spectrum.

Hassler Whitney, Harvard, February, The coloring of graphs.

Kamcheung Woo, California (Berkeley), May, Projective-transformation group on a hyperquadric in four-dimensional space.

Marguerite L. Zeigel, Missouri, August, minor in physics, Some invariant properties of a two-dimensional surface in hyperspace.

Dr. H. M. Bacon, of Stanford University, California, would like to secure one or more copies of a book by H. S. Carslaw, *The Elements of Non-Euclidean Plane Geometry and Trigonometry* (Longmans, Green and Company, 1916) for use in connection with a course which he is giving. The book has recently gone out of print. Anyone having a copy for sale is requested to communicate with Dr. Bacon.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merges with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 2.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Southern Methodist University in Dallas, Texas, on Saturday, February 11th, 1933. Those attending the meeting numbered thirty-one at the morning session, and ninety-seven at the afternoon session; among these were the following fourteen members of the Association: L. W. Blau, L. M. Blumenthal, H. E. Bray, J. E. Burnam, Alice C. Dean, Nat Edmonson, G. C. Evans, L. R. Ford, C. M. Howard, E. H. Jones, P. K. Rees, W. A. Rees, B. P. Reinsch, C. R. Sherer.

The proceedings took place in the Hyer Physics Laboratory of Southern Methodist University. Lunch was served in the Virginia Dining Hall of the University. In the evening the members of the Texas Section and their guests met at a banquet served in Virginia Hall. The principal speaker at this dinner was Professor J. M. Bledsoe of the East Texas State Teachers' College, who gave by invitation a paper entitled "The outlook of mathematics in secondary education." Professor Bledsoe reached the conclusion that secondary school authorities are steadily tending to minimize the importance of mathematics in the secondary school curriculum in Texas. The quantity of mathematics required is becoming less and less, while the quality of the teaching deteriorates. The result is that the college mathematics staffs of the state are facing a problem which is becoming more severe.

The Texas Section of the Mathematical Association of America is attempting, as part of its policy, to establish greater contact with the teachers of mathematics in the secondary schools of Texas. As a result of special efforts in this direction a large number of such teachers attended this meeting.

The following papers were presented:

1. "Some applications of mathematics in geophysics" by Dr. L. W. Blau, Director of Geophysical Research, Humble Oil and Refining Company.
2. "A geometric study of the equation

$$dy/dx = (l_1x + m_1y)/(l_2x + m_2y)$$

by Professor L. R. Ford, The Rice Institute.

3. "Certain criteria in the interpretation of statistical coefficients" by Professor Lonnie Langston, Texas Technological College, by invitation.
4. "A determinantal characterization of d-cyclic and pseudo d-cyclic sets of points" by G. A. Garrett, The Rice Institute, by invitation.
5. "Psychological principles applied to the learning process and to testing in mathematics" by Professor B. P. Reinsch, Southern Methodist University.
6. "Does the present high school curriculum in mathematics prepare students for college?" by Professor C. R. Sherer, Texas Christian University.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Geophysics is the physics of the earth. The geophysicist applies the methods of physics in making measurements on the earth. The geophysicist in commercial work makes measurements on the surface of the earth in an effort to locate commercial mineral deposits. Magnetic, electrical, gravitational, radio-active, and seismic methods are used in the search for structures favorable to the accumulation of oil. The work involves potential theory, differential geometry, least squares, the calculus of variations, differential equations, the theory of vibrating systems under forcing, and the theory of the propagation of seismic waves through elastic media. Further mathematical research is needed in the field of forced vibrations of systems described by differential equations of higher orders.

2. In this paper Professor Ford gives a geometric study of the equation

$$\frac{dy}{dx} = \frac{l_1x + m_1y}{l_2x + m_2y}.$$

This equation, which can be solved by elementary methods, has an amazingly rich variety of solutions. A geometrical study, without solving, brings out the properties of the various types of solution. A fundamental result is the following: For each such equation there is a unique family of central conics which are cut at a constant angle by all the integral curves. This theorem, together with a knowledge of the rectilinear solutions, enables one to sketch the various types of integral curves.

3. In this paper Professor Langston discusses some of the misinterpretations of statistical results, and shows that the greater part of erroneous conclusions could be eliminated if proper checks and tests were made. The errors may be due to sampling, probability, or a misunderstanding of the data. The use of a group of minor coefficients (termed by the writer as 'coefficients of investigation') including the Yule coefficients of association and colligation, the Pearsonian coefficients of contingency, and the coefficient of concurrent deviations, is treated. The main body of the paper deals with the determination, use, and relation of the Gamma, Beta, and Mu functions. The Beta and Mu coefficients can be found by a mechanical process, which is used in most elementary text books; they can also be determined by integration. Given a set of data the equation of best fit can be found by the method of least squares, and the Beta and Mu coefficients can be computed directly by integration from this equation. The Chi-square test is introduced as a check for the goodness of fit in the study of a frequency distribution. The values of β_1 and β_2 are shown to be 0 and 3 respectively for a normal distribution. The type of frequency curve to be used can be determined by the values of β_1, β_2 and κ_2 where $\kappa_2 = F(\beta_1\beta_2)$. A proper interpretation of statistical coefficients depends on the use and understanding of sampling, probability, the data, and the coefficients connected with the particular problem.

4. In two recent papers (L. M. Blumenthal: *A Complete Characterization of Proper Pseudo D-Cyclic Sets of Points*, American Journal of Mathematics, vol. 54 (1932), pp. 387–396; and *Concerning Regular Pseudo D-Cyclic Sets*, American Journal of Mathematics, vol. 54, pp. 729–738) theorems characteriz- d-cyclic and pseudo d-cyclic sets of points were proved by synthetic methods. In this paper the characterization is obtained by means of determinants. Apart from interpreting analytically the theorems of the papers referred to, the methods used here have the advantage of being readily extended to effect a characterization of the n-dimensional spherical surface S_n. We designate by $\Delta(1, 2, \cdots n)$ the following determinant:

$$\begin{vmatrix} 1 & \cos\dfrac{p_1p_2}{r} & \cos\dfrac{p_1p_3}{r} & \cdots & \cos\dfrac{p_1p_n}{r} \\ \cos\dfrac{p_2p_1}{r} & 1 & \cos\dfrac{p_2p_3}{r} & \cdots & \cos\dfrac{p_2p_n}{r} \\ \cos\dfrac{p_3p_1}{r} & \cos\dfrac{p_3p_2}{r} & 1 & \cdots & \cos\dfrac{p_3p_n}{r} \\ \vdots & \vdots & \vdots & & \vdots \\ \cos\dfrac{p_np_1}{r} & \cos\dfrac{p_np_2}{r} & \cos\dfrac{p_np_3}{r} & \cdots & 1 \end{vmatrix}$$

where the distance $p_ip_j(i, j=1, 2, \cdots n)$ is measured as the length of the shorter arc joining p_i and p_j. The following theorems are proved:

Theorem 1: Three points p_1, p_2, p_3 are congruent to three points of a circle of radius r if and only if $p_ip_j/r \leqq \pi(i, j=1, 2, 3)$ and $\Delta(1, 2, 3)$ equals zero.

Theorem 2: Four points p_1, p_2, p_3, p_4 are congruent to four points of a circle of radius r if and only if each three points is congruent to three points of the circle and $\Delta(1, 2, 3, 4)$ equals zero.

Theorem 3: If all four third-order principal minors of the determinant $\Delta(1, 2, 3, 4)$ are equal to zero, and $p_ip_j/r \leqq \pi$, then $\Delta(1, 2, 3, 4)$ either vanishes or has the value

$$-4\sin^2\frac{p_1p_2}{r}\sin^2\frac{p_1p_3}{r}\sin^2\frac{p_2p_3}{r},$$

in which case $p_ip_j/r \neq \pi$ and

$$\cos^2\frac{p_1p_2}{r} = \cos^2\frac{p_3p_4}{r}, \quad \cos^2\frac{p_1p_3}{r} = \cos^2\frac{p_2p_4}{r}, \quad \cos^2\frac{p_2p_3}{r} = \cos^2\frac{p_1p_4}{r}.$$

It is shown that the three kinds of pseudo d-cyclic quadruples may be obtained by reflections in a circle.

5. In this paper Professor Reinsch showed that the fundamental psychological principles and laws of thought, memory, habit, and development of

abilities govern in the methods of teaching. By means of specific illustrations in geometry, algebra, trigonometry, and calculus, he demonstrated how these psychological principles are to be employed in achieving some of the important aims in teaching mathematics as: effective acquisition of knowledge; thorough understanding of principles; ability to use mathematics as a tool; ability to think carefully and correctly; ability to do independent thinking; development of judgment; appreciation of beauties in mathematics; development of desirable habits of alert attention, questioning attitude, self-activity.

6. In view of the proposed change in secondary mathematics in Texas, a study of the efficiency of the present system might serve as a basis for reorganization. A study of failures in freshman mathematics for the State of Texas during the years 1925–26, 1927–28, 1928–29, and 1929–30 (See Committee of Deans' Report—Proceedings of the Secondary Schools and Colleges in the Southern States) shows that 26.2 per cent of those students who went directly from high school to college were not able to receive a passing grade in their first attempt. In 1929–30, 41.6 per cent either failed or received a barely passing grade, both of which were unsatisfactory.

It is the consensus of opinion among college teachers of mathematics (See J. Seidlin, Mathematics Teacher, Dec. 1932) and science that the secondary schools produce graduates with the following general characteristics:

(1) Worn out or weary of mathematics,
(2) No inspiration for individual investigation,
(3) No appreciation of accuracy,
(4) Not able to place a decimal point in its proper place,
(5) Direct and inverse proportions are meaningless.

The following suggestions might make secondary mathematics more effective:

(1) Reorganization of Junior and Senior High School mathematics so that it will be useful and inspirational, and at the same time contain the fundamental notions for scientific study.

(2) College teachers should recommend only those students for teaching positions in secondary mathematics who have had two, and preferably three years of college mathematics.

NAT EDMONSON, *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE OHIO SECTION

The eighteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, Thursday, April 6, 1933. An afternoon session, dinner, and evening session were held, with the chairman of the Section, Professor O. L. Dustheimer, presiding.

The Section had as its guests Professor and Mrs. H. T. Davis of the University of Indiana.

In spite of the prevailing unfavorable economic conditions, the attendance at this meeting, so far as members of the Association were concerned, was the largest in the history of the Section, while the total registration was the largest since 1920. Eighty-six persons were registered as being in attendance, including the following fifty-seven members of the Association: R. B. Allen, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, L. T. Black, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, R. S. Burington, O. E. Brown, F. E. Carr, T. F. Cope, F. F. Crandell, Rufus Crane, Wayne Dancer, H. T. Davis, O. L. Dustheimer, P. S. Dwyer, F. J. Feinler, C. W. Foard, T. M. Focke, B. C. Glover, Harris Hancock, R. C. Hildner, E. J. Hirschler, F. C. Jonah, Margaret E. Jones, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, R. H. MacCullough, C. C. MacDuffee, R. E. Manchester, Florentina Mathias, C. N. Moore, C. C. Morris, Max Morris, J. R. Musselman, R. L. Newlin, Jesse Pierce, Tibor Radó, S. E. Rasor, Y. K. Roots, S. A. Rowland, W. G. Simon, Mary E. Sinclair, Ruth B. Smyth, H. E. Stelson, C. F. Thomas, C. L. Weaver, J. H. Weaver, R. B. Wildermuth, F. B. Wiley.

The following officers were elected for the coming year: Chairman, I. A. Barnett, University of Cincinnati; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, F. R. Bamforth, Ohio State University; Member of Program Committee, J. R. Musselman, Western Reserve University.

It is expected that the next meeting of the Section will be held at the Ohio State University on Thursday, April 5, 1934.

The afternoon session was given to the reading of papers. The evening session was given to the discussion of the report of a committee that has been working for three years on a study of the mathematical preparation of students entering our colleges from the high schools of Ohio.

The following papers were read at the afternoon session:

1. "Applied mathematics in a liberal arts college" by the Chairman of the Section, Professor O. L. Dustheimer, Baldwin-Wallace College.

2. "The life and work of Eliakim Hastings Moore" by Professor W. G. Simon, Western Reserve University.

3. "Undergraduate preparation for graduate work in analysis" by Professor C. N. Moore, University of Cincinnati.

4 "The expanding universe" read by title on account of the absence of the author, Professor J. J. Nassau, Case School of Applied Science.

5. "The mathematical representation of a score in statistics" by Professor H. A. Toops, Ohio State University.

6. "Approximate methods for finding simple interest on installments" by Professor H. E. Stelson, Kent State College.

7. "The sum of the mth powers of the first n natural numbers" by C. L. Weaver, Kent State College.

8. "Some applications of complex numbers to real geometry of the plane" by M. G. Boyce, Western Reserve University.

9. "The predictable element in economic series" by Professor H. T. Davis, Indiana University, by invitation of the Program Committee.

Abstracts of these papers follow:

1. Since mathematics is no longer compulsory in our schools, teachers of mathematics must "sell" mathematics to the student. They must make their classes so interesting and beneficial that students will elect mathematics.

Too many of our educators have no conception of the role which mathematics has played in the development of our present civilization and its possibilities for future applications. Chemistry and biology are rapidly becoming exact sciences. All chemical phenomena are now being interpreted in terms of physical chemistry, which can be mastered only when the calculus is applied. In biology differential equations are used in the study of metabolic activities, digestion, growth, genetic equilibrium and surface tensions. Mathematical methods are rapidly being applied in education, psychology, psychophysics, economics, and even sociology.

The modern teacher of mathematics has a real opportunity in the training of students in the use of mathematical tools that will help him in so many other departments. We must constantly recall that our major task is to educate our students in the concepts, methods, and applications of the "queen of the sciences."

2. In his paper Professor Simon gave a personal appreciation of Professor E. H. Moore, based on impressions gained from Moore's reputation as a student in Woodward High School in Cincinnati, and also from his own contact with Moore while a student of his in Chicago.

3. In this paper Professor Moore stressed the importance of avoiding an excess of formalism in such courses as introductory and advanced calculus. He urged that the spirit of analysis be preserved, even in elementary courses in the subject, and that so-called proofs based on geometric intuition be avoided. For students who concentrate on analysis in their undergraduate work it would seem desirable, and in many cases feasible, to introduce into their work some of the more recent developments of analysis, such as the Lebesgue integral and the use of divergent series.

4. One may summarize our knowledge of the expanding universe by answering the following questions: What is it that expands? What are the observational evidences of the expansion? What is the rate of expansion? What are the theories regarding it? The last of these questions will be considered here.

In 1917, Professor Einstein modified the law of gravitation, $G_{\mu\nu}=0$ to $G_{\mu\nu}=\lambda g_{\mu\nu}$ in order to avoid the difficulties of the boundary solution at infinity. This new equation makes the world static, finite, and saturated with mass.

Another solution of the same equation was put forward in the same year by Professor de Sitter which makes the world expanding but without mass. In 1923, Professor Eddington and Professor Weyl established independently the fact that the λ was necessary and that it is a factor that makes measurements of length relative. Assuming a non-static world, Professor LeMaitre brought forward in 1928 a solution of the same equation which makes the world expanding but this time with a mass of uniform distribution.

In 1931, Eddington bridged this with the quantum theory, and by utilizing the idea that distances within atoms must be functions of λ to produce relative measures he established the equation $\sqrt{N}/R = mc^2/e^2$ where N is the number of electrons in the universe, R is the initial radius of the universe, c is the velocity of light, and m and e are the mass and charge of an electron. That is, from laboratory values only, he is able to determine a relation between N and R, from which the value of the rate of recession could be established.

A joint paper by Einstein and de Sitter in 1932 brings out the fact that the factor λ may be equal to zero and we may still be able to explain the recession of the galaxies. The latest contribution (1933) on the theoretical side of this problem is the work of Professor Tolman who introduces a pulsating universe.

The story of the Greek letter λ is a fascinating one, and is admirably told in Eddington's new book, "The Expanding Universe."

5. A series of statistical classes represented by their face-values, X_F, may be represented for computational purposes by a series of coded scores, X', defined by the relationship, $X_F = IX' + F_0'$, where I is signed, (positive if the X' series ascends with ascending values of X_F, and vice versa), the width of each class; F_0' is the face-value of the class corresponding to the origin of the X' series. Since only 40 such variables may be punched into a Hollerith card, some 240 variables may be punched into a Hollerith card by employing multiple geometrical punching, where any one such X' score is defined as

$$X_{F_1}' = a_1 2^0 + b_1 2^1 + c_1 2^2 + d_1 2^3,$$

where the coefficients, a_1, b_1, etc., are restricted to zero and one (zero for no punch, one for punch). All possible moments may be computed from the formulae resulting from a manipulation of the above basic relationship.

6. The practice of using simple interest over short time intervals justifies the use of approximation formulae for computing simple interest in short time installment purchases. This may be as much as ten to fifteen percent below the compound interest in actual cases. Three approximation formulae can be used to give fairly satisfactory results. The value of i as given by the equation

$$K = \sum_{x=1}^{n/r} (1 + xri)^{-1}$$

can be approximated by Newton's and the Euler-Maclaurin formulae. The simple interest rate can be computed from the discount rate by use of average time.

7. An expansion of x^m may be secured by Newton's advancing difference formula. The finite integral of this expression may be used to sum the mth powers of the first n natural numbers.

$$\sum_1^n x^m = \Delta^{-1}x^m\Big]_1^{n+1} = \Delta 0^m(n+1)_2 + \Delta^2 0^m(n+1)_3 + \Delta^3 0^m(n+1)_4 + \cdots$$

The terms of this expansion are symmetrical and the coefficients are easily determined. This expression is somewhat easier to use than the expansions usually found.

8. The principal object of this paper was a classification of triads of points by means of invariants of the cubics whose roots are co-ordinates of the points in the Argand diagram. The cases treated included those of coincident and collinear points, and points that are the vertices of equilateral, isosceles, and right triangles. A brief discussion was also given of certain parametric equations of central conics.

9. In an address illustrated by lantern slides, Professor Davis showed how the periodic structure of economic series (for example, stock market averages, pig iron production indices, price averages, etc.) can be studied by means of periodogram analysis, difference equations, and the harmonic analysis of lag-correlations. It was shown that high correlations between such series as the stock market average and the pig iron production index are due in large part to coincident periods in the two series. The analysis of lag-correlations reveal these coincident periods, while the analysis of systems of difference equations reveal the non-coincident periods as elastic interactions. In discussing the dangers of too confident prediction in economic matters, Professor Davis showed that a random series smoothed by a twelve-months' moving average had a period almost exactly equal to the fundamental period of the Bradstreet commodity price index. He also touched on the significance of straightline and logistic trend lines and showed how the present decline in the price index was a phenomenon probably predictable on the basis of long-time trend lines. The computations involved in the paper were furnished by the laboratory of the Cowles Commission for Research in Economics.

RUFUS CRANE, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The seventeenth annual meeting of the Rocky Mountain Section was held at Colorado Agricultural College, Fort Collins, Colorado, on Friday and Saturday, April 14–15, 1933. Professor A. G. Clark presided at each of the three sessions.

The attendance was thirty-five, including the following twenty-one members of the Association: C. F. Barr, Jack Britton, A. G. Clark, I. M. DeLong,

J. C. Fitterer, G. W. Gorrell, I. L. Hebel, C. A. Hutchinson, Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, S. L. Macdonald, A. S. McMaster, W. K. Nelson, Greta Neubauer, E. D. Rainville, O. H. Rechard, A. W. Recht, Mary S. Sabin, C. H. Sisam.

Members and friends of the Association were guests of the College on the evening of April 14. At the business session, Professor C. H. Sisam of Colorado College, was elected Chairman for the coming year, and Professor J. C. Fitterer, of Colorado School of Mines, was elected Vice-Chairman.

The following ten papers were read:

1. "A solution of a system of linear matrix equations in two unknowns," by Miss Rachel Achenbach, University of Wyoming, by invitation.

2. "On foci of algebraic curves with applications to cubic curves" (Thesis presented by Ethel A. Rice for M. A. University of Colorado) by Professor Claribel Kendall, University of Colorado.

3. "Entropy, strain and the Pauli exclusion principle" by Professor Guy Berry, Colorado Agricultural College, by invitation.

4. "Notes on Riccati's differential equations" by E. D. Rainville, University of Colorado.

5. "A theorem on point-wise discontinuous functions" by Professor O. H. Rechard. University of Wyoming.

6. "On Graeffe's method of solution of algebraic equations" by Professor C. A. Hutchinson, University of Colorado.

7. "The use of calculators in solving Kepler's problem" by Professor A. W. Recht, University of Denver.

8. "The 'Zig' function of Wirth" by Professor C. F. Barr, University of Wyoming.

9. "The solution of algebraic equations by infinite series" by Professor A. J. Lewis, University of Denver.

10. "Symmetric functions and resultants" by Professor C. H. Sisam, Colorado College.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. The purpose of this paper is to develop methods of solution for the sixteen systems of linear matrix equations in two unknowns which result from the corresponding ordinary algebraic system when the constants and unknowns in these latter equations are replaced by matrices of the nth order. Two of the systems yielded to solution by the methods of elimination by addition and by substitution. The other fourteen systems were reduced to a single standard form which was solved by the use of the Hamilton-Cayley equation.

2. In this paper, Miss Rice was particularly interested in the graphical representation of the location of the real foci of certain cubic curves. For practical purposes in the matter of computation the cubic curves chosen were sym-

metric with respect to a point or an axis. The foci of some twenty-four such cubics of varying types were given. The discussion of these special cubics was preceded by a summary of the known results concerning the number of foci of algebraic curves in general.

3. The author shows, by using the geometrical weight method developed by Kimball for an ideal gas, that the entropy of a real gas is proportional to the strain, and that the equations for maximum entropy are equilibrium equations between stress and strain. The velocity distribution function is the same as that of the Fermi-Dirac statistics. This method offers an explanation of the second law of thermodynamics and the Pauli exclusion principle.

4. In the general Riccati Equation $dy/dx=A_0(x)+A_1(x)y+y^2$, it is most desirable to obtain a simple particular solution. If this is available, the complete solution follows almost at once. When A_0 and A_1 are polynomials, it is reasonable to search for polynomial solutions. In these notes Mr. Rainville shows that never more than one or two polynomials, for which simple formulas are given, need be tested as trial solutions. Some extensions varying the functional form of A_0 and A_1 are treated.

5. In this paper there is presented a statement and proof of the following theorem: Given a function, $f(x)$, continuous over a residual point set on an interval (a, b); if the function is defined at the remaining points of the interval by the "closest approximating function" method, it will be point-wise discontinuous on the interval.

6. This is the paper read by title at the Los Angeles Meeting of the Association. An abstract appeared in this MONTHLY, November 1932, p. 503; and the complete paper will appear in a later issue.

7. A demonstration showing how J. Peter's 7-place table of natural trigonometric functions, in which the argument is given in decimals of a degree, combined with the modern calculator, greatly facilitate the solution of Kepler's well-known equation $M=E-e\sin E$.

8. This paper presents the function

$$\operatorname{zig} u = 2\left\{[R_{u/2}] + (-1)^{[R_{u/2}]}R_{u/2}\right\}$$

and its derivative; in which $u=f(x)$, $[s]$ means the greatest integer in s, and R_u means $u-[u]$. Its adaptibility to special configurations was demonstrated by using it in the polar equation

$$\rho = a\cos\frac{\pi}{n}\Big/\cos\left\{\frac{\pi}{n}\left(1-\operatorname{zig}\frac{n\theta}{\pi}\right)\right\}$$

to map a regular n-sided polygon in a circle of radius a. The function zig u was developed and named by Mr. Don Wirth.

9. This paper outlines methods of expressing all the roots of an algebraic equation by infinite series and formulates conditions of convergence for these series.

10. This paper deals with a proof of the possibility of representing a given integral symmetric function in terms of the elementary symmetric functions and with a method of determining this representation by the use of resultants.

A. J. LEWIS, *Secretary*

THE TWENTY-SECOND MEETING OF THE IOWA SECTION

The twenty-second meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at Coe College, Cedar Rapids, Iowa, on Friday and Saturday, April 21 and 22, 1933. The meetings were held in room 117, Science Hall.

The attendance was about fifty, including the following twenty-one members of the Association: R. P. Baker, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, C. W. Emmons, Cornelius Gouwens, M. E. Graber, I. J. Gwinn, Gertrude A. Herr, Dora E. Kearney, O. C. Kreider, F. M. McGaw, Arthur Ollivier, J. F. Reilly, H. L. Rietz, W. J. Rusk, E. R. Smith, C. W. Strom, John Theobald, L. E. Ward, Roscoe Woods.

The Section Chairman, Professor L. M. Coffin, presided at both the Friday afternoon and Saturday morning sessions. Dinner was enjoyed together Friday evening in the Jefferson Room, Hotel Roosevelt. The officers elected for 1933–1934 are as follows: Chairman, J. F. Reilly, University of Iowa; Vice-Chairman, M. E. Graber, Morningside College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College.

A committee consisting of Professors R. P. Baker and Roscoe Woods prepared the following statement relative to the death of Daniel Kreth: "Daniel Kreth, engineer and surveyor, of Wellman, Iowa, a charter member of the Association, died in 1932. From 1914 to 1924 Mr. Kreth was an active contributor to the Monthly of problems and solutions. His interest in mathematics showed itself not only by his activities in the Association but also in the collection of a library. The Iowa Section laments his passing as a member and feels keenly the loss of inspiration which comes from knowing a man who derived a great deal of pleasure from his study of mathematics."

The program consisted of fourteen papers, as follows:

1. "On the resolution of $4X=Y^2-(-1)^{(p-1)/2}\,pZ^2$ for $p=67$, 71, and $X=(x^p-1)/(x-1)$" by Professor Cornelius Gouwens, Iowa State College.

2. "Some properties of the logarithmic potential" by Professor J. J. Westemeier, Des Moines Catholic College, by invitation.

3. "The teaching of the trigonometric functions of $2x$ and of $x/2$" by Professor Roscoe Woods, University of Iowa.

4. "Sophus Lie's geometry of imaginaries" by Professor M. E. Graber, Morningside College.

5. "A problem in simple harmonic motion of a particle moving in a medium of varying density" by Robert MacAllister, Wartburg College, by invitation.

6. "Runge's method" by Professor N. B. Conkwright, University of Iowa.

7. "A system of circles inscribed in an annulus" by Professor C. W. Emmons, Simpson College.

8. "An Iowa journal of mathematics" by Professor Gertrude A. Herr, Iowa State College.

9. "Zeros of the Hermitian polynomials" by Professor E. R. Smith, Iowa State College.

10. "An approach to a class in freshman mathematics" by Father John Theobald, Columbia College.

11. "A test for significance in a unique sample" by Professor A. E. Brandt, Iowa State College, by invitation.

12. "Devices in mathematical instruction" by Professor F. M. McGaw, Cornell College.

13. "The nature of probability" by Professor E. S. Allen, Iowa State College, by invitation.

14. "Note on the purchase price of a bond" by Professor J. F. Reilly, University of Iowa.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In 1930 Professor Gouwens reported on the cases for $p=41$ and $p=43$ but later learned that those had been given by Pocklington in Nature, vol. 107, (1921). In this paper the resolutions for $p=67$ and $p=71$ are presented.

2. In this paper Professor Westemeier presented the elegant elementary methods developed by Erhard Schmidt (*Bemerkung Zur Potentialtheorie*, H. A. Schwarz Gewidmet, 1914, p. 364–383) to obtain the properties near the acting masses of the integrals defining the logarithmic potentials of simple plane curve, double plane curve, and plane area distributions.

6. Professor Conkwright gave an exposition suitable for class room use of Runge's method of numerical approximation to the solution of the differential equation $y'=f(x, y)$.

7. In his paper Professor Emmons developed the relations among R, R', r, n, and α, where R and R' are radii of the inner and outer bounds of an annulus, r, the radius of a circle inscribed in the annulus, n, the number of circles tangent consecutively to each other and to the annulus, and 2α, the angle at the center of the annulus subtended by the circles. Special consideration was given to integral and non-integral values of n.

8. The Analyst, A Monthly Journal of Pure and Applied Mathematics, was published in Des Moines from January 1874 to November 1883 by Dr. Joel E. Hendricks, a self taught mathematician. From the correspondence of Dr. Hendricks, from his scrap-book, and from interviews with his daughters and grandchildren, Miss Herr has collected some interesting facts concerning the author and the publication. The character of the journal is discussed from the standpoint of the contributions and the types of articles.

9. If one substitutes $e^{x^2/2}\mu_n(x)$ for $P_n(x)$ in the differential equation $P_n''(x) - 2xP_n'(x) + 2nP_n(x) = 0$, which the Hermitian polynomials satisfy, there is obtained a new equation $\mu_n''(x) + (1+2n-x^2)\mu_n(x) = 0$. Professor Smith showed how the values of μ could be approximately determined by extrapolation based on corresponding values for polynomials of lower order, and how the values of the roots thus obtained could be corrected by the Newton-Raphson method. Professor Smith presented a table of roots computed to five decimal places.

10. Experience shows that it is unwise to expect much mathematical background in the case of the average student entering college. Many dread mathematics. These should be assured that mathematics is not so difficult, and that it will prove interesting if carefully studied. Father Theobald believes that in the opening session of a freshman course the class should be told that the daily paper exercise is the recognized way toward mastery in mathematics, and that for this reason adequate credit will be allowed for daily work satisfactorily completed.

11. If p_O is the observed proportion of positive differences between the reactions of two organisms to the same treatment or between the effects of two treatments on the same organism, p_E the expected proportion, n the number of observations, N_1 the number of observations necessary for highly significant results (R. A. Fisher's 1% point) and N_5 the necessary number of observations for significance (Fisher's 5% point), the ratio of the difference of the observed proportion and the expected proportion to the standard deviation is

$$\frac{Np_O - Np_E}{\sqrt{Np_E q_E}}.$$

Professor Brandt makes use of this in testing the significance of the deviation of the observed proportion from the expected. If n is equal to N_1 the deviation is highly significant and if equal to N_5 it is significant.

12. Professor McGaw stressed the difference between two methods of presenting mathematics especially to young students at and before college level. He differentiated these into Static and Dynamic methods. The Static is the usual method in which subjects are taught with the aid of fixed figures and ideas; the Dynamic method makes use of changing figures. The ideas were illustrated with string diagrams, strings being drawn around pegs on a board. Changes in the positions of the pegs, marking points or vertices, were followed by appropriate and visual changes in the relations of the several parts of the figure. Generalization would seem to be favored by the adoption of such an idea in presentation. The applicability of this notion to topics in plane geometry and calculus and to special objects was illustrated.

13. Professor Allen in his paper considers various definitions of probability —psychological (both topological and numerical); axiomatic; definition by "equipossibility"; and by frequency. The latter is preferred, but there are difficulties, particularly in connection with physical applications. Some of these difficulties deal with the existence of the limit of relative frequency in an em-

pirical sequence: possible solutions may be obtained by restriction of the type of admissible sequence, by postulate, or by extension of the definition of probability.

14. In his paper Professor Reilly showed how the Makeham and the "excess" formulas for determining the purchase price of a bond could be generalized for the case when dividends are payable p times a year, and the investment rate is nominal convertible m times a year, $p\neq m$. By the use of a computing machine the computation of the purchase price can be readily found without logarithms.

J. F. Reilly, *Secretary*

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section of the Mathematical Association of America was held in conjunction with the annual session of the Nebraska Academy of Sciences at the University of Nebraska in Lincoln, on Friday afternoon, April 28, 1933, with Professor T. A. Pierce of the University of Nebraska as chairman.

Several visitors were present as guests, and the following seventeen members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. L. Candy, D. C. Dearborn, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, A. L. Hill, J. M. Howie, R. M. McDill, G. D. Nichols, T. A. Pierce, Lulu Runge, Mabel F. Schmeiser, R. B. Thompson.

Officers were elected for the ensuing year as follows: Chairman, J. M. Earl, University of Omaha; Secretary-Treasurer, J. M. Howie, Nebraska Wesleyan University; and as member of executive committee, T. A. Pierce, University of Nebraska.

The following papers and reports were presented:

1. "The possible number of magic squares" by Professor A. L. Candy, University of Nebraska.
2. "Dynamical trajectories" by D. C. Dearborn, University of Nebraska.
3. "The absolute regression line" by Professor C. C. Camp, University of Nebraska.
4. "Solutions of certain Diophantine equations" by R. B. Thompson, University of Nebraska.
5. "On a certain class number recurrence of Kronecker" by Professor M. A. Basoco, University of Nebraska.
6. "The generalized rational field" by H. B. Roberts, University of Nebraska.
7. "Polynomials of best approximation over infinite regions" by Professor J. M. Earl, University of Omaha.

Abstracts of some of these papers follow, numbered in accordance with their numbers in the list of titles:

1. This is a continuation of the paper presented by Professor Candy one year ago. The purpose of this further study is to get some idea of the number of approximately symmetric 12×12 squares that can be constructed by the method of "Current Groups."

Starting with the 3×3 square he constructs twelve 6×6 squares, by selecting the groups in 6 different ways, and balancing the rows and columns in slightly different ways, when the numbers in each group have a common difference. From each of these squares 64 different squares can be formed by reversing (interchanging columns) one or more pairs of symmetric groups, or by inverting pairs of symmetric rows. All of these squares are symmetric with respect to the center except the two end numbers in the two middle columns and the two middle rows.

In the case of the 12×12 square the groups can be selected in 33 different ways. By substituting these 33 different sets of groups in the twelve 6×6 squares mentioned above, he has actually written down 151 different 12×12 squares in which the groups in the 6×6 squares have remained unchanged, and the groups in the 12×12 squares are all written in the cyclic order. By reversing (interchanging columns) and inverting (interchanging rows) one or more groups of the fundamental 6×6 squares, (as mentioned above) he gets 9248 12×12 squares in which the order of their groups remain unchanged. Finally, by writing columns of groups in different orders, namely, cyclic and crossed, and reversing one or more pairs of symmetric groups, and inverting double pairs of symmetric groups which are balanced as to rows, he gets a grand total of 249,048,103,172,096.

By varying the order of the groups in the separate columns, as well as in the rows, and disregarding symmetry, this number of possible 12×12 squares can be vastly increased.

2. Professor Kasner, (Transactions of the American Mathematical Society, vol. 7, p. 401) has shown the necessary and sufficient conditions that a triply infinite set of plane curves may be the trajectories due to a positional field of force. This paper is a consideration of the restrictions on an otherwise arbitrary surface such that the system of curves derived by section and orthogonal projection from the surface will satisfy these conditions. It is found that the surface must be a developable and further that it must satisfy a certain partial differential equation. It is possible to find the components of the force field associated with any surface satisfying these two conditions.

3. Ordinarily in statistics one has the two regression lines

$$Y-\overline{Y} = r\frac{\sigma_y}{\sigma_x}(X-\overline{X}) \text{ and } (X-\overline{X}) = r\frac{\sigma_x}{\sigma_y}(Y-\overline{Y})$$

depending on the choice of independent variable. In case one wishes to treat both variables alike one is led to a line such that the sum of the squares of its distances from the original points is a minimum. This gives the absolute regression line

$$Y-\overline{Y}=\frac{\sigma_y^2-\sigma_x^2+\sqrt{(\sigma_y^2-\sigma_x^2)^2+4p^2}}{2p}(X-\overline{X}),$$

where $p=r\sigma_x\sigma_y=\sum(X_iY_i)/n-\overline{X}\overline{Y}$. The median regression line which makes the sum of the absolute values of the distances from the line a minimum is unsuitable because it is not always unique. If unique it usually passes through one or more of the original points with about the same number on either side. By holding one point fixed one may easily estimate by using horizontal distances whether rotation to right or left will diminish the sum of the perpendicular distances. Weighted points count as multiple points. However, this line does not fit in so well with ordinary probability theory as the absolute regression line defined above.

4. The purpose of Mr. Thompson's paper was to present parametric solutions of two very general Diophantine equations,

$$X_1^2+X_2^2+X_3^2+\cdots+X_p^2=W^n$$

and

$$X_1^2+a_1X_2^2+a_2X_3^2+\cdots+a_{p-1}X_p^2=Y_1^2+b_1Y_2^2+b_2Y_3^2+\cdots+b_{n-1}Y_n^2.$$

The solutions were obtained directly from the stated problem in terms of independent parameters, thus obtaining multiple infinitudes of solutions, according to the number of parameters involved.

6. This is a field formed by taking one-rowed matrices of three elements each with multiplication $\cdot$, and addition $+$, defined as follows: $(a, b, c)+(x, y, z)=(az+2by+cx, bz+cy, cz)$; and $(a, b, c)\cdot(x, y, z)=(ax, by, cz)$; $(a, b, c)=(x, y, z)$, if $x/a=y/b=z/c$.

7. Professor Earl's paper considers the approximation to a given function of two variables by means of a sequence of polynomials which are determined so as to minimize the integral over an unbounded region of the product of a non-negative weight function and the mth power of the magnitude of the error.

A. L. HILL, *Secretary*

THE TENTH MEETING OF THE INDIANA SECTION

The tenth meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, May 5 and 6, 1933, at Indiana University, Bloomington, Indiana.

There were forty-two registered for the meetings on Saturday including the following twenty-one members of the Association:

Gladys L. Banes, H. T. Davis, W. E. Edington, P. D. Edwards, E. D. Grant, E. H. C. Hildebrandt, F. H. Hodge, E. L. Klinger, Mayme I. Logsdon, Florence Long, Juna M. Lutz, H. A. Meyer, T. W. Moore, J. A. Reising, C. K. Robbins, Fred Robertson, D. A. Rothrock, L. S. Shively, H. E. Slaught, R. O. Virts, K. P. Williams.

On Friday evening an informal reception was held in the East Parlors of the Student Building for arriving members and their guests. At six o'clock a dinner was held in the Grill room of the Indiana Union Building. Professor K. P. Williams acted as toastmaster. Short talks were given by Professor Fernandus Payne, Dean of the Graduate School and head of the department of zoology, and by Professor H. E. Slaught of the University of Chicago. At eight o'clock Professor Slaught gave a public lecture on the subject: "The lag of mathematics behind literature and art in the early centuries." This lecture was later given on the program of the national meeting of the Mathematical Association of America held in Chicago in June and an account of it will appear in the report of that meeting.

The meeting on Saturday was presided over by Professor K. P. Williams of Indiana University, chairman of the Section. At the business meeting the following officers were elected: Chairman, Professor Juna M. Lutz, Butler University; Vice-Chairman, R. O. Virts, Central High School, Fort Wayne; Secretary-Treasurer, Professor P. D. Edwards, Ball State Teachers College.

The retiring chairman's address was given by Professor Williams on the subject, "Early theories of comet orbits." Professor Williams presented the important phases of the early theories and gave especial attention to the very great difficulties involved in the calculation of orbits. Many of the solutions by first rate mathematicians have been all but useless to the practical astronomer because of the extreme difficulty of carrying out the necessary calculations.

The following papers were presented:

1. "Dynamic symmetry" by Professor S. A. Cain, Department of Botany, Indiana University, by invitation.
2. "The Mathematical Association and a decade of mathematics in Indiana" by Professor W. E. Edington, De Pauw University.
3. "Concerning modular functions" by Professor W. E. Maier, Purdue University, by invitation.
4. "Some elementary formulas suggested by an elementary equation in trigonometry" by Professor F. H. Hodge, Purdue University.
5. "The expansion of an arbitrary operative function in successive derivatives" by Professor Fred Robertson, Iowa State College.
6. "Polar line coordinates" by Professor C. K. Robbins, Purdue University.
7. "Some applications of periodogram analysis" by P. W. Overman, Indiana University, by invitation.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The term "dynamic symmetry" was coined by Professor Jay Hambidge to describe some of the relationships he discovered in Greek art of the Classical Period. His later researches led to its discovery in ancient Greek writings. It had failed of translation because moderns had no concept of the meaning until Hambidge had developed it, by an analysis of art objects, as a system of pro-

portion probably used consciously by the Greeks in the construction of art and architectural objects. The underlying theme is that of root-rectangular areas and a peculiar area, the whirling-square rectangle, which are manipulated so as to form *commensurate areas* and gnomens by diagonals and perpendiculars—the construction being such as largely to control the "idea" of the art object as to form, and, to a certain extent, ornament. The end and side of the whirling-square rectangle are as 1 to 1.618. This has been called the ϕ ratio and is found in many diversified phenomena. The root rectangles have 1 as their end and $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, as their sides.

The Greeks probably got the proportions from the ancient Egyptians as the ϕ ratio has been found in the pyramids dating back as far as 4700 B.C. The ϕ ratio rectangle and its reciprocal equal the root-five rectangle, both of which, along with the other root-rectangles, could have been obtained from a square by at least two simple methods of construction used by Egyptians in the "cording of the temple." Why these proportions are "good" in art it is difficult to say, but many modern artists and craftsmen have found it profitable to employ the methods in manipulation of their ideas for designs.

It is truly a remarkable phenomenon that the ϕ ratio should be found as a fundamental underlying theme in the architecture of plants, especially in phyllotaxy, the arrangement of leaves on the stem. Most leaves are arranged in a spiral system the relations of which form a numerical series known as the Fibonacci summation series, running 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, the ratio of which converges on 1.618, the ϕ ratio. This series has been known for centuries but the explanation has only recently been found by A. H. Church of Oxford in continued proportionate growth of leaf primoidia arranged in curve systems on a conic-ovoid growing point. The ubiquitous spiral in nature has been the subject of many interesting speculations and studies and is frequently logarithmic in nature, due in the main to growth phenomena.

The ϕ ratio is also that of Euclid's proposition, to cut a line in extreme and mean proportion, the so-called Golden Section. There have been many formulae of beauty, perfect figures and more or less mystical obsessions held by philosophers of the past and almost any good art will permit of analysis leading to formulae and approaching scientific exactness, but that is certainly neither the whole explanation nor the sole need for creative work. That the ϕ ratio should have been used by the ancient Egyptians and the classical Greeks in the production of good art, constitute the Golden Section, be inherent in the root-five rectangle and other geometrical figures, and constitute one of the most wide spread of botanical phenomena is without any correlating explanation.

2. The history of the Mathematical Monthly and the problems of its publication were sketched briefly in order to show how the problem of its continuation led up to the conception and organization of the Mathematical Association through the combined efforts of the founder of the Monthly, B. F. Finkel, and the three active organizers, H. E. Slaught, E. R. Hedrick and W. D. Cairns. Some original correspondence concerning the charter membership was shown.

The Indiana Section of the Association was reorganized on an active basis on October 16, 1924, at Indianapolis, and has since met regularly in the spring at the various universities and colleges over the state. The programs have always included addresses by speakers of note among whom have been Jacques Hadamard, of Paris, F. R. Moulton, Jacob Kunz, D. W. Moorehouse, Warren Weaver, L. C. Karpinski, T. C. Fry, G. A. Bliss, Cornelius Lanczos, and H. E. Slaught.

The membership of the Section has increased from 30, in 1916, to 65 according to the last directory. A total of 20 research papers by members of the Section have been published in reputable mathematical journals during the past decade. The interest aroused by the meetings of the Section has led to the formation of several undergraduate mathematical clubs in the colleges of Indiana. The advances made in recent years in the development of graduate study in the state was pointed out and the fact that six Ph.D. degrees with mathematics as a major were conferred in the state during the past four years was presented as evidence of general quickening in mathematical interest no little part of which may be attributed to the work of the Indiana Section of the Association.

3. Let $0<F(\omega)$ and $e^{\pi i\omega}=q$. The classical identity

$$\left(\sum_{v=-\infty}^{\infty} q^{v^2}\right)^2 = 2\sum_{n=-\infty}^{\infty} \frac{q^n}{1+q^{2n}}$$

was proved in a new way due to the fact that, generally,

$$\lim_{l\to\infty} \sum_{0<h^2+k^2<l} \frac{(-1)^h}{2h+1+2k\omega}$$

$$= \lim_{l\to\infty} \sum_{0<h^2+k^2<l} \frac{(-1)^k}{2h+(2k+1)\omega} \cdot \begin{cases} i \\ -i, \end{cases} \; if \; \begin{cases} 0<F(\omega) \\ 0>F(\omega). \end{cases}$$

4. Starting with the identity cos 20° cos 40° cos 80° = 1/8 this paper gives several methods of proving this identity with several general formulas together with proofs suggested by the methods of proof for the original identity.

5. The problem is the determination of a method of expanding an arbitrary operative function $f(x, z)$ in a series of successive powers of z. The symbol z shall be interpreted to mean the operational derivative d/dx and its powers the corresponding successive derivatives.

The components z^i define the function $f(x, z)$. These component functions are grouped according to the formula of Schmidt for the discovery of a normalized orthogonal system of functions by linear combinations of the given set.

The function is then expanded in the Fourier manner in terms of these functions $\phi_i^{(z)}$ for the range $0<z\leqq 1$ of the independent variable. The resulting series is then rearranged in terms of the successive powers of z.

The expansion is stated thus,

$$f(x, z) = \sum_{n=1}^{\infty} A_n(x) \sum_{i=1}^{n} \sum_{\alpha} (-1)^{\beta+i+1} \frac{1}{N_{n\ldots\beta\ldots i}} z^i$$

where

$$A_n(x) = \int_0^1 f(x, z)\phi_i(z)dz$$

and α is every possible combination of $(n-1) \cdots (i+1)$ in the order given, β is the number of indices omitted, $(\beta=0, 1, \cdots, n-i-1)$ and $1/N_i$ is the reciprocal of the square root of the norm if the index is not repeated but is the product of these expressions by $\int_0^1 \phi_i(z)z^i dz$ when the index is repeated.

6. The ordinary polar coordinates of a point also determine a line perpendicular to the radius vector at its end point and may, therefore, be thought of as the coordinates of this line.

To find the polar line coordinate equation of any curve, express its pedal curve in polar coordinates. This type of coordinates proved to be extremely cumbersome in practise but some of the by products turned out to be both new and interesting. For example, if, to three lines through a point perpendiculars of lengths a, c and b are dropped from another point then $c \sin(A+B) = a \sin A + b \sin B$ where A is the angle between b and c, and B the angle between a and c.

7. In this paper the theory of the Schuster periodogram was developed. Applications were shown to such divergent subjects as sun-spots, weather changes, market fluctuations, recurrence of earthquakes, stellar variations, magnetic disturbances, etc. The speaker announced the completion of an elaborate set of tables for the computation of periodograms. Values of $\sin\theta$ and $\cos\theta$, $\theta=2\pi s/u$, have been computed to 8 decimal places for s from 0 to u, and u from 5 to 75 by integers.

At the afternoon session of the Section a resolution was adopted expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the members of the mathematics department of Indiana University.

P. D. EDWARDS, *Secretary*

THE MAY MEETING OF THE MARYLAND—DISTRICT OF COLUMBIA—VIRGINIA SECTION

The May meeting of the Maryland—District of Columbia—Virginia Section of the Mathematical Association of America was held at the University of Virginia on Saturday, May 13, 1933.

Sixty-three persons attended the meeting including the following forty-two members of the Association: O. S. Adams, M. W. Aylor, Archie Blake, J. W. Blincoe, W. E. Byrne, Paul Capron, Orpha A. Culmer, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, W. H. Echols, Mary Ewin, P. J. Federico,

Isabel Harris, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, Gil-, lie A. Larew, B. Z. Linfield, J. J. Luck, Florence M. Mears, Ethel I. Moody Eugenie M. Morenus, F. D. Murnaghan, E. J. Oglesby, E. K. Paxton, W. T. Puckett, K. S. Purdie, O. J. Ramler, Beulah Russell, L. W. Smith, C. M. Sparrow, H. E. Stelson, J. H. Taylor, Mildred E. Taylor, H. W. Tyler, T. L. Wade, F. M. Weida, C. H. Wheeler, G. T. Whyburn, Evelyn P. Wiggin.

The annual Fall meeting will be held at the George Washington University on Saturday, December 2, 1933.

The following officers for the year 1933–34 were elected: Chairman, B. Z. Linfield, University of Virginia; Secretary, F. M. Weida, George Washington University; additional members of executive committee, Abraham Cohen, Johns Hopkins University and Alexander Dillingham, U. S. Naval Academy.

Professor Lipót Fejér, of the University of Budapest, an outstanding figure in the very significant school of mathematics in Hungary, was the invited speaker. He addressed the afternoon session on "The characterization of some remarkable systems of points of interpolation by means of conjugate points, of the Cotes's numbers, and of certain external properties."

The following papers were presented at the morning session:

1. "On the frequency of high correlation coefficients in small samples" by Archie Blake, U. S. Coast and Geodetic Survey.

2. "A problem concerning the homogeneity of continua" by C. H. Wheeler, Johns Hopkins University.

3. "Four-dimensional orthogonal matrices" by Professor F. D. Murnaghan, Johns Hopkins University.

4. "Projective geometry and vector analysis" by Professor Tobias Dantzig, University of Maryland.

5. "A paradox in differential geometry" by Professor B. Z. Linfield, University of Virginia.

6. "The Taylor series construction of a function using only the derivative process" by Professor W. H. Echols, University of Virginia.

7. "Syzygies for Weitzenböck's irreducible complete system of Euclidean concomitants for the conic" by T. L. Wade, University of Virginia.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Given an integer $n>1$ and a number r such that $|r| \leqq 1$, we inquire what is the probability that two random sets of n numbers each will have a correlation coefficient at least as large in absolute value as $|r|$. We base the discussion not upon the usual procedure of drawing the variates from a normal population, but on the assumption that vectors in all directions in the hyperplane $\sum_{i=1}^{n} x_i = 0$ occur with equal frequency. For statistical problems in which this hypothesis is approximately satisfied, we obtain useful formulas for low as well as high values of n, and not only in the real number-system, but also for complex numbers and quaternions.

To extend the definition of a correlation coefficient to these number-systems, we consider two vectors $\xi\equiv(x_1\cdots x_n)\neq 0$ and $\eta\equiv(y_1\cdots y_n)\neq 0$. In the statistical case, $\sum x=\sum y=0$. We define the right correlation coefficient of η with respect to ξ as the value of the fraction $\bar{\xi}\eta/(M\xi\, M\eta)$, where M denotes the modulus of a vector. This definition leads to an orthogonality property and a least norm property analogous to the orthogonality and least square properties in the real case. We distinguish between right and left correlation coefficients because multiplication is not commutative for quaternions.

2. A continuum M is said to be homogeneous provided that for any pair of points a and b of M there exists a homeomorphism T of M into itself such that $T(a)=b$. In this paper conditions are studied in order that a locally connected continuum M should be "cyclic element homogeneous," i.e., in order that for any two non-degenerate cyclic elements C_1 and C_2 of M there should exist a homeomorphism of M into itself sending C_1 into C_2.

5. It is well known that for a real surface to have an umbilic at a point we must impose two conditions upon the partial derivatives at that point. Still, if in the quadratic equation giving the principal radii at that point we impose the single condition that its roots be equal, the surface will have an umbilic at that point. Professor Linfield pointed out that when the surface is given, the discriminant of the quadratic equation for the principal radii is the product of $(1+p^2)(1+q^2)$ by

$$\begin{vmatrix} 1+p^2 & pq \\ pq & 1+q^2 \end{vmatrix}\left(\frac{r}{1+p^2}-\frac{t}{1+q^2}\right)^2+(pq)^2\left(\frac{2s}{pq}-\frac{r}{1+p^2}-\frac{t}{1+q^2}\right)^2,$$

whose vanishing will impose clearly two conditions on the first and second partial derivatives p, q and r, s, t. More generally, if A and B are second order real symmetric matrices and the determinant of one of them is positive, the condition that the roots of $\det(A-tB)=0$ be equal will always amount to two conditions, and the matrices A and B will then be linearly dependent. For, when $\det(A)>0$, the discriminant is the product of $A_{11}A_{22}$ by

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}\left(\frac{B_{11}}{A_{11}}-\frac{B_{22}}{A_{22}}\right)^2+A_{12}^2\left(\frac{2B_{12}}{A_{12}}-\frac{B_{11}}{A_{11}}-\frac{B_{22}}{A_{22}}\right)^2.$$

6. (1). *Prerequisites.* (a) If the derivative of a function is zero in a region C the function is constant in the region. (b) If a series of functions and the series of their derivatives are both absolutely and uniformly convergent in C then the derivative of the sum of the functions is equal to the sum of the derivatives. (c) A bounded infinite assemblage assigns at least one limit-number. In particular, a bounded defined sequence assigns one and only one limit-number.

(2). *Definition.* A function $f(z)$ is regular in a region C when there can be assigned an absolute number M such that

$$\left|\frac{f^{(n)}(z)}{n!}\right|<M^n,$$

M independent of n, for each z in C, $n=0.\ 1,\ 2,\cdots$. Then the bounded sequence $|f^{(n)}(z)/n!|^{1/n}$ assigns in $(0, M)$ a limit α. Then for $1/R=\alpha+\epsilon$, there can be assigned a y such that $y-z<R$; then the sequence

$$\left| (y-z)^n\frac{f^{(n)}(z)}{n!} \right|$$

is summable and the series

$$\sum (y-z)^n\frac{f^{(n)}(z)}{n!}$$

is absolutely and uniformly convergent in a circle with center at z and radius R.

(3). If a is the center and R the radius of convergence of

$$\sum (x-a)^n\frac{f^{(n)}(a)}{n!}$$

then when x is any point in circle R_a and y is any point in the ellipse, foci a and x and major axis R, the series

$$\phi(y) \equiv \sum (x-y)^n\frac{f^{(n)}(y)}{n!}$$

for all values of y is absolutely and uniformly convergent in the ellipse and its derivative as to y is zero in the ellipse and the series is constant in this region. The points a and x are in the ellipse. Therefore $\phi(x)=\phi(a)$ or

$$f(x) = \sum (x-a)^n\frac{f^{(n)}(a)}{n!}$$

for any assigned x in the circle R_a.

7. By means of the fundamental identities of the symbolic method an algebraically complete set of nine syzygies are found connecting the eighteen irreducible concomitants of the conic, as established by Dr. Roland Weitzenböck in *Über Bewegungsinvarianten*, Wiener Berichte 122 (1913) and 123 (1914). One syzygy is on invariants and covariants, two on invariants and contravariants, and the remaining six on all types of concomitants.

F. M. WEIDA, *Secretary*

A PHOTO-ELECTRIC NUMBER SIEVE

By D. H. LEHMER, Altadena, California

Until recently, the mathematician has been considered the only scientist fortunate enough not to need any laboratory equipment to carry out his researches. The last decade, however, has shown a tendency among mathematicians to adapt the existing commercial calculating machines to their computa-

tions,[1] and in rare cases to invent devices of their own to perform special operations.[2]

There is an important class of problems in the theory of numbers which cannot be solved readily by any commercial calculating machine, so that the number-theorist has had to resort to a kind of graphical method (similar to the celebrated Sieve of Eratosthenes) in handling problems of this sort. By the generous cooperation of the Carnegie Institution of Washington it has been possible to construct a new kind of calculating machine, applicable to this class of problems, in which modern physics has made its contribution to the oldest and least practical branch of mathematics.

In order to make clear the details of this machine, it is desirable to say a few words about the kind of problems it can solve and the underlying principles by means of which it solves them. We shall not attempt to describe completely this class of problems, since this would require the introduction of various concepts and notations unfamiliar to the general reader who is not well acquainted with the theory of numbers. It will suffice to illustrate with the following problem which is a typical representative.

Let it be required to find an integer x for which ax^2+bx+c is a square number y^2. Here a, b, and c are given integers and determine the problem.[3] Simple as this problem seems at first sight, it contains as special cases problems of extreme difficulty. Thus $a=1$, and $b=0$ gives $c=y^2-x^2=(y-x)(y+x)$. This problem, then, is equivalent to factoring the number c and is one of the central problems of the theory of numbers. From the fact that a, b, and c are small we must not conclude that x is small. In fact, when $a=1549$, $b=0$, and $c=1$, we find the smallest positive value of x to be

$$x = 12223\ 09542\ 82674\ 74959\ 34242\ 68334\ 63805$$
$$08818\ 07626\ 31786\ 81966\ 09867\ 28279\ 63220.$$

To give an elementary explanation of the method by means of which the machine solves such problems, we may introduce the idea of a finite arithmetic which deals only with the numbers $0, 1, 2, 3, \cdots, p-1$. We may perform addition, subtraction and multiplication in the ordinary manner, but in every case the result is divided by p and the remainder alone preserved. In this way we remain inside our system of numbers. The number of multiples of p which we discard in reducing the answer to a number in our system is as immaterial to us as the number of complete revolutions of the wheel is to the roulette player.

[1] Dr. L. J. Comrie of Great Britain's Nautical Almanac Office is an exponent of this procedure. See Monthly Notices of the Royal Astronomical Society, 92, (1932), 523–541.

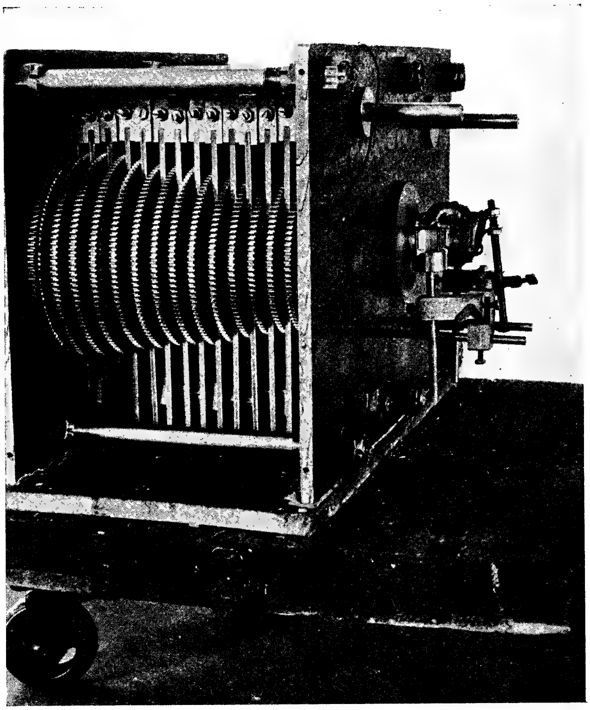
[2] The mechanical integraphs recently developed by Dr. V. Bush of the Massachusetts Institute of Technology may be cited in this connection. See Journal of the Franklin Institute, vol. 212, (1931), 447–488.

[3] The problem is not solved by showing that x exists, nor by giving inequalities which x must satisfy. The problem is to exhibit x.

Let us imagine a problem in ordinary arithmetic requiring the solution in integers of a certain equation. This equation may be translated bodily into our finite arithmetic by merely replacing the coefficients of the equation by their remainders on division by p. To solve the problem in the finite arithmetic is simple enough because the values of the unknown are restricted to lie among the p numbers $0, 1, 2, \cdots, p-1$. If p is small we can find all the solutions by actual trial. But why is it useful to consider a real problem in one of these artificial arithmetics? What interpretation can we give to the solutions of the problem in such a system? To answer these questions let us consider one of the answers to the original problem. If we divide this answer by p, the remainder will be one of the solutions of the problem in the finite arithmetic. The desired answer then is some one of these artificial answers plus a certain multiple of p. This is not much information, but it is easy to obtain. Moreover we have not committed ourselves to the choice of p. In fact, we may choose as many different p's as we like, and if they have no common factor, the information offered by each arithmetic will be independent. The combination of these various bits of information leads to the solution of the original problem. It is this combination which the machine is designed to effect.

In order to obtain a mechanical picture of the situation let us imagine a disk gear with p teeth having a small hole opposite each tooth. These holes are at a constant distance from the periphery of the gear and are numbered from 0 to $p-1$. Let us plug up all the holes except those which correspond to answers to our problem considered in the finite arithmetic. If a light is set behind the gear, then, as the gear rotates, it will transmit the light when and only when the number of teeth turned past has a chance of satisfying the problem as far as this particular gear is concerned. To get the combined effect of using several p's, we may set up several gears parallel to each other. If the gears are mounted so that they have a common line of tangency and are driven at the same linear speed, then the light which shines through a gear will be transmitted or blotted out by the next gear. In this way the gears are allowed to pass judgment upon the eligibility of each number as it turns past. The decision of each judge is not influenced by the rulings of the other judges and his rejection of a candidate is final. When a number is a solution of our problem, however, there will be no dissention among the judges and the beam of light will succeed in running the gauntlet. It is true that other numbers may be thus unanimously elected without being answers to our problem. If enough gears are used, however, these undesirables can be eliminated altogether. In the kind of problems that the machine solves about one half of the holes are stopped up in each gear, and there are 30 gears. This means that we can expect an extraneous alignment of holes about once in $2^{30} = 1073741824$ numbers. By this time the reader has grasped the essential features of the mechanical system. It is clear that the gears may be driven at great speed and still there will be ample time for the light to pass through every gear when the answer arrives. In fact this time interval is one ten-thousandth of a second.

So much for the mechanical system. All that is needed now, is an alert and untiring eye behind the last gear to observe and report the appearance of the tiny flash of light. This eye is a photo-electric cell. The energy produced by the small amount of radiation falling on the retina of the cell is very small indeed. It must be sufficiently magnified to operate a circuit breaker for the electric motor driving the machine. This is done by means of a six stage amplifier and a three stage relay system. The amplifier magnifies the energy from the photo-electric cell about 700 million times and transmits it to a delicate vacuum tube relay, the first of a series of three which finally stop the machine.



Courtesy of the Carnegie Institution

View of the mechanical system showing one series of gears.

To give a concrete example of how the machine works, let us consider the following problem: to find a value of x for which

$$(1) \quad 91894770302976x^2 + 2877225288670 21824x + 256527596541064768$$

is a perfect square. These large coefficients are not arbitrary numbers. In fact they arise quite naturally in an investigation into the possible factors of the Mersenne number $2^{79}-1$. The numbers 2^n-1 where n is a prime have been the subject of investigation since the time of Euclid. Twelve of these numbers have been proved prime and twelve composite ones have been completely factored. Since 1924 it has been known that if a value of x exists for which (1) is a

square then $0 < x < 39110012$. This problem was considered in each of the finite arithmetics corresponding to a prime or a power of a prime $p < 127$, and the appropriate holes in the corresponding gears were stopped up. This presents the problem to the machine, which, canvassing numbers at the rate of 300000 a minute, can cover the above range for x in about two hours without attention. As a matter of fact the power was automatically shut off in 12 seconds, and the machine coasted to a stop. Reversing the machine slowly and substituting the human eye for the photo-electric cell, the light was seen to shine through at $x = 56523$ according to the reading on the revolution counter. Substituting this value of x in (1) we obtain at once the number 309853160646773276521024, which is the square of 556644555032. Hence our problem is solved. Incidentally this leads to the factorization

$$2^{79} - 1 = 2687 \cdot 202029703 \cdot 1113491139767.$$

We take this opportunity to answer a few questions that are almost always asked by those inspecting the machine. The first of these has to do with the period of the machine. Since each gear goes through the same position over and over again, the same is true of the machine as a whole so that the light will pass through the machine periodically. This is quite true in theory. The period of the machine is clearly the least common multiple of the periods of the gears. This means that the machine will return to its original position after

$$2497\ 19431\ 65929\ 91015\ 26347\ 12970\ 31949\ 33696\ 87002\ 72210$$

teeth have turned past. Even though the machine runs at the speed of 300000 teeth per minute it would wear out long before it got really started on its period.

Another question that is always asked is: Do you know approximately what the answer is before you start? In the problems we consider, the unknown is lost among millions of consecutive numbers. If we could tell in advance just which million contained the answer it would not have been necessary to construct the machine.

As a final question: What is the exact relation between the machine and mathematics itself? We do not hesitate to apply mathematics to physics, but in applying physics to mathematics there arises the question of the reliability and accuracy of the results. Fortunately, in this case, this question is not a vital one. The accuracy of the mechanical system is absolute. The machine does not depend upon measurements or estimated values. The gears arrive inexorably at their appointed positions with absolute certainty. The photo-electric system, however, with its enormous amplification and its elaborate safeguards against outside disturbances, would not have been possible a few years ago, and even to-day it is easy for the unintiated to underestimate its reliability. This system is indispensable, however, since any substitute for the weightless beam of light could not be depended upon at the high speeds that some problems demand.

After all, such questions are for the technician. As for the mathematician, he is glad to obtain immediately verifiable answers to his problem no matter

how unreliable their source may be. For what sources are more unreliable and yet more indispensable than imagination, intuition or inspiration?

The successful completion of the machine has been made possible first of all by the Carnegie Institution of Washington, which once more came to the aid of the number-theorist by a grant of the necessary funds. We also wish to acknowledge the assistance of Mr. T. J. Palmateer of Stanford University for valuable advice in constructing the mechanical parts of the machine. We are also much indebted to Dr. R. C. Burt of Pasadena who not only constructed the photo-electric system but, on account of his interest in the project, has generously given space in his laboratory where the machine is now operating.[1]

LAGRANGE'S QUINTIC FOR THE NEUTRAL HELIUM ATOM

By W. E. COX, JR., Tulane University

In a paper investigating the straight line solutions of the problem of three bodies, Lagrange[2] arrived at a quintic equation, the real solutions of which determined the relative positions of the body along the straight line. The purpose of this paper is to discuss the corresponding quintic for the case of the neutral helium atom.[3]

Let the mass of the nucleus be m_2 and the masses of the electrons m_1 and m_3. Assume a positive charge, e_2, on m_2 and negative charges, $-e_1$ and $-e_3$, on m_1 and m_3 respectively. Let ξ_i, η_i, ζ_i $(i=1, 2, 3)$ be the coordinates of the three bodies with the center of gravity of the three masses as the origin and with axes rotating uniformly in the $\xi\eta$-plane with angular velocity ω. Let r_{ij} $(i, j=1, 2, 3)$ denote the distance between m_i and m_j. Then the following differential equations follow from Coulomb's law,

$$
(1)\qquad
\begin{aligned}
&\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} - \omega^2\xi_i - \frac{k^2}{m_i}\frac{\partial U}{\partial \xi} = 0,\\
&\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} - \omega^2\eta_i - \frac{k^2}{m_i}\frac{\partial U}{\partial \eta} = 0,\\
&\frac{d^2\zeta_i}{dt^2} - \frac{k^2}{m_i}\frac{\partial U}{\partial \zeta} = 0, \quad i = 1, 2, 3,
\end{aligned}
$$

where

$$U = \frac{e_1e_2}{r_{12}} - \frac{e_1e_3}{r_{13}} + \frac{e_2e_3}{r_{23}} \cdot$$

We seek a particular solution of these equations in which the three masses lie in a straight line and rotate in circles about the center of gravity of the sys-

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[1] Since the above was written the apparatus has been put on exhibition at A Century of Progress.

[2] Lagrange's *Collected Works*, vol. VI, p. 277.

[3] See H. E. Buchanan, this MONTHLY, vol. 38 (1931), pp. 511–521.

tem. The coordinates of the three bodies are, then, constants and their derivatives zero. Equations (1), therefore, become

$$
(2)\qquad
\begin{aligned}
&- m_1\omega^2\xi_1 - \frac{k^2e_1e_2(\xi_2 - \xi_1)}{r^3_{12}} + \frac{k^2e_1e_3(\xi_3 - \xi_1)}{r^3_{13}} = 0,\\
&- m_2\omega^2\xi_2 - \frac{k^2e_1e_2(\xi_1 - \xi_2)}{r^3_{12}} - \frac{k^2e_2e_3(\xi_3 - \xi_2)}{r^3_{23}} = 0,\\
&- m_3\omega^2\xi_3 + \frac{k^2e_1e_2(\xi_1 - \xi_3)}{r^3_{13}} - \frac{k^2e_2e_3(\xi_2 - \xi_3)}{r^3_{23}} = 0,\\
&- m_1\omega^2\eta_1 - \frac{k^2e_1e_2(\eta_2 - \eta_1)}{r^3_{12}} + \frac{k^2e_1e_3(\eta_3 - \eta_1)}{r^3_{13}} = 0,\\
&- m_2\omega^2\eta_2 - \frac{k^2e_1e_2(\eta_1 - \eta_2)}{r^3_{12}} - \frac{k^2e_2e_3(\eta_3 - \eta_2)}{r^3_{23}} = 0,\\
&- m_3\omega^2\eta_3 + \frac{k^2e_1e_3(\eta_1 - \eta_3)}{r^3_{13}} - \frac{k^2e_2e_3(\eta_2 - \eta_3)}{r^3_{23}} = 0,\\
&- \frac{e_1e_2(\zeta_2 - \zeta_1)}{r^3_{12}} + \frac{e_1e_3(\zeta_3 - \zeta_1)}{r^3_{13}} = 0,\\
&- \frac{e_1e_2(\zeta_1 - \zeta_2)}{r^3_{12}} - \frac{e_2e_3(\zeta_3 - \zeta_2)}{r^3_{23}} = 0,\\
&+ \frac{e_1e_3(\zeta_1 - \zeta_3)}{r^3_{13}} - \frac{e_2e_3(\zeta_2 - \zeta_3)}{r^3_{23}} = 0.
\end{aligned}
$$

The last six of equations (2) are satisfied by $\eta_i = \zeta_i = 0$ ($i = 1, 2, 3$). We will have found a solution of equations (1) in which the masses lie in a straight line along the ξ-axis if we find constants ξ_1, ξ_2, ξ_3, which satisfy the first three of equations (2) with $\eta_i = \zeta_i = 0$. Assume $\xi_3 > \xi_2 > \xi_1$; since for any position of the nucleus other than between the two electrons there are no solutions as can be ascertained by going through the following operations for the other two cases. The unit of distance is arbitrary, so let $r_{12} = \xi_2 - \xi_1 = 1$. Then equations (2) become

$$
(3)\qquad - m_1\omega^2\xi_1 - k^2e_1e_2 + \frac{k^2e_1e_3}{(\xi_3 - \xi_1)^2} = 0
$$

$$
(4)\qquad - m_2\omega^2(\xi_1 + 1) + k^2e_1e_2 - \frac{k^2e_2e_3}{(\xi_3 - \xi_2)^2} = 0
$$

$$
(5)\qquad - m_3\omega^2\xi_3 - \frac{k^2e_1e_3}{(\xi_3 - \xi_1)^2} + \frac{k^2e_2e_3}{(\xi_3 - \xi_2)^2} = 0.
$$

Equation (5) may be replaced by

$$m_3\xi_3 + m_2\xi_1 + m_2 + m_1\xi_1 = 0, \tag{6}$$

obtained by adding equations (3), (4), and (5).

Solving equations (3) and (6) for ω^2 and ξ_3 in terms of ξ_1 and substituting in equation (4), the following equation in ξ_1 is obtained,

$$\begin{aligned}
&e_1e_2M^4(m_1+m_2)\xi_1^5 + e_1e_2M^3(m_2M + 4m_2^2 + 2m_2m_3 + 4m_1m_2 + 2m_1m_3)\xi_1^4\\
&\quad + M^2[e_1e_2(m_1+m_2)(m_2+m_3)^2 + 4e_1e_2m_2^2M + 2e_1e_2m_2m_3M\\
&\quad + 4e_1e_2m_2(m_1+m_2)(m_2+m_3) + e_1e_2m_2^2(m_1+m_2) - e_3m_3^2(e_1m_2+e_2m_1)]\xi_1^3\\
&\quad + m_2M[e_1e_2(m_2+m_3)^2(M+2m_2+2m_1) + 2e_1e_2m_2(m_1+m_2)(m_2+m_3)\\
&\quad + 4e_1e_2m_2M(m_2+m_3) + e_1e_2m_2^2M - e_3m_3^2(e_1M+2e_2m_1)\\
&\quad - 2e_1e_3m_3^2(m_2+m_3)]\xi_1^2 + m_2[2e_1e_2m_2M(m_2+m_3)^2\\
&\quad + e_1e_2m_2(m_1+m_2)(m_2+m_3)^2 + 2e_1e_2m_2^2M(m_2+m_3)\\
&\quad - e_1e_3m_3^2(m_2+m_3)^2 - e_2e_3m_1m_2m_3^2 - 2e_1e_3m_3^2M(m_2+m_3)]\xi_1\\
&\quad + e_1m_2(e_2m_2^2 - e_3m_3^2)(m_2+m_3)^2 = 0,
\end{aligned} \tag{7}$$

where

$$M = m_1 + m_2 + m_3.$$

Since this is a fifth degree equation in ξ_1, there is at least one real root. The coefficients of ξ_1^5 and ξ_1^4 are definitely positive and the constant term is $e_1m_2(e_2m_2^2 - e_3m_3^2)(m_2+m_3)^2$. Thus, if $e_3m_3^2 > e_2m_2^2$, there is at least one real positive root. Similarly, there is at least one real negative root if the constant term is positive. In the case of the helium atom $e_2 = 2e_3$ and m_2 is much greater than m_3, and hence there is at least one real negative root.

If we let $A = \xi_3 - \xi_2$ and substitute the resulting value of ξ_1,

$$\xi_1 = -\frac{m_3A + m_2 + m_3}{M},$$

in equation (7), a fifth degree equation in A is obtained,

$$\begin{aligned}
&e_1e_2m_3(m_1+m_2)A^5 + e_1e_2m_3(3m_1+2m_2)A^4\\
&\quad + m_3(e_1e_2m_2 + 3e_1e_2m_1 - e_2e_3m_1 - e_1e_3m_2)A^3\\
&\quad + m_1(e_1e_2m_3 + e_1e_3m_2 - 3e_2e_3m_3 - e_2e_3m_2)A^2\\
&\quad - e_2e_3m_1(3m_3+2m_2)A - e_2e_3m_1(m_2+m_3) = 0.
\end{aligned} \tag{8}$$

This corresponds to Lagrange's quintic for the case under consideration. A definite distribution of the three bodies along the ξ-axis, and, therefore, a definite particular solution of our problem, is given by every real positive root of equation (8). It is seen that the coefficients of A^5 and A^4 are both positive and the coefficient of A and the constant term are both negative, but that the coef-

ficients of A^3 and A^2 may be either positive or negative for arbitrary values of e_i and m_i; and, in particular, the coefficient of A^3 may be negative and that of A^2 positive, as, for example, when $e_1=4$, $e_2=1$, $e_3=5$, $m_i=1$. In any case there is an odd number of changes of sign in the coefficients and, hence, at least one real positive root. In every case, except that in which the coefficient of A^3 is negative and the coefficient of A^2 positive, there is only one change of sign in the coefficients and, therefore, only one real positive root. In the particular case in which the coefficient of A^3 is negative and that of A^2 positive there are three changes of sign in the coefficients and, hence, a possibility of three positive roots. The author has been unable to determine whether or not there can actually be three positive roots of this equation.

In the case of the helium atom, $m_3=m_1=m$ and $2e=e_2=2e_1=2e_3$. Making these substitutions in (8), we obtain

$$(9)\quad 2(m+m_2)A^5+2(3m+2m_2)A^4+(4m+m_2)A^3 - (4m+m_2)A^2-2(3m+2m_2)A-2(m+m_2)=0.$$

There is but one change of sign in the coefficients of this equation, and therefore, there is only one real positive root. It is obvious that this root is $A=1$, which was to be expected from the way in which the unit of measurement was chosen.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

QUESTION CONCERNING THE MAXIMUM TERM IN THE DIATOMIC SERIES—PROPOSED[1] BY A. A. BENNETT

A REPLY BY ALFRED BRAUER, University of Berlin

Given k, let π_k denote the continued product of the first k primes, p_i, starting with $p_i=2$. For example, $\pi_5=2.3.5.7.11=2310$. Let p_0 be 1.

Question: *Given k, what is the maximum number of consecutive integers each of which is divisible by one of the primes $p_1, p_2, \cdots, p_k$?*

This question had already been studied by Legendre in his work *Théorie des Nombres*[2] when he tried to prove the known theorem (first proved by Dirichlet) that there exist an infinite number of primes in any arithmetical progression $mx+n$, $(m, n)=1$. Legendre showed, as did Mr. Bennett, the existence of at

[1] This MONTHLY, vol. 39 (1932), pp. 27–28.

[2] Legendre, *Théorie des nombres*, Tome II, Paris (1830), pp. 71–79.

least $2p_{k-1}-1$ consecutive integers each of which is divisible by some factor of π_k. But Legendre thought this number also to be the maximum number of the problem in question. This however is incorrect, though the fallacy was for a long time unnoticed.

Dirichlet was the first who recognized the incorrectness of the proof Legendre gave for his statement,[1] but the statement itself he considered to be true and tried in vain to find a valid proof for it.

In 1858, the Academy of Paris proposed as a problem for competition:[2] *To prove the statement of Legendre or to find the actual maximum number of the problem.* None of the papers delivered contained a satisfactory solution of the question. The prize was awarded to A. Dupré who showed[3] that the statement of Legendre is correct for $p_k \leqq 19$ and for $p_k = 29, 31, 41$, but that it does not hold generally because it is incorrect for $p_k=23$, 37 and for $43 \leqq p_k \leqq 113$.

Later on, Moreau[4] and Piltz[5] independently pointed out the incorrectness of the statement of Legendre. Piltz supposed the statement to be wrong for all sufficiently great values of p_k.

Recently, Mr. H. Zeitz and I gave the first general proof[6] that *the statement of Legendre is incorrect for all primes* $p_k \geqq 43$. Morever, given any small number $\epsilon \leqq 0$, there is always a number $k_0=k_0(\epsilon)$ such that for all integers $k>k_0$ there exist

$$N = (4-\epsilon)p_k$$

consecutive integers each of which is divisible by one of the primes $p_1, p_2, \cdots, p_k$.

This result was improved in a paper of E. Westzynthius[7] who for the above number N substitutes the value

$$N = (2-\epsilon)e^C \frac{\log\log p_k}{\log\log\log p_k} p_k,$$

where C denotes the constant of Euler.

---

[1] Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Faktor sind, unendlich viele Primzahlen enthält*, Werke I, pp. 313–342.

[2] *Rapport sur le concours pour le grand prix de sciences Mathématiques*, Comptes Rendus 48 (1859), pp. 487–488.

[3] Dupré, *Examen d'une proposition de Legendre relative à la théorie des nombres, ouvrage placé en première ligne par l'Académie des Sciences dans le concours pour le grand prix de Mathématiques de 1858*, Paris (1859) Mallet-Bachelier.

[4] Extrait d'une lettre de M. Moreau, Nouvelles Annales (2) XII (1873), pp. 323–324.

[5] Piltz, *Ueber die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze*, Habilitationsschrift Jena 1884.

[6] A. Brauer und H. Zeitz, *Ueber eine zahlentheoretische Behauptung von Legendre*, Sitzungsberichte d. Berliner Mathematischen Gesellschaft, 29 (1930), pp. 116–125.

[7] E. Westzynthius, *Ueber die Verteilung der Zahlen, die zu den n ersten Primzahlen teilerfremd sind*, Commentationes Physico-Mathematicae, Societas Scientiarum Fennicae, vol. 25 (1931).

A NOTE ON THE ROOTS OF A CUBIC

By E. C. Kennedy, College of Mines, El Paso, Texas

If the cubic

$$X^3 + AX^2 + BX + K = 0$$

(A, B, and K rational) has one or more rational roots it may often be solved very quickly by the scheme illustrated below. If c is a rational root and $a \pm \sqrt{b}$ are roots, a and b rational, it can be shown that

$$2a + c = -A \tag{1}$$

$$b = K/c + a^2 \tag{2}$$

$$2ac^2 - Bc - K = 0. \tag{3}$$

From (3), $D = B^2 + 8Ka$ must be made a perfect square by some rational value of a. This almost instantly gives us the three roots of our cubic. To illustrate the brevity of our method let us examine, for integral roots,

$$X^3 - 304X - 1920 = 0 \tag{4}$$

Here,

$$2a + c = 0 \tag{5}$$

$$b = -1920/c + a^2 \tag{6}$$

$$D/4 = 23104 - 3840a \tag{7}$$

To find a we arrange the work as follows:

$$\begin{array}{r} 23104 \\ \underline{3840} \\ 19264 \\ \underline{3840} \\ 15424 \\ \underline{3840} \\ 11584 \\ \underline{3840} \\ 7744 \end{array} = \text{a perfect square.}$$

Thus $a = 4$. From (5) $c = -8$ and from (6) $b = 256$, and the roots of the cubic are $X_1 = 4 + 16 = 20$, $X_2 = 4 - 16 = -12$, $X_3 = -8$.

This scheme is often much quicker than finding the roots by trial, especially if the constant term has a large number of factors as does 1920. The method will always work if the roots are integers—though a may be negative or zero (the latter if $A = K/B$).

Rational roots present no difficulty. For example, to solve

$$6X^3 - 21X^2 + 2X - 56 = 0, \tag{8}$$

$$2a + c = 21/6$$

$$b = -56/c + a^2$$

$$D = (2/6)^2 - \frac{8(56)a}{6} = \frac{4 - 224a'}{36}. \qquad (a' = 12a)$$

Evidently $a' < 0$ (otherwise c would be complex since a' is an integer) so we write

$$\begin{array}{r} 4 \\ \underline{224} \\ 228 \\ \underline{224} \\ 452 \\ \underline{224} \\ 676 \end{array} = \text{a perfect square.}$$

Thus $a' = -3$, $a = -3/12$, $c = 4$, $b = -109/48$. This gives us immediately all three of the roots. We set a' equal to $12a$ instead of equal to $6a$ because the equation might conceivably be resolved into two factors one of which is a quadratic with the coefficient of X (in the quadratic) odd.

An interesting fact may be observed at this stage. By Descartes' Rule there are either three or one positive real root of (8). There are no negative real roots. Now if there are three real roots, it follows that $a > 0$. But we must have $a < 0$ in order to make $4 - 224a > 0$ which it must be in order for c to be real. Therefore, (8) has exactly one real root. This is quicker than computing the discriminant and determining the number of real roots in that way.

As a final illustration let us test

$$X^3 - 14X^2 + 9X - 12 = 0. \tag{9}$$

Here $a > 0$ and we write mechanically $D = 81 - 48a$, $81 - 48 = 33$, not a perfect square. Therefore no rational roots.

Note: The scheme described above will always work and is often quite short. It seems that every method for finding rational roots has some drawbacks. In our method we cannot always determine the sign of a immediately, though we may often find very restricted limits for it. Again it sometimes happens that we obtain a value of a that will not serve. In such a case we quickly eliminate that possibility by considering (1), (2), or (3). If the roots are all rational then there are three acceptable values of a, any one of which will serve. The scheme has one big advantage over the usual method in that the amount of labor involved is wholly independent of the number of factors in the constant term. It is not expected to displace other methods of determining rational roots, but it is often useful.

LINE REPRESENTATION OF THE HYPERBOLIC FUNCTIONS

By C. A. HUTCHINSON, University of Colorado

The following construction for representing the values of the hyperbolic functions by lines may be of interest. That part of it which concerns the hyperbolic sine and cosine is old.

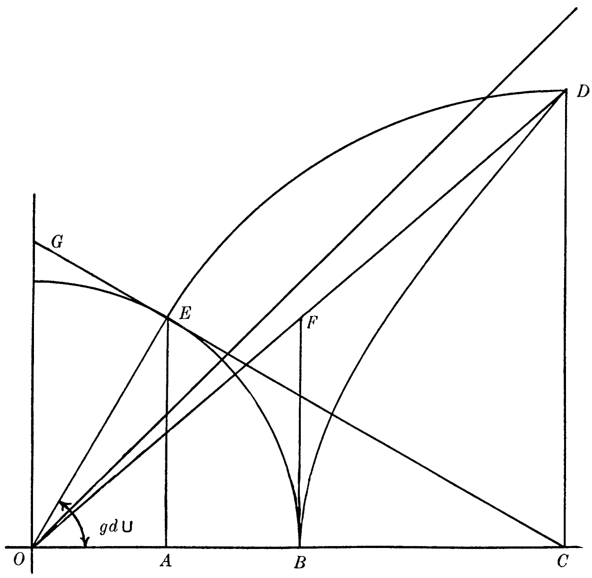

Fig. 1

Draw the "unit" equilateral hyperbola

$$x^2 - y^2 = 1,$$

and the corresponding circle

$$x^2 + y^2 = 1.$$

Let D be any point on the hyperbola, and

$$u = (2 \times \text{area of hyperbolic sector } OBD)/(OB)^2.$$

Draw DC perpendicular to the transverse axis OC. From C draw CG tangent to the circle at E. This can be done easily by intersecting the circle at E by a circular arc with C as center and CD as radius; or E can be located as the intersection of the circle with the line joining D to the other vertex of the hyperbola, H (not shown in Fig. 1). OG, AE and BF are perpendicular to OC. Then

$\sinh u = CD = CE$;	$\coth u = OG$;
$\cosh u = OC$;	$\operatorname{sech} u = OA$;
$\tanh u = AE = BF$;	$\operatorname{csch} u = EG$;

angle $AOE = \text{gd } u =$ gudermannian angle of u.

Note the relations read from right triangles:

from triangle OCE: $\sinh^2 u + 1 = \cosh^2 u$;

from triangle OAE: $\operatorname{sech}^2 u + \tanh^2 u = 1$;

from triangle OEG: $1 + \operatorname{csch}^2 u = \coth^2 u$.

The same diagram lends itself also to the representation of the functions of $u/2$ (see Fig. 2). Let the line DH intersect OP at J. Draw JK parallel to OC,

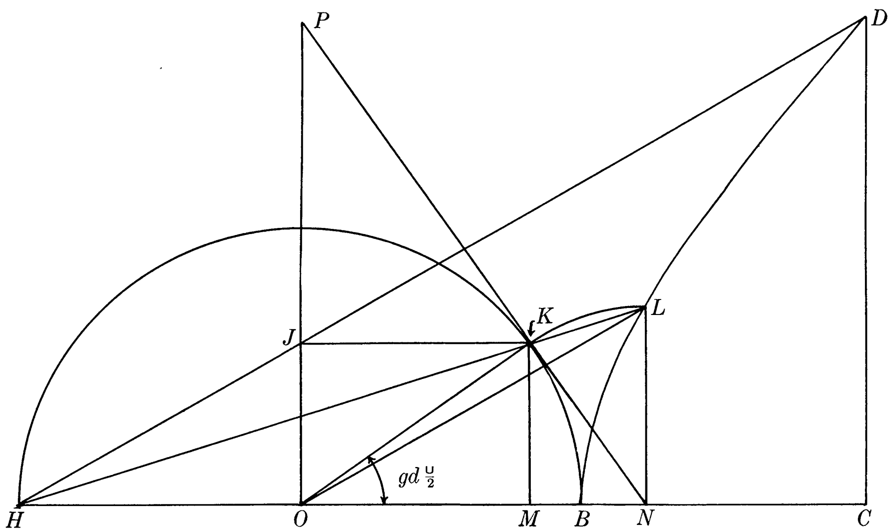

Fig. 2

intersecting the circle at K. At K, draw NP tangent to the circle. Through K and H, draw HL, intersecting the hyperbola at L. Then

$$\sinh u/2 = NL = NK; \qquad \coth u/2 = OP;$$
$$\cosh u/2 = ON; \qquad \operatorname{sech} u/2 = OM;$$
$$\tanh u/2 = OJ = MK; \qquad \operatorname{csch} u/2 = PK;$$
$$\text{angle } NOK = \operatorname{gd} u/2.$$

It will be noticed that line OL gives a bisection of the hyperbolic sector OBD. This line also bisects angle AOE, or gd u. This fact gives an even easier construction of point L. (To make the second figure clearer, the unnecessary lines of Fig. 1 have been omitted.)

Note. After sending the manuscript of this note to the MONTHLY, the author received a copy of Franklin's *Differential Equations for Electrical Engineers*, and noted that it contains a very similar construction for some of the hyperbolic functions.

REVIEWS

Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur. By Heinrich Dörrie. Breslau, Ferdinand Hirt, 1933. vii+386 pages.

This is a remarkable and unique book. The author has selected from the entire field of mathematics one hundred interesting and significant problems and gives a complete elementary solution for each, elementary in the sense that no use is made of the calculus. The nature and unusual scope of these problems will be clear from a brief summary.

The first section of arithmetic problems includes preliminary tid-bits such as Archimedes' cattle problem, Newton's cattle problem, Berwick's problem of the seven sevens, Bachet's weights problem, and Kirkman's school-girls problem. Then after a clever introduction of the infinite series representing the elementary functions, there are proofs of the fundamental theorem of algebra, of the impossibility of algebraic solution of the general algebraic equation of degree greater than four, and of the transcendence of e and of π.

The second section of problems in plane geometry includes synthetic proofs for the Euler line, nine-point circle, Malfatti's problem, Apollonius's problem of the construction of circles tangent to three given circles, analytic proofs of the impossibility of the trisection of the angle and duplication of the cube, and a construction of the regular seventeen-sided polygon.

The third section deals with conics and cycloids. Included in the twenty-four problems of this section are various construction problems, Steiner's theorem that the envelope of the Wallace lines of a triangle is a three-cusped hypocycloid, the determination of the curvature of a conic, quadrature of the conics, and rectification of the parabola. The chapter ends with a closely-knit condensed treatment of projective geometry including Pascal's and Brianchon's theorems, and applications to the construction of conics.

The fourth section deals with problems in three-dimensional geometry. It includes a few properties of tetrahedra, the determination of the five regular solids, a proof that any quadrilateral can be projected into a square, stereographic projection, and Mercator's projection.

The fifth section deals with nautical and astronomical problems such as the length of a loxodrome on a sphere, the determination of position at sea, duration of polar night, construction of sun-dials, and the retrograde motion of the planets.

The sixth and final section deals with problems of extrema. As typical problems we may cite Fagnano's problem of the triangle of minimum perimeter inscribed in an acute-angled triangle, the maximum brilliance of Venus, determination of shortest twilight, and Steiner's theorem that of isoperimetric curves the circle has maximum area.

There is, of necessity, throughout the book, a certain amount of development, so that reference to earlier problems is frequent, and many problems do

not stand alone. The proofs are, however, in the main those by the great mathematicians whose names are associated with the problems. The extent to which reference to the literature should be made was probably a difficult question for the author; as it is for the reviewer. In general the very brief historical references are confined to author, nationality, and dates; original memoirs are sometimes cited but not in the majority of cases. The reviewer would have preferred to see an exact reference to an original paper for each problem, and a carefully selected, but not exhaustive, group of references which would allow the interested reader to approach the subject from other points of view.

The references given are, in the main, accurate. But problem 36 on the seventeen-sided polygon contains an error. Gauss did prove the possibility of construction of a regular polygon of n sides where n is a prime Fermat number, but he did not give a proof for the impossibility for all other primes, as stated by Dörrie. Here, for example, the single reference to the *Disquisitiones arithmeticae*, 1801, seems totally inadequate for such a historically significant problem. In problem 57 on the rectification of the parabola, credit should be given to Huygens.

The author has high standards of accuracy and rigor which are maintained throughout. It is of course difficult to say what theorems may be considered as well-known and assumed without reference. The reviewer would prefer in the proof of the transcendence of e to have a definite statement that the number of primes is infinite, a theorem necessary for the proof as given. Again in the fundamental theorem of algebra, page 107, it is questionable whether the existence of the minimum should be assumed without comment.

The proof-reading is of the highest order, the only slip in the entire book which was noted being in the tenth decimal place of the approximation to e given on page 52, where 1 should replace 2. A word should be said with regard to the *sans serif* type employed. This may prove a satisfactory type for mathematical work if contrasting type is used for letters in equations. This is not done, and some pages (page 277, for example) are harder reading than is necessary.

This book should be widely circulated. Members of mathematics clubs will here find excellent material for presentation at meetings. It should be read as much for pleasure as for profit. We are under obligations to the author for this discriminating selection of the brilliant problems of mathematics. It is regrettable that there is nothing like it in English.

B. H. Brown

Philosophischer Versuch über die Wahrscheinlichkeit, a translation of Essai philosophique sur les Probabilités, by P. S. de Laplace, edited by R. v. Mises. (No. 233 Ostwald's Klassiker der exakten Wissenschaften). Akademische Verlagsgesellschaft M.B.H., 1932. vii+211 pages.

Two previous translations into German of Laplace's well known essay have appeared, one in 1819 based on the original third edition, the second in 1886

based on the fourth edition. Both editions have long been out of print. The present translation is by Dr. Heinrich Löwy and follows the older translation of 1886 and the fifth edition (1825) of the original. The new translation presents the essay in more modern German, corrects slips and gives more precision to many statements.

The main value of this edition lies in the forty pages of "Anmerkungen" by H. Pollaczek-Geiringer. In general these notes are of two kinds. First, many of the more difficult mathematical passages of the original are explained and put into more modern symbolism. Second, the ideas of Laplace are compared with those of the more modern theory of probability. These comparisons are subjective, and, von Mises being editor of the book, we are not surprised to find in these "Anmerkungen" a very clear presentation of the relative frequency theory and the usual arguments centering about "Gleichmöglichkeit."

A. R. CRATHORNE

Arithmetic for Teachers. By Harriet E. Glazier. New York, McGraw-Hill Book Company, 1932. xv+291 pages. $2.00.

This book might well be called *Elementary Arithmetic from an Advanced Standpoint*. Subject, rather than method, is emphasized throughout the volume.

The author traces the development of arithmetic as an outgrowth of simple natural conditions involving number ideas, through stages of crude number symbolism, into our present scheme of notation and processes of operations with these symbols. The basic principles underlying the fundamental operations are set forth. The book is written as a text for teachers courses in colleges and universities and serves this purpose admirably well. An attractive feature is the list of readings and exercises at the end of each chapter. The typographical set-up of the book is unusually good.

Chapters II and VII should be of interest to the general reader. Chapter II deals with the number concept, clearly explains number bases and number systems, and summarizes the history of notation. Chapter VII, which is on measurement, gives the essentials of a good unit of measure and compares the different standard systems. In Chapters VIII and IX the subjects of geometry and algebra are introduced. The entire book is rich in historical references.

MAY M. BEENKEN

Introduction to Practical Astronomy. By Dinsmore Alter. New York, Thomas Y. Crowell Company, 1933. VIII+80 pages. $2.00.

This book might well be called a laboratory manual, since it treats the material as experiments with very little theoretical discussion. The book is divided into four sections beginning with "preliminary experiments" which include experiments with the vernier, systems of coordinates, and lenses. Section II considers time, relations between coordinate systems, experiments concerning the planets and precession. The greater part of Section III is devoted to the de-

termination of positions on the earth's surface using the engineers transit. Section IV takes up a method of determining the errors of adjustment of an equatorial telescope and the right ascension and declination of objects. There are a liberal number of sheets of coordinate paper in the back of the book.

The experiments cover a wider field than I believe can be found in other elementary books of this nature. It seems however that additional material might profitably be added in the first two sections. A set of good star maps would also be desirable. The sections on the engineer's transit and the equatorial telescope should prove especially interesting for those students who do not expect to take more than an elementary course in astronomy.

The chief criticism of the book is the rather frequent use of poorly constructed sentences especially in the introduction. The material on the whole is well selected and the book will be a welcome addition to the rather limited number of manuals of laboratory astronomy.

M. F. JORDAN

C. A. Bjerknes, sein Leben und seine Arbeit. By V. Bjerknes. Aus dem Norwegischen ins Deutsche übersetzt von E. Wegener-Köppen. Berlin, Julius Springer, 1933. iv+218 pages. RM 8.60.

In this biography V. Bjerknes gives a sympathetic picture of the difficulties and struggles encountered by a young gifted man without means desirous of becoming a scientist. C. A. Bjerknes, well known for his hydrodynamical investigations, was born in Norway in 1825. The economic situation both in his family and in the country was none too good at the time. The School of Mines afforded the only opportunity for scientific study and the school required active service in the mines even for students wanting to specialize in more abstract subjects like mathematics and astronomy. After completing his studies Bjerknes had to accept a position as overseer in the copper mines and the study of mathematics had to be confined to his spare moments. After several disappointments he obtained a small position at the university and shortly afterwards a fellowship for foreign study. He came to Göttingen in 1855 and worked with Lejeune-Dirichlet, to whom he had been recommended, and also attended the lectures of Riemann, where Dedekind and Schering were the only other students. Next year he continued his studies in Paris with Cauchy, Liouville, Bertrand and Lamé.

Upon his return to Norway Bjerknes was appointed professor at the University and started the long series of investigations and experiments, which he continued for the rest of his life, on hydrodynamical phenomena and their analogy to electricity.

The book gives a quaint narrative of the life of a typical scientist. It is a biography of a father by his son, it consequently eulogizes his life and works and is adorned by a large number of personal details.

OYSTEIN ORE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1932–1933

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Alabama

Our chapter has forty active members. On October 11, 1932, five were initiated and on April 25, 1933, seven were initiated.

The meetings and programs were as follows:

September 27, 1932: "The minimum of a definite integral with respect to one sided variation" by Julian D. Mancill.

October 25, 1932: "The four color problem" by William Sell.

November 29, 1932: "Infinite products" by Edna Jane Cofield.

January 31, 1933: "Some properties of a set of polynomials whose zeros are all real" by H. S. Thurston.

February 28, 1933: "Fourier's theory" by G. N. Carmichael.

March 28, 1933: "On the properties of a certain class of collineation groups" by Fred A. Lewis.

April 25, 1933: "Analytic proofs of some theorems concerning a Triape and its related circles" by Boyce Garrett: "Some properties and applications of continued fractions" by Mary Catherine Granade; "Some special roulettes" by Winston M. Scott.

In addition to the above meetings, two socials were given: on December 8, 1932, a bridge party and on May 5, 1933, the annual picnic.

The following members of Pi Mu Epsilon were elected to Phi Beta Kappa this year: Sarah Bloodworth, Elizabeth Shirley, Jane Ward.

One of our members, Mr. William Sell, has revived the Newtonian Club, Freshman Mathematics Society, and reports a very successful year.

The Chapter, this year, has voted to use some of its funds in subscribing to mathematical magazines for use in the general library.

WILLIAM F. ADAMA, *Secretary*

Pi Mu Epsilon of Brooklyn College

The Delta Chapter of Pi Mu Epsilon in New York was installed at Brooklyn College of the College of the City of New York on February 6, 1933. The installation was held at the Hotel Pierrepont, Brooklyn, and was conducted by Professor Lao G. Simons assisted by Professors Marie Whalen and Mina S. Rees, all of the Hunter College Chapter (Beta of New York). Fifteen instructors and thirty-three graduates and undergraduates were initiated. The chapter grew out of the Delta Society which was organized in June, 1931, and held regular meetings since that time. Miss Alma M. Hauland, who has been President of the Delta Society, presided at the initiation banquet.

The officers of the new chapter are: Professor Roger A. Johnson, Director; Miss Elsie Kardonsky, Vice Director; Miss Irene Silverstein, Secretary.

During the first semester, the meetings of the Delta Society were devoted to the study of differential geometry. In the present semester, geometrical transformations have been studied.

R. A. Johnson, *Director*

LOCAL MATHEMATICS CLUBS

The Euclid Club of the College of William and Mary

The officers for the year 1932–1933 were: C. S. Sherwood, III, President; Louise Lang, Vice President; Elizabeth Toler, Secretary; Ethel Hartman, Treasurer; Dr. J. M. Stetson and Miss Beulah Russell, Faculty Advisers.

The aim of the club is to stimulate a creative interest in mathematics. Membership is open to all persons who intend to major or minor in mathematics and who have completed nine semester hours and are registered for at least three hours more. A grade of 91 is required in one course and an average of 85 in the other courses. This year we had forty members.

The meetings and programs were as follows:

October 21, 1932: "The influence of mathematics on civilization." Papers were read by Douglas Matthew, Elizabeth Potterfield, Susie Brittle and Eleanor Berger.

November 18, 1932: "The structure of the aeroplane wing" by Dr. Stetson.

December 16, 1932: "Simple aspects of Einstein's theory of relativity" by Alfred Armstrong.

February 17, 1933: "Probability" by Marianne Norris and Douglas Matthew.

March 17, 1933: Business meeting.

April 21, 1933: Election of officers.

April 26, 1933: Banquet. Dr. J. A. C. Chandler was our guest and speaker of the evening.

Elizabeth Toler, *Secretary*

The Mathematics Club of Brown University

The meetings and programs were as follows:

November 7, 1932: Scott Read Chatterton, '33, presiding; "Centennial dates" by Mary Helena Quirk, '34; "The puzzler's perplexities" by Charles Roland Eddy, '35.

December 12, 1932: Tina Codianni, '33, presiding; "Parabolic fit" by Vernon Franklin Kenyon, '35; "Areas by machine" by Cyril Garbutt Sargent, '33.

January 16, 1933: Dean Richardson, presiding; "When stars shone in the afternoon" by Professors Currier and Smiley.

February 20, 1933: Garland Balch Russell, '33, presiding; "Brilliant points" by William Fuller Branch, '34; "The path of the pendulum" by Walter Harris Porter, '34.

March 20, 1933: Elizabeth Alma Partridge, '33; presiding; "Archimedes" by Harriet Amy Legg, '34; "Omar Khayyam" by Jean Esther Smith, '33; "Galileo" by Mary Stella Barao, '34.

April 17, 1933: Professor Gilman, presiding; "Repeating decimals" by Professor Widder of Harvard University.

Our committee on program was: Professor Bennett; Professor Oakley; Mary Stella Barao, '34; Scott Read Chatterton, '33; Charles Roland Eddy, '35; Mary Helena Quirk, '34.

Our committee on arrangements was: Mr. Tucker; William Fuller Branch, '34; Edgar Gage Hotaling, '35; Edith Viola Janson, '34; Rhoda Madden, '35.

R. C. Archibald, *Professor of Mathematics*

The Mathematics Club of the University of Buffalo

The mathematics club of the University of Buffalo is open to all students interested in mathematics whether they are majors or not. The attendance at the meetings this year averaged about thirty. Some of the meetings were social gatherings at the home of the professors and others were business meetings at which student papers were read.

The officers for the year 1932–1933 were: John Wrench, President; Lois Plummer, Vice President; Charlotte Houck, Secretary and Treasurer.

The meetings and programs were as follows:

October 1932: Opening supper. Motion pictures of "The Span Supreme."

November 1932: The club was entertained at the home of Professor and Mrs. Pound with a mathematical spelling bee and games.

January 1933: "A sketch of Einstein" by John Wrench; "A review of 'The Queen of the Sciences' by E. T. Bell" by Alice Link.

March 1933: "Systems and properties of axioms" by Charles Strobel; "Axes and centers of symmetry" by Robert R. Lyle. (The latter was taken from the thesis submitted by Mr. Lyle in partial requirement for the degree of Master of Arts, granted February, 1933.

April 1933: The club was entertained at a Monte Carlo party at the home of Professor and Mrs. Harrington.

May 1933: The retiring president, John Wrench, was host to the club at his home. The Wilfred H. Sherk Memorial Prize in Mathematics was awarded to Charles Strobel for his paper submitted on "Axioms and postulates and their properties."

HARRIET F. MONTAGUE, *Department of Mathematics*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 48. *Proposed by Norman Anning, University of Michigan.*

If the squares of the sines of a set of angles are in harmonic progression, show that the squares of the tangents of the same angles are also in harmonic progression.

E 49. *Proposed by Arnold Dresden, Swarthmore College.*

In the following problem in long division it is required to determine the non-zero digits, which are designated by x's. The zeros are properly shown.

$$\begin{array}{r@{}ccccccccccc@{}l}
x\,0\,x\,0\,x) & x & x & x & 0 & x & 0 & x & 0 & 0 & x & 0 & (x\,0\,x\,0\,0\,x \\
 & x & x & 0 & x & 0 & x & & & & & & \\ \cline{2-7}
 & & & & x & 0 & x & x & 0 & & & & \\
 & & & & x & 0 & x & 0 & x & & & & \\ \cline{5-9}
 & & & & & & x & 0 & x & 0 & x & 0 & \\
 & & & & & & x & 0 & x & 0 & x & 0 & \\ \cline{7-12}
 & & & & & & & & x & 0 & 0 & 0 &
\end{array}$$

E 50. *Proposed by H. T. R. Aude, Colgate College.*

Two fractions, F_1 and F_2, when written in a certain scale of notation, are $0.3737 \cdots$ and $0.7373 \cdots$ respectively. When written in a second scale of notation, these same fractions are $0.2525 \cdots$ and $0.5252 \cdots$ respectively. Find the fractions and the bases used for the two scales.

E 51. *Proposed by J. Rosenbaum, The Milford School, Milford, Conn.*

Prove that there are just three pairs of non-negative integers (x, y), which satisfy the equation, $3\cdot 2^x+1=y^2$, and determine them.

E 52. *Proposed by Moshe Abrahami, Case School of Applied Science.*

Find the area of a triangle in terms of the altitude, interior angle bisector, and median, all from the same vertex of the triangle.

E 53. *Proposed by Morgan Ward, California Institute of Technology.*

If s is a positive integer, prove that

$$D_x^s(x^s \ln x) = s!(\ln x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 1/s).$$

SOLUTIONS

E 15 [1932, 607]. *Proposed by Mrs. Pearl C. Miller, Washington University.*

Prove that if two external angle-bisectors of a scalene triangle are equal, then the sines of the three interior half-angles form a geometric progression. By external angle-bisector is here meant that segment of the line bisecting the exterior angles at a vertex of a triangle, intercepted between that vertex and the opposite side of the triangle.

Solution by Roy MacKay, Albuquerque High School.

Let $AU=s$ bisect the exterior angle A of the triangle ABC, and $BV=t$ bisect the exterior angle at B. Then angle $AUC=(B-C)/2$ and angle $BVA=(C-A)/2$. Now by the law of sines, $s=b \sin C/\sin \frac{1}{2}(B-C)$ and $t=c \sin A/\sin \frac{1}{2}(C-A)$. Equating $s=t$, and $b/c=\sin B/\sin C$ gives

$$\sin A \sin \tfrac{1}{2}(B - C) = \sin B \sin \tfrac{1}{2}(C - A).$$

By a series of easy reductions, there now results

$$2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \sin \tfrac{1}{2}(B - C) = 2 \sin \tfrac{1}{2}B \cos \tfrac{1}{2}B \sin \tfrac{1}{2}(C - A),$$
$$\sin \tfrac{1}{2}A\{- 2 \sin \tfrac{1}{2}(B + C) \sin \tfrac{1}{2}(B - C)\} = \sin \tfrac{1}{2}B\{- 2 \sin \tfrac{1}{2}(C + A) \sin \tfrac{1}{2}(C - A)\},$$
$$\sin \tfrac{1}{2} A(\cos B - \cos C) = \sin \tfrac{1}{2}B(\cos C - \cos A).$$

Changing the cosines to one minus twice the square of the sine of the half-angle, rearranging and dividing out $\sin \frac{1}{2}A+\sin \frac{1}{2}B$, which is not zero, we have the desired result:

$$\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B = \sin^2 \tfrac{1}{2}C.$$

Solved also by Simon Vatriquant and the proposer.

E 22 [1933, 490]. *Proposed by R. M. Winger, University of Washington.*

As a western version of problem E 7, find the digits represented by the various letters in the following problem in addition, and determine whether or not the solution is unique. (Except that obviously R and L and S and G are interchangeable.) No two different letters represent the same digit.

$$\begin{array}{ccccc} & S & E & N & D \\ & M & O & R & E \\ & G & O & L & D \\ \hline M & O & N & E & Y \end{array}$$

Solution by Simon Vatriquant, Athénée Royal d'Ixelles, Brussels, Belgium.

1. Obviously $M=1$ or 2. If $M=2$, since at most $S+G=17$, $O=0$ or 1. If $O=0$, then in the second column E would have to be 9, which would reduce $S+G$. If $O=1$, the carry from the second to the first column would be insufficient. Consequently, $M=1$.

2. The sum of the ten digits is 45, and the sum of the digits of the three numbers to be added $\equiv M+O+N+E+Y$ (mod. 9).

3. The alternate sum of the digits of the three numbers to be added $\equiv M-O+N-E+Y$ (mod. 11).

4. $2D+E\equiv Y$ (mod. 10). Hence D is neither 0 nor 5.

Therefore we may write

$$2Y+N+1=D, \text{ or } D+9, \text{ or } D+18.$$

$$3D+4E+4O\equiv N+2 \pmod{11}.$$

Cancelling the values which give a repeated digit, there remain the following systems: (Note that the word $MONEY$ is determined in each case, and we may thus calculate the sums $S+G$ and $R+L$)

Y	N	D	E	O	$R+L$	$S+G$	R,L	S,G
0	5	6	8	3	11	11	4,7	2,9
0	5	6	8	3	11	11	2,9	4,7
0	7	8	4	6	5	14	2,3	5,9
2	9	3	6	0	impossible			
6	3	7	2	5	8	13	0,8	4,9
7	0	6	5	2	impossible			
6	9	4	8	0	8	9	3,5	2,7
7	5	2	3	0	impossible			
8	4	3	2	0	impossible			
8	5	4	0	7	5	15	2,3	6,9
9	5	6	7	4	impossible			
	6	7	5	0	impossible			

Permuting the R, L and S, G in all possible ways gives twenty-four different solutions, of which the six fundamental ones are

2856	4856	9478	9237	2894	6054
1348	1328	1624	1502	1038	1720
9376	7396	5638	4587	7054	9734
13580	13580	16740	15326	10986	17508

Solved also by M. A. Heaslet.

E 23. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In the following sum and product, the digits from 1 through 9 are represented in some order by the letters A through I. Determine the representation and prove that it is unique.

$$AB = CD + EF, \; EF = G \times HI.$$

Solution by W. E. Buker, Leetsdale, Pa.

1. Since $EF = G \times HI$, $G \neq 1$ or 5. $I \neq 1$ or 5.

2. Since $AB < 99$, $11 < CD$, it follows that $EF < 87$. But the solution $98 = 12 + 86$ is ruled out by the repeated 8, and $EF \neq 85$ since neither G nor $I = 5$. Hence $EF < 85$.

3. Since the problem demands that no two letters have the same value, we are reduced to the following possible values for G and HI.

$G=2$, $HI=17, 18, 19, 34, 38, 39$
$G=3$, $HI=16, 18, 19, 26$
$G=4$, $HI=13, 17, 18, 19$
$G=6$, $HI=13$
$G=7$, $HI=12$

Trial of these values shows that the only set which completely fulfills the conditions of the problem are $G=4$, $H=1$, $I=7$. Consequently, the only solution is $93 = 25 + 68$ and $68 = 4 \times 17$.

Solved also by Simon Vatriquant and the proposer.

E 24 [1933, 111]. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

There are just three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately canceling a digit. One of these is

$$\frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}.$$

Find the other two and confirm the statement that there are no others.

Solution by Pincus Schub, Gratz College, New York City.

If, after cancelling the digit, n, from numerator and denominator, the resulting proper fraction in lowest terms is a/b, then the original fraction must have had one of these four forms:

(A) $(10a+n)/(10b+n)$, (B) $(10n+a)/(10n+b)$, (C) $(10n+a)/(10b+n)$, or (D) $(10a+n)/(10n+b)$. Here a, b and n are each positive integers <10, and $a<b$.

It is easily seen that (A) and (B) each lead to a contradiction. (C) is also impossible, since if $a/b=(10n+a)/(10b+n)$, then $n=9ab/(10b-a)$, $<a$. This last is true because $9b<10b-a$. From this it would follow that $b=an/(10n-9a)<0$, which is impossible. Hence only (D) remains possible.

From (D), $n=9ab/(10a-b)$, which is larger than $9ab/(10a-a)=b$. That is, $b<n$. Again from (D), $a=bn/[9(n-b)+n]\leqq\frac{1}{2}b$, since $9(n-b)\geqq n$. Hence $a<5$.

Since $b=10a-9ab/n$, and $a<b$, it follows that n is a multiple of 3. This leaves just four pairs of values for a and n to make b an integer, namely: 1,6; 1,9; 2,6; and 4,9. The resulting fractions are 16/64, 19/95, 26/65 and 49/98. But the last is not reduced to lowest terms by the cancellation, and hence there are only three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately cancelling a digit.

Solved also by W. E. Buker, L. G. Butler, L. S. Johnston, Theodore Lindquist, E. Mrock, C. W. Munshower, Simon Vatriquant and the proposer.

E 25. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Find the only ten-place integer which is both a square and a triangular number.

Solution by H. T. R. Aude, Colgate University.

The desired number must be of the form $\frac{1}{2}n(n+1)$ to be triangular, and must also be equal to some square, m^2. Replacing $2n$ and $2m$ by $x-1$ and y, respectively, gives the Pellian equation: $x^2=2y^2+1$. Its primary solution is $x=3$, $y=2$, which through the relation

$$x + y\sqrt{2} = (3 + 2\sqrt{2})^k, \qquad [k = 0, 1, 2, 3, \cdots]$$

yields all positive integral solutions. We next examine the values of k to see which, if any, will make m^2 a ten-place integer. The only value is $k=7$, whence $y=80782$, and the desired number is

$$1{,}631{,}432{,}881=40{,}391^2=(57{,}121)(57{,}122)/2.$$

Solved also by W. E. Buker, W. R. Ransom, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

A Note by Otto Dunkel. This problem can be easily solved by elementary geometry. As is seen from the above solution there are triangles having a negative area analytically, and the analysis gives a maximum of zero for such triangles. It is obvious from a figure that in such triangles the fixed point does not lie within the segment of the side which passes through that point. Setting aside this case which is of no interest, we may state the problem as follows:

Given that P is a fixed point lying within the fixed angle $MON < 180°$, and that MN is any straight line through P cutting the sides OM and ON of the angle in M and N, it is required to determine the position of MN so that the area of the triangle OMN is a minimum.

Through P draw a line parallel to MO cutting ON in Q. Then for an actual triangle N must lie on OQ produced. In order to compare the lengths of the segments PN and PM lay off on OM the length $OA = 2QP$, and draw AP cutting ON in B. Then $PB = PA$. Suppose first that N lies on QB produced. The exterior angle at B of the triangle OAB is greater than the interior angle at A. Hence if we rotate the triangle PAM through 180° about P, A will fall upon B and PM will fall along the line of PN. Since M is within OA, AM will fall within the interior of triangle PBN. Thus $PN > PM$, and $PBN > PAM$ where areas are compared in the last inequality. If N lies within QB, M lies on OA produced. Also angle $PAM >$ angle PBN. Hence after rotation of PAM the point M falls on PN produced. Thus $PM > PN$, and $PAM > PBN$.

Therefore in the first case I, and the second case II

I. $OMN - OAB = PBN - PAM > 0$,
II. $OMN - OAB = PAM - PBN > 0$,

where again areas are designated. This shows that OAB has the smallest possible area.

3569. [1932, 490] *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given line s to draw a plane cutting the faces of a given dihedral angle along the lines a, b, and the bisecting planes of this dihedral angle along the lines c, d, so that the lines c, d shall be the bisectors of the angles formed by a, b. The problem has, in general, two solutions. What are the special cases?

Solution by S. Vatriquant, Athênée Royale d'Ixelles, Brussels, Belgium.

Let α, β be the faces of the given dihedral angle, γ, δ, the two bisecting planes, and A, B, C, D, the points of intersection of these planes with the given line s. If the required plane cuts in X the edge of the dihedral angle, angle CXD is 90°, and a locus of X will be the sphere described on CD as diameter.

We have, in general, two solutions, namely, if the distance of the mid-point of CD to the edge of the dihedral angle is less than $CD/2$; one solution, if the same distance $= CD/2$, and no solution if that distance is more than $CD/2$.

But this last case cannot occur, for the plane through CD and the shortest distance cuts the right dihedral angle $\gamma\delta$ along an angle$>90°$, and the median of the obtained triangle is less than or equal to $CD/2$. The second case occurs when s and the edge of the dihedral angle are orthogonal (1 sol.).

No solution, if s is parallel to the edge of the dihedral angle.

If s is parallel to a single face of the given dihedral angle, e.g. α, A is at infinity, and B is the mid-point of CD. The construction is the same as above.

If s is parallel to a single bisecting plane, e.g. δ, D is at infinity and we have $XA=XB$. Hence a locus of X is the mediating plane of AB, namely, the perpendicular plane to AB through C; we have in this case generally one solution, but if s is perpendicular to the bisecting plane γ, we have ∞ solutions.

Solved also by W. B. Campbell, J. M. Feld, A. Pelletier, and W. H. Rasche.

3571. [1932, 491] *Proposed by Lester R. Ford, Rice Institute.*

What should one pay in order to receive 1, 8, 27, $\cdots$, n^3, $\cdots$, dollars at the end of 1, 2, 3, $\cdots$, n, $\cdots$, years, the interest rate being six percent?

Solution by Robert E. Moritz, The University of Washington.

It is assumed that what is sought in this problem is the limit of the sum

$$S_n = v + 2^3v^2 + 3^3v^3 + \cdots + n^3v^n,\ v = (1.06)^{-1},$$

as n approaches infinity. The existence of the limit may be inferred from the fact that the corresponding infinite series is convergent, the Cauchy test ratio approaching v in the limit.

The sum of the first n terms is readily obtained by multiplying both sides of the above equality by $(1-v)^4$. We find

$$\begin{aligned}(1-v)^4S_n = {} & v+2^3v^2+3^3v^3+4^3v^4+\cdots+n^3v^n\\ & -4v^2-4\cdot 2^3v^3-4\cdot 3^3v^4-\cdots-4n^3v^{n+1}\\ & +6v^3+6\cdot 2^3v^4+\cdots+6(n-1)^3v^{n+1}+6n^3v^{n+2}\\ & -4v^4-\cdots-4(n-2)^3v^{n+1}-4(n-1)^3v^{n+2}-4n^3v^{n+3}\\ & v^5+\cdots+(n-3)^3v^{n+1}+(n-2)^3v^{n+2}+(n-1)^3v^{n+3}+n^3v^{n+4}.\end{aligned}$$

The coefficient of v^k for $k=5, 6, 7, \cdots, n$ is equal to

$$k^3-4(k-1)^3+6(k-2)^3-4(k-3)^3+(k-4)^3$$

which vanishes for all values of k. Collecting the remaining coefficients we have

$$S_n = \frac{v+4v^2+v^3}{(1-v)^4} - \frac{(n+1)^3v^{n+1}-(3n^3+6n^2-4)v^{n+2}+(3n^3+3n^2-3n+1)v^{n+3}-n^3v^{n+4}}{(1-v)^4}.$$

L'Hospital's Theorem shows that as n approaches infinity the second part of the expression on the right approaches 0 as a limit. Therefore

$$S = \lim S_n = \frac{v + 4v^2 + v^3}{(1 - v)^4}.$$

Since $v = (1.06)^{-1}$, we readily find $S = 520{,}479.63$, the number of dollars of the present value sought.

Solved also by W. B. Campbell, A. G. Clark, J. D. Leith, F. L. Manning, A. S. Merrill, H. A. Meyer, and F. Underwood.

A Note by Otto Dunkel. The finite series of this problem belongs to a more general class which can be summed by a process which may be of interest since it is quite analogous to integration by parts. Let $\sum$ denote the operation of summing, and let Δ denote the difference operator with the unit difference of unity. Then for any two functions $u(x)$ and $v(x)$ the formula

$$\Delta u(x)v(x) = u(x + 1)\Delta v(x) + v(x)\Delta u(x) \tag{1}$$

is obvious. Hence by summation we have

$$\sum^n v(x)\Delta u(x) = u(n + 1)v(n + 1) - \sum^n u(x + 1)\Delta v(x), \tag{2}$$

where a constant is to be added on the right depending upon the lower limit for x. This is our formula for summation by parts.

The example in the problem is of the type in which the terms to be summed are of the form $P(x)a^x$, where $P(x)$ is a polynomial in x of degree k and a is a constant not unity. Since $\Delta\, a^x = (a - 1)a^x$, we have from (2)

$$\sum^n P(x)a^k = \sum^n \frac{P(x)}{a - 1}\Delta a^x = \frac{P(n + 1)}{a - 1}a^{n+1} - \sum^n \frac{\Delta P(x)}{a - 1}a^{x+1}. \tag{3}$$

Repeating the process (3) on the second term on the right, and continuing in this way using (3) as a recursion formula, we have

$$\begin{aligned}\sum^n P(x)a^x = {} & \frac{P(n + 1)}{a - 1}a^{n+1} - \frac{\Delta P(n + 1)}{(a - 1)^2}a^{n+2} + \frac{\Delta^2 P(n + 1)}{(a - 1)^3}a^{n+3} - \cdots \\ & + (-1)^k \frac{\Delta^k P(n + 1)}{(a - 1)^{k+1}}a^{n+k+1}.\end{aligned} \tag{4}$$

Setting $P(x) = x^3$, we have for the given example

$$\begin{aligned}&\sum_0^n x^3 a^x \\ &= \frac{a^{n+1}}{(a - 1)}\left[(n + 1)^3 - \frac{a}{a - 1}(3n^2 + 9n + 7) + \frac{6a^2}{(a - 1)^2}(n + 2) - \frac{6a^3}{(a - 1)^3}\right] + C\end{aligned}$$

The value of C is found by setting $n = 0$, and we obtain

$$C = \frac{a^3 + 4a^2 + a}{(a - 1)^4}.$$

Finite series of the form

$$\sum^{n} P(x) \sin (ax + b),$$

where a and b are constants and $P(x)$ is again a polynomial of the kth degree, may also be summed by the same method. For we may write

$$\sin (ax + b) = \frac{\Delta \sin \left[ax + b - \frac{1}{2}(a + \pi)\right]}{2 \sin (a/2)},$$

and the summation proceeds as before.

3572. [1932, 549] *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The sum of the three bimedians of a tetrahedron (i.e., the lines joining the mid-points of the pairs of opposite edges) is less than one half and greater than one fourth of the sum of the edges of the tetrahedron.

Solution by J. Rosenbaum, Milford, Conn.

Let the sides of a face be a, b, c, and the opposite edges, x, y, z. Also let the mid-points of a, b, c be P_1, P_2, P_3, and the mid-points of x, y, z; Q_1, Q_2, Q_3. From the triangle $P_2Q_2Q_3$ we have $P_2Q_2 < Q_2Q_3 + Q_3P_2$; and there are two other inequalities which follow in the same manner. These may be written

$$P_2Q_2 < (a + x)/2, \; P_3Q_3 < (b + y)/2, \; P_1Q_1 < (c + z)/2.$$

The addition of these three inequalities yields the first part of the required inequality.

The second part is proved by making use of the theorem that the bimedians of a tetrahedron are concurrent at a point O which bisects each bimedian. Thus from the triangle OP_2P_3 we have $OP_2 + OP_3 > P_2P_3$, or $(P_2Q_2 + P_3Q_3)/2 > a/2$. There are two other inequalities which are obtained in a similar manner: and these may be written

$$P_2Q_2 + P_3Q_3 > a, \; P_3Q_3 + P_1Q_1 > b, \; P_1Q_1 + P_2Q_2 > c.$$

Adding these three inequalities we have

$$P_1Q_1 + P_2Q_2 + P_3Q_3 > (a + b + c)/2.$$

In the same manner by using triangles such as OP_1Q_3, we find

$$P_1Q_1 + P_2Q_2 + P_3Q_3 > (x + y + z)/2.$$

The addition of these last two inequalities gives the second required inequality.

Solved also by A. D. Bradley, Rufus Crane, L. S. Johnston, M. Markowitz, J. B. Meyer, H. D. Ruderman, C. A. Rupp, and F. Underwood.

3573. [1932, 549] *Proposed by Samuel I. Jones, Nashville, Tenn.*

A hawk, eagle, and sparrow are in the air. The eagle is 50 feet above the sparrow and the hawk is 100 feet below the sparrow. The sparrow flies straight forward in a horizontal line. Both hawk and eagle fly directly towards the sparrow. The hawk flies twice as fast as the sparrow. The hawk and eagle reach the sparrow at the same time. How far does each fly and at what rate does the eagle fly?

Solution by Eugene M. Berry, Lynchburg College.

First let us consider the curve of pursuit followed by either the hawk or the eagle. Let the sparrow start at the origin with a speed of unity and let $1/k$ be the speed and s the distance traveled by the pursuer. At the time $t=0$, the position of the pursuer is $(0, b)$; at any time, t, the position of the sparrow is $(t, 0)$, while that of the pursuer is (x, y). Then $t=ks$.

Since the hawk or eagle flies directly toward the sparrow we have $dx/dy = (x-t)/y = (x-ks)/y$. Differentiate this with respect to y and put q for dx/dy, and, using the fact that $ds/dy = -\sqrt{1+q^2}$, we get

$$y(dq/dy) = k\sqrt{1 + q^2}.$$

The solution of this is

$$q + \sqrt{1 + q^2} = Ay^k, \tag{1}$$

A is positive although q is negative. From the initial conditions, $q=0$ when $y=b$, we find $A = b^{-k}$. Using this and solving (1) for q we get

$$q = dx/dy = (b^{-k}y^k - b^k y^{-k})/2.$$

Integrating, we get

$$x = \tfrac{1}{2}\left[b^{-k}y^{1+k}/(1 + k) - b^k y^{1-k}/(1 - k)\right] + c. \tag{2}$$

Since $y=b$ when $x=0$, $c=bk/(1-k^2)$. The conditions at the end are $x=a, y=0$. Using these values and the value for c in equation (2) we get

$$a = bk/(1 - k^2). \tag{3}$$

It is evident that

$$s = a/k. \tag{4}$$

Solving equation (3) for k we get

$$k = (-b + \sqrt{b^2 + 4a^2})/2a. \tag{5}$$

Only the positive sign is used before the radical since k is positive and numerically less than 1, since we assume the sparrow is to be caught.

For the case of the hawk $k=1/2$, $b=100$. Then from equation (3) we get $a=66\ 2/3$ and equation (4) gives $s=133\ 1/3$.

For the case of the eagle, putting $a=66\ 2/3$ and $b=50$ in equation (5) gives $k=.693$ and $1/k=1.443$; then equation (4) gives $s=96.2$.

Thus, the sparrow travels 66 2/3 ft., the hawk 133 1/3 ft. and the eagle 96.2 ft. The speed of the eagle is 1.443 times that of the sparrow.

Solved also by J. A. Calderhead, Paul Capron, J. P. Howe, J. D. Leith, M. Markowitz, W. H. Turney, and F. Underwood.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

THE ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION

The eighteenth annual meeting of the Mathematical Association of America will be held in Cambridge, Massachusetts, December 27–30, 1933, in conjunction with the meetings of the American Association for the Advancement of Science and the American Mathematical Society. The sessions of the Society will be held on Wednesday and Thursday, those of the Mathematical Association on Friday and Saturday, and there will be a joint session with Section A of the American Association and the Society on Friday morning.

Arrangements have been made for members to be accommodated at the Commander Hotel and the Continental Hotel, Cambridge. Headquarters for mathematicians and their families will be at the Commander Hotel. The scientific sessions of the Society and the Association will be held in Longfellow Hall, Radcliffe College, Cambridge.

A joint dinner of the mathematical organizations will be held on Thursday evening at the Walker Memorial Building, Massachusetts Institute of Technology, for which the charge will be $1.50 per person. Before the dinner, there will be an exhibition of machines, charts, and the electric integraph.

Through the kindness of Professor and Mrs. J. L. Coolidge, mathematicians and their friends will be entertained at tea on Wednesday afternoon at Lowell House, Harvard University.

Special rates will be in force at the Commander Hotel and the Continental Hotel. Every room is equipped with a private bath. The charges are: single rooms, $3.00 per day; double rooms, $2.25 per day, per person. Meals are available in the hotels at reasonable prices. Those desiring accommodations at more modest prices will be able to obtain them nearby.

The usual railway rates (to Boston) of a fare and a half for return will doubtless be available.

The "Annali della R. Scuola Normale Superiore di Pisa-mathematical and physical sciences," announces a competition on the following themes:

I. Prepare a rapid digest of the theory of partial differential equations of elliptic and parabolic types.

II. Prepare a rapid digest of the theory of analytic functions of several complex variables, and of particular classes of especially remarkable functions of that kind.

To the best work on either of these two themes, which is received by the board of editors of the "Annali" (Direzione degli Annali, Pisa, Scuola Normale Superiore) not later than June 15, 1934 there will be given a prize of two thousand Lire (2000 L), of which one thousand were given expressly for this purpose by a former student of the Scuola who wishes to remain incognito. For this gift the board of directors expresses its profound thanks.

Only works not previously published will be accepted. They must be typewritten and three duplicate copies submitted, and they must be written in Italian, French, English or German, and not exceed thirty printed pages of the "Annali."

The memoir receiving the prize must be translated into Italian under the auspices of its author. This translation will be published in the "Annali."

The competition will be conducted by a commission, which will be appointed at the proper time by the board of directors of the "Annali."

Professors Lipot Fejér and Tullio Levi-Civita arrived in this country several weeks before the Society's meeting at Chicago in June, at which they both lectured. Professor Fejér lectured during April and May at Brown University, Harvard, Columbia, Princeton, Ohio State, the University of Pennsylvania, Cornell, and the University of Cincinnati, and before a sectional meeting of the Mathematical Association of America at the University of Virginia. His topics were "On new properties of the arithmetical means of the partial sums of Fourier Series" and "On the characterization of some remarkable systems of points of interpolation by means of conjugate points of the Cotes' numbers; and of certain extremal properties." His lectures at the Chicago meeting were "On the infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures." Professor Levi-Civita lectured at Princeton Harvard and Brown during the last week in May and the first week in June, on "Secular effects of the tides on the motion of a planetary system" and "On adiabatic invariants." His lectures at Chicago were "On some mathematical aspects of the new mechanics" and "Nets on a surface and extension of trigonometry."

Dr. Felix Bernstein, director of the Institute of Mathematical Statistics of the University of Göttingen, spoke before the Sigma Xi chapter at the University of Cincinnati and the mathematics club of the Ohio State University on April 10 and 11 on the subject "A solution of mathematical problems in physics and engineering by new mechanical means."

Professor Albert Einstein has resigned his membership in the Prussian

Academy of Sciences. He has accepted professorial appointments at the Universities of Madrid and Paris in addition to his professorship at the Institute for Advanced Study at Princeton.

Dr. Irving Langmuir of the General Electric Company was made an honorary member of the School of Engineering Alumni Association at Columbia University on April 26.

Dr. Henry Norris Russell, research professor in astronomy and director of the observatory at Princeton, president this year of the American Association for the Advancement of Science, delivered the Halle lecture at the University of Oxford, June 1.

At the April meeting of the National Academy of Sciences in Washington Professors G. C. Evans of Rice Institute and J. F. Ritt of Columbia University were elected members of the academy. At this meeting the following mathematical papers were read: *Geometry of the Laplace equation*, by Professor Edward Kasner; *The geometry of spinors*, by Oswald Veblen.

Professors Jesse Douglas and M. H. Stone have been elected fellows of the American Academy of Arts and Sciences.

Professor E. V. Huntington, of Harvard University, has been elected a member of the American Philosophical Society.

Professor W. H. Roever, of Washington University, has received a grant from the Rockefeller Science Research Fund for the publication of some of his work in descriptive geometry. Part of this work was done for the Committee on Standards for Graphical Presentation, on which Professor Roever served as representative of the American Mathematical Society.

The council of the Royal Society of London has recommended Professor J. H. M. Wedderburn of Princeton University for membership in that society.

Professor R. L. Wilder of the University of Michigan has been granted leave of absence for the next academic year in order to continue work at the Institute for Advanced Study at Princeton.

Professor T. C. Esty of the department of mathematics at Amherst College has been made vice-president of the college.

Associate Professor W. C. Graustein of Harvard University has been promoted to a professorship of mathematics.

Associate Professor Einar Hille of Princeton University has been appointed professor of mathematics at Yale University.

Assistant Professor M. H. Stone of Yale University has been appointed associate professor of mathematics at Harvard University.

Dr. H. F. Bohnenblust has been appointed assistant professor of mathematics at Princeton.

Dr. J. J. Gergen has been appointed to an assistant professorship at the University of Rochester.

The following appointments to instructorships in mathematics are announced:

Harvard University—D. P. Adams, J. H. Curtiss, T. L. Downs, Jr., Z. I. Mosesson, Arthur Sard, F. H. Steen, D. H. Ballou, Walter Leighton, Jr., G. B. VanSchaack, Hassler Whitney. In addition to these, Wladimir Seidel and A. E. Currier have been appointed Benjamin Pierce Instructors in Mathematics and Tutors in the Division of Mathematics.

Princeton University—E. J. McShane and A. W. Tucker.
University of Rochester—G. B. Price.

Dr. M. F. Deuring and Mr. Saunders MacLane have been awarded Sterling fellowships in mathematics at Yale.

Professor L. M. Defoe, professor emeritus of mechanics at the University of Missouri, died April 3, 1933, at the age of 72.

Mr. R. A. E. C. Paley, international research fellow at Massachusetts Institute of Technology, was killed near Banff, Alberta, by an avalanche April 7, 1933. Mr. Paley was one of the most promising of the young English mathematicians. He had done considerable work with Professor Littlewood and was continuing his work with Professor Wiener at the Institute.

Dr. J. G. Porter, Professor of astronomy at the University of Cincinnati and director of the observatory from 1884 to 1931, died April 15, 1933.

Professor F. B. Williams of Clark University died March 7, 1933. He was a charter member of the Mathematical Association.

A NOTE

Mr. S. A. Corey has called my attention to the fact that, in my *Reading List in the Elementary Theory of Equations*, published in this MONTHLY, vol. 40, 1933, pp. 77–84, I omitted a paper of his. The title is *A Method of Solving Numerical Equations*, and it was published in this MONTHLY, vol. 21, 1914, pp. 290–92. It should be listed in part V of my classification. Mr. Corey has a more recent paper of the same general type, and bearing the same title, in the MONTHLY for March 1933, pp. 163–64.

RAYMOND GARVER

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merges with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 2.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE SEVENTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The seventeenth summer meeting of the Mathematical Association of America was held, by invitation, at A Century of Progress Exposition, the University of Chicago and Northwestern University, during the week of June 19, 1933, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Nearly five hundred persons were present at the meetings, including the following one hundred ninety-three members of the Association:

H. M. ACKLEY, State Teachers College, Kalamazoo, Mich.
C. R. ADAMS Brown University
R. P. AGNEW, Cornell University
R. B. ALLEN, Kenyon College
NOLA L. ANDERSON, Sophie Newcomb College

R. W. BABCOCK, Kansas State Agricultural College
FRANCES E. BAKER, Iowa City, Iowa
G. A. BAKER, Shurtleff College
R. W. BARNARD, University of Chicago
I. A. BARNETT, University of Cincinnati
WALTER BARTKY, University of Chicago
W. D. BATEN, University of Michigan
ETHELWYNN R. BECKWITH, Milwaukee-Downer College
THEODORE BENNETT, University of Wisconsin
B. A. BERNSTEIN, University of California
H. R. BEVERIDGE, Monmouth College
G. D. BIRKHOFF, Harvard University
H. F. BLICHFELDT, Stanford University
G. A. BLISS, University of Chicago
JULIA W. BOWER, University of Chicago
C. F. BOWLES, South Dakota State School of Mines
FANNIE W. BOYCE, Wheaton College
W. E. BROOKE, University of Minnesota
C. T. BUMER, Kenyon College
O. Z. BURDICK, Chicago, Illinois
R. S. BURINGTON, Case School of Applied Science
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
HELEN CALKINS, Pennsylvania College for Women
G. A. CAMPBELL, American Tel. and Tel. Co.
J. W. CAMPBELL, University of Alberta
R. D. CARMICHAEL, University of Illinois
J. E. CASE, St. Louis University
W. F. CHENEY, JR., Connecticut Agricultural College
R. V. CHURCHILL, University of Michigan
G. R. CLEMENTS, U. S. Naval Academy
A. B. COBLE, University of Illinois
L. M. COFFIN, Coe College
L. W. COHEN, University of Kentucky
J. B. COLEMAN, University of South Carolina
JULIA T. COLPITTS, Iowa State College
A. R. CRATHORNE, University of Illinois
D. R. CURTISS, Northwestern University
E. H. CUTLER, Lehigh University
L. H. CUTTING, High School, Kansas City, Mo.

W. M. DAVIS, Armour Institute of Technology
L. E. DICKSON, University of Chicago
L. L. DINES, University of Saskatchewan
B. F. DOSTAL, University of Florida
J. E. DOTTERER, Manchester College
ARNOLD DRESDEN, Swarthmore College
W. L. DUREN, JR., Tulane University

W. E. EDINGTON, DePauw University
MARGARET C. EIDE, State Teachers College, River Falls, Wis.
E. B. ESCOTT, Oak Park, Illinois
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

F. J. FEINLER, St. Peter's Church, Loudonville, Ohio
B. F. FINKEL, Drury College
C. H. FISCHER, University of Minnesota
A. H. FOX, Schenectady, N. Y.
T. C. FRY, Bell Telephone Laboratories

M. G. GABA, University of Nebraska
W. H. GARRETT, Baker University
B. P. GILL, College of the City of New York

D. C. GILLESPIE, Cornell University
C. H. GINGRICH, Carleton College
CORNELIUS GOUWENS, Iowa State College
W. C. GRAUSTEIN, Harvard University
L. M. GRAVES, University of Chicago
T. N. E. GREVILLE, Highlands, N. C.
F. L. GRIFFIN, Reed College

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MARTHA HILDEBRANDT, High School, Maywood, Ill.
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F. H. HODGE, Purdue University
T. F. HOLGATE, Northwestern University
W. R. HOLLOWAY, Manhattan, Illinois
JEWELL C. HUGHES, Hunter College
E. V. HUNTINGTON, Harvard University

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
R. D. JAMES, University of Chicago
C. M. JENSEN, Macalester College
S. A. JOFFE, Mutual Life Insurance Co.
MARIE M. JOHNSON, Oberlin College
OLIVE M. JONES, Queens-Chicora College
E. M. JUSTIN, Case School of Applied Science

L. C. KARPINSKI, University of Michigan
LOIS KARR, Lindenwood College
EVELYN M. KENNEDY, University of Cincinnati
A. E. KENNELLY, Harvard University
H. J. KERSTEN, University of Cincinnati
J. M. KINNEY, Crane Junior College
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DOROTHY MCCOY, Belhaven College
EDITH A. MCDOUGLE, University of Delaware
JAMES MCGIFFERT, Rensselaer Polytechnic Institute
MORRIS MARDEN, University of Wisconsin
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C. O. OAKLEY, Brown University
H. L. OLSON, Michigan State College
F. W. OWENS, Pennsylvania State College
Mrs. F. W. OWENS, State College, Pa.

Z. M. PIRENIAN, University of Florida
H. H. PIXLEY, College of the City of Detroit

D. W. QUERFELD, U. S. Bureau of Standards

TIBOR RADÓ, Ohio State University
W. R. RANSOM, Tufts College
W. T. REID, University of Chicago
C. E. RHODES, University of Cincinnati
R. G. D. RICHARDSON, Brown University

H. L. Rietz, University of Iowa
J. F. Ritt, Columbia University
W. E. Roth, University of Wisconsin
D. A. Rothrock, Indiana University
Lulu L. Runge, University of Nebraska

R. G. Sanger, University of Chicago
Hazel E. Schoonmaker, Hartwick College
E. W. Schreiber, State Teachers College, Macomb, Ill.
W. A. Shewhart, Bell Telephone Laboratories
C. Grace Shover, Connecticut College
E. B. Skinner, University of Wisconsin
H. E. Slaught, University of Chicago
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V. B. Teach, Armour Institute of Technology
Mrs. H. L. Titsworth, Sophie Newcomb College
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E. B. Wilson, Harvard School of Public Health
Elizabeth W. Wilson, Cambridge, Mass
Margarete C. Wolf, University of Wisconsin
F. E. Wood, Northwestern University
Meta A. Wood, Lincoln School, New York, N. Y.

B. F. Yanney, College of Wooster
C. H. Yeaton, Oberlin College
E. I. Yowell, University of Cincinnati

The Science Congress, sponsored jointly by the American Association for the Advancement of Science and A Century of Progress, was held from June 19 to June 30, the first week being devoted to pure science and the second to applied science. The notable feature of the first week, within which the mathematics meetings were held, was the presence of twenty-seven distinguished foreign men of science as guests of A Century of Progress and the American Association. It was the great pleasure of the mathematics group to entertain Professor Lipót Fejér of the University of Budapest, Professor Niels Bohr of the University of Copenhagen, and Professor Tullio Levi-Civita of the University of Rome, as well as Professor Enrico Bompiani of the University of Rome who was associated with the science exhibits of the Italian Government. A reception was given to the foreign guests on the evening of June 19 in the Hall of Science at which welcoming addresses were made by Rufus Dawes, president of A Century of Progress, Professor A. H. Compton of the Local Committee, Professor J. J. Abel, president of the American Association for 1932, and Professor Henry Crew who was in charge of science for the Exposition; and a response was made by Professor Bohr in behalf of the foreign guests. These guests formed the nucleus for a number of symposia held during the week in connection with the meetings of the various science organizations. A gala dinner was given on Thursday evening at the Hotel Stevens in honor of the foreign guests.

Numerous mathematical exhibits were shown in the northern part of the Hall of Science. Here, in the center of an octagonal room, were displayed a set of slides, illustrating the history of mathematics, by Professor L. C. Karpinski of

the University of Michigan, shown by means of four projection lanterns, one for each of the four great fields into which mathematics is divided. Near this were shown striking mathematical surfaces made by Mr. C. E. Johansson, and elaborate geometrical models by Doctor Saul Pollock. Nearby were seen Michelson's harmonic analyzer and mathematical instruments used in the calculations involved in his notable scientific investigations. Here also were to be found the Galton Quincunx in which probability curves are formed by steel balls deviated in their falling by steel pegs in "penny-slot-machine" fashion, another exhibit in which the probability of a rod falling on any one of a group of parallel lines is used to determine experimentally the value of π, a machine for the computation of simple harmonic motions, numerous geometrical and kinematical devices, and machines in the nature of mathematical recreations. In this same section was the number-theory machine of Doctor D. H. Lehmer which was described by him on the program of the Mathematical Association. Neighboring halls and corridors exhibited in popular form many of the striking attainments in physics, chemistry and the other sciences. A notable exhibit in the Hall of Social Sciences was the collection of early American arithmetics and other text-books from the well-known collection of Mr. George A. Plimpton of New York City.

The American Mathematical Society held sessions throughout the week with the largest summer attendance in its history. On Tuesday afternoon a joint session of the Society and Section A of the American Association, with the American Physical Society as their guests, was addressed by Professor Tullio Levi-Civita on "Some mathematical aspects of the new mechanics," and by Professor G. D. Birkhoff on "Quantum mechanics and asymptotic series." A second symposium of the Society and Section A on Wednesday morning was addressed by Professor Lipót Fejér on "The infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures," by Professor C. N. Moore on "The use of Cesàro means in determining criteria for Fourier's constants," and by Professor Dunham Jackson on "Certain problems of closest approximation." A symposium on geometry was held on Wednesday afternoon at the Italian Building on the Exposition grounds; addresses were delivered by Professor Levi-Civita on "Nets on a surface and extension of trigonometry," by Professor W. C. Graustein on "Invariant methods in differential geometry," and by Professor Enrico Bompiani on "Deformations of higher species of surfaces and manifolds." The Society also held a session devoted to number theory on Friday afternoon at which Professor L. E. Dickson spoke on "Recent progress in additive number theory," followed by a number of contributed papers. Sessions were held on Monday afternoon and Thursday and Friday mornings for the reading of short contributed papers.

Mathematicians and their guests were housed in very comfortable quarters in Judson Court on the south side of the Midway, opposite the University campus; spacious lounges and social rooms were available for their use. The Common Room at Eckhart Hall proved to be very convenient for social purposes,

tea being served several afternoons at the close of the sessions by the ladies of the faculty. There was a certain loss of time in the wide separation in the University campus and the Exposition grounds, but with the convenience of transportation this was more than offset by the freedom which the mathematicians had to choose their own personal combination of meetings and pleasure trips. The good services of the committee on local arrangements were recognized in a suitable vote by the Trustees including the recognition of the extensive plans made in advance by Professor Everett, the chairman of the committee.

About two hundred fifty members and guests attended the joint dinner of the mathematical organizations Friday evening at Judson Court. After the delightful dinner had been served, Professor Bliss as toastmaster introduced the various speakers. Professor Fejér expressed his appreciation of the cordial welcome with which he had been received. At the request of Professor Bliss, he delivered this in his native language and this was followed by a translation ably given by Mr. Medgycy, the Hungarian Consul-General at Chicago. Professor Levi-Civita then expressed his pleasure in being the guest of A Century of Progress and his good fortune in being associated with American mathematicians. His speech, again at Professor Bliss's request, was delivered in Italian and was translated by Professor Bompiani. Professor Bohr spoke of the present status of university professors in Europe, particularly those in mathematics and physics. Professors Bompiani, Coble and Dresden spoke more briefly, the latter two representing the Society and Association. All these addresses, together with the jovial introductions by Professor Bliss, constituted a delightful evening.

The program of the Mathematical Association consisted of a joint session with Section A on Tuesday morning at Eckhart Hall, University of Chicago, and a second joint session with Section A Thursday afternoon in South Hall of the Hall of Science on the Exposition grounds, and a joint session with Section A and the American Mathematical Society Saturday afternoon at Northwestern University, Evanston. President Dresden presided at the first two sessions and Professor Slaught at the third. The Trustees recognized by a resolution the very effective organization of a good program of speakers, as prepared by the program committee which consisted of Professors Louis Brand, Mayme I. Logsdon, E. R. Smith, and E. J. Moulton, chairman. The program follows, together with abstracts of some of the papers numbered in accordance with their place on the program.

First Joint Session of the Association with Section A of the American Association

1. "The lag of mathematics behind literature and art in the early centuries" by Professor H. E. Slaught, University of Chicago.
2. "Mathematics and art" by Professor G. D. Birkhoff, Harvard University.

3. "The postulational method in mathematics" by Professor E. V. HUNTINGTON, Harvard University.

1. The address by Professor Slaught will appear in an early issue of the MONTHLY.

2. The possibility of a connection between mathematics and art was considered by Professor Birkhoff from the point of view of the theory of aesthetic measure, first presented to the Association at Ithaca in 1925. In its applications to polygonal forms, ornaments and tilings, vases, harmony, melody, and the musical quality in poetry, certain elements of order in the aesthetic object are taken to be of fundamental importance. According to his theory the enjoyability of such an object depends primarily on the density of these elements of order. It is an interesting fact that these elements correspond in general to simple mathematical relations, and this suggests that aesthetic enjoyment may arise in part from the intuitive appreciation of such relations. For example, primitive man doubtless enjoyed the beauty of form of the moon or sun because of an intuitive appreciation of the geometric relation of circularity. In the applications considered, the most important relations of this kind were found to involve geometry, the theory of groups and of periodic functions. It was only in this special sense that a connection was made out between mathematics and art, despite the fact that many writers from the time of antiquity have endeavored to establish a mystical connection between the two fields.

3. This paper will appear in an early issue of the MONTHLY. It outlines three "levels of classification" of a given universe: (1) primary observations; (2) secondary observations, which are generalized into "hypothetical laws"; and (3) interrelations among these hypothetical laws. It is only on this third level that "mathematical propositions" arise. Thus, a set of n laws divides the given universe into 2^n compartments, and the assertion that one of these compartments is "empty" is what is meant by a mathematical proposition.

SECOND JOINT SESSION OF THE ASSOCIATION WITH SECTION A OF THE AMERICAN ASSOCIATION

1. "Fundamental concepts in the theory of probability" by Doctor T. C. FRY, Bell Telephone Laboratories.

2. "Applications of mathematics to real estate problems" by Doctor H. A. BABCOCK, Evanston, Illinois.

3. "A number theory machine" by Doctor D. H. LEHMER, Altadena, California.

1. Doctor Fry presented in a very concrete fashion the simpler concepts in the theory of probability and led up through numerous illustrative examples to exact definitions of "probability" and related terms. His paper will appear in full in an early issue of the MONTHLY.

2. In the valuation of land and the improvements thereon to determine what price a prospective purchaser is warranted in paying and the prospective seller

is warranted in accepting for any specific property, the fair price may be estimated, in the case of income producing properties, by estimating the future series of net receipts from the property and computing the present, or initial, worth of the series at an interest rate which reflects the uncertainties involved in the prediction of future events. Also, in the case of an aggregate of parcels of land to be sold over a period of time to those who wish to utilize the parcels the series of net returns from future sales may be estimated and the initial value of the series computed. The initial value of a series of net receipts $r_1, r_2, \cdots, r_n, \cdots r_N$ may be written

$$V_0 = \sum_1^N r_n v^n, \qquad v = 1/(1+i).$$

Doctor Babcock applied the principles given in this MONTHLY for March 1933 to the real estate problems appearing in a community in which the population is increasing and hence bringing about demands for additional land to serve as sites for additional buildings. The difference in prices paid for land arises from the fact that certain sites are more desirable for the intended use than others, the type of intended use being a factor in determining the maximum price. Moreover the owner with land to sell will hold land in some cases for many years, refusing offers to buy at low prices and waiting for a demand at a higher price. Assume an aggregate of land with a total area A, with a single ownership, assume that there are annual demands for various parts of this land by prospective purchasers and therefore at various unit prices, the annual demand in each price group being the same in each year. The owner of area A thus has an opportunity to sell over a period of time, parcel by parcel. The initial value of the area, subject to tax on the unsold portions each year, if the demand in one price group only is accepted, will be

$$V_0 = R_1 w_1 a_{\overline{N|}\, i+\rho},$$

where R_1 is the demand rate, w_1 the unit price, N the area A divided by the rate R_1, i the rate of return, ρ the tax rate and $a_{\overline{N|}}$ the annuity factor at the rate $(i+\rho)$ for N years.

The question arises whether the owner would not produce a greater initial value for himself by selling some of his land at a lower price provided he disposes of it faster. If the various unit prices are designated by $w_1, w_2, \cdots, w_s$ and the corresponding demand rates by $R_1, R_2, \cdots, R_s$, and if the number of years that he will accept these demands are designated by $N_1, N_2, \cdots, N_s$, the initial value of the area will be

$$V_0 = R_1 w_1 a_{\overline{N_1|}} + R_2 w_2 a_{\overline{N_2|}} + \cdots + R_s w_s a_{\overline{N_s|}},$$

subject to the condition that

$$R_1 N_1 + R_2 N_2 + \cdots + R_s N_s = A.$$

The condition that V_0 be a maximum gives the values of the several N's and the amount of area to be sold at each price.

Under certain circumstances some of the partial areas thus determined will be negative, which is physically inadmissible. It is found, however, that if the price groups are arranged in descending order, all the negative areas appear below a certain price group, which the author has called the critical price, and that the value of the area produced by accepting just those demands which occur at prices ranging down to the critical price, is greater than any other combination which does not involve negative areas, regardless of the relative sizes of the demand rates. It appears also that the length of time the demand at any price will be accepted is less the lower the price. If we assume that the rejected demands are filled somewhere outside the area A and that the prospective purchasers in all price groups accept land in an area B only because they cannot secure land in A at a sufficiently low price, the important result is found that the distribution of building development in the two areas will be the same when the owners of A and B are in competition as when the total area $(A+B)$ is in one ownership. Similarly for a third area C, and so on.

The speaker derived various consequences from this theory, illustrating it by the instance of Los Angeles where, under the belief that the population would continue to increase at a rapid rate, the building development had been pushed out to a great extent and a large amount of vacant land left behind, and of Baltimore, a less rapidly growing city, where the building development is more compact and there is a smaller percentage of vacant land left behind. He described his method as a start toward a more comprehensive theory which will take account of variations in these demand rates, and indicated paths for development on the practical and theoretical side.

3. The essentials of the explanation of Doctor Lehmer's very interesting number-theory machine have appeared in this Monthly, for August-September 1933. At the instance of A Century of Progress, Doctor Lehmer spent the summer at Chicago and showed each day the workings of his machine as a part of the mathematical exhibit in the Hall of Science.

Following this, Professor L. C. Karpinski of the University of Michigan showed a considerable portion of the slides in the history of mathematics which were being exhibited in the mathematical section of the Hall of Science. He commented briefly on these, pointing out the features of mathematical interest.

Joint Session of the Association with Section A of the American Association and the American Mathematical Society

On Saturday afternoon more than one hundred of the mathematicians went by automobile on an excursion along the North Shore and returned to Harris Hall of Northwestern University, where an address on "Mathematical reminiscences of the World's Fair of 1893" was given by Professor T. F. Holgate of Northwestern University.

Various congresses were organized at the time of the World's Fair under the auspices of the World's Columbian Exposition Auxiliary. The strength of the

mathematical congress, he said, arose from the action of the German government which had an elaborate exhibit of German mathematical books and apparatus, and from the appointment of Professor Felix Klein by the German government as their representative. Professor Klein organized a program of papers by eminent European mathematicians describing the status of mathematical study at that time. About forty papers were given on the program, thirteen or fourteen being by American mathematicians. The Congress met in what is now the Art Institute of Chicago. In connection with this, a smaller colloquium was held at Evanston, these papers comprising the first of the series of colloquium publications of the Mathematical Society. Professor Holgate showed in the form of slides two photographs of the members of the Congress and of the colloquium. As the printed reports of these meetings show, these included numerous well-known mathematicians and mathematical physicists.

After the lecture the ladies of the department of mathematics of Northwestern University served an elaborate tea at one of the attractive dormitories of the University. A resolution of appreciation of the hospitality of the friends at Northwestern University was adopted by rising vote at the afternoon session.

Meeting of the Board of Trustees

Eleven members of the Board of Trustees were present at the meeting at Judson Court on Friday afternoon.

The following thirty-two applicants for membership were elected:

Emily E. Calkins, A.B. (Columbia; William and Mary) Instr., Coll. of William and Mary, Williamsburg, Va.

Rev. Arthur Danzl, A.M. (Columbia) Instr., St. John's Univ., Collegeville, Minn.

Rachel Davison, A.M. (Oberlin) Instr., Houghton Coll., Houghton, N. Y.

L. J. Deck, A.M. (Pennsylvania) Prof., Muhlenberg Coll., Allentown, Pa.

F. A. Downing, Freight Rate Clerk, Atlantic Coast Line R.R., Wilmington, N. C.

D. C. Duncan, Ph.D. (California) Instr., Univ. of California, Berkeley, Calif.

E. W. Franz, A.M. (California) Instr., Junior Coll., Ventura, Calif.

T. N. E. Greville, Ph.D. (Michigan) Highlands, N. C.

W. N. Hallett, Ph.D. (Pennsylvania) Prof., Cedar Crest Coll., Allentown, Pa.

G. M. Hayes, Ph.D. (Fordham) Asst. Prof., Coll. of the City of New York, New York, N. Y.

J. J. Hayes, B.S. (Utah) Instr., Univ. of Utah, Salt Lake City, Utah

E. Marie Hove, M.S. (Iowa) Instr., State Teachers Coll., Wayne, Nebr.

G. J. Kalcik. Senior, St. Nobert Coll., West DePere, Wis.

S. H. Kimball, Ph.D. (Harvard) Instr., Univ. of Rochester, Rochester, N. Y.

M. S. Kovalenko, Ph.D. (Princeton) Asst. Prof., Math. and Astr., Swarthmore Coll., Swarthmore, Pa.

C. H. Lady, A.M. (Southern California) Prof., State Teachers Coll., Slippery Rock, Pa.

T. J. Love, A.M. (Woodstock) Prof., Physics and Math., Loyola Coll., Baltimore, Md.

L. E. Loveridge, Ph.D. (California) Prof., Seton Hall Coll., South Orange, N. J.

R. E. Norris, A.M. (Illinois) Head of Dept., State Teachers Coll., Milwaukee, Wis.

Mabel I. Nowlan, M.S. (Michigan) Head of Dept., Bethel Woman's Coll., Hopkinsville, Ky.

A. S. Peters, M.S. (New York Univ.) Instr., New York Univ., New York, N. Y.

Gladys M. Quigg, A.M. (Penna. State Coll.) Instr., Pennsylvania State Coll., State College, Pa.

C. B. Read, A.M. (Princeton) Asst. Prof., Univ. of Wichita, Wichita, Kans.

W. H. ROBINSON, A.M. (Boston) Head of Dept., Brick Jr. Coll., Bricks, N. C.
L. M. SKOFIELD. Reading, Mass.
C. E. SMITH, A.M. (Swarthmore) Instr., Math. and Astr., State Teachers Coll., Fresno, Calif.
E. R. STABLER, Ed.M. (Harvard) Cambridge, Mass.
M. L. VEST, M.S. (West Virginia) Instr., West Virginia Univ., Morgantown, W. Va.
K. J. WAIDER, A.B. (San Francisco) Instr., Math. and Physics, Univ. of San Francisco, San Francisco, Calif.
C. L. WEAVER, Senior, Kent State Coll., Kent, Ohio
A. MARIE WHELAN, Ph.D. (Johns Hopkins) Asst. Prof., Hunter Coll., New York, N. Y.
MARGARETE C. WOLF, A. B. (Wisconsin) Grad. Student, Univ. of Wisconsin, Madison, Wis.

The Secretary-Treasurer made a report to the Trustees by letter on June 7 and at this meeting concerning the status of part of the funds of the Association which had been tied up in the Oberlin Savings Bank since the "bank holiday" in February. The bonded investments of this bank, as of other banks and investors, have shrunk during the past three years but, to speak briefly, it is expected that this bank will soon resume its activities, make 70% of the frozen deposits available for legitimate business or personal needs of depositors, and issue participation certificates for the remaining 30%, the assets covered by these certificates to remain in the hands of three trustees in the hope that as they become more fluid these participation certificates may be redeemed. By mail vote in June the Trustees authorized the Finance Committee to agree to the plan of reorganization and to accept the participation certificates.[1] The Finance Committee regret that any of the Association funds should be in any way tied up or have an uncertain status, but it seemed to be agreed in the Trustees' discussion that our investments and deposits are widely scattered and that the Association is faring better than many other business organizations. Other than this there has been no default in the income from investments except in one $1000 land trust certificate, quarterly dividends of $13.75 having been passed for the past four quarters.

The Trustees voted to designate The Northern Trust Company of Chicago as one of the depositaries of the Association. The sum of $500.00 recently received on account of the Chace Fund has been deposited there in the form of a certificate of deposit.

Professor Slaught gave an oral report concerning the sets of the Chace Papyrus, a limited number of which are still available. He also reported concerning the sales of the Carus Monographs and the plans of the committee for the future; an announcement will be made to the Association members as soon as more definite plans for the publication of the fifth monograph are formed.

Suggestions as to changes in the conditions of the award of the Chauvenet Prize having been made earlier in the year, a committee on the rules for this prize was appointed in February by President Dresden, as follows: W. B. Ford, chairman; E. T. Bell, R. D. Carmichael, H. E. Slaught, and J. L. Walsh. At the

[1] The bank resumed full activities at the end of August, and a distribution of part of the frozen funds is expected soon.

Chicago meeting a partial report was sent by the chairman, and the Trustees as a committee of the whole adopted a number of tentative conditions and referred these to the committee as suggestions in formulating a new statement of rules.

Growing out of the request of Professor Walter W. Hart that the Association appoint "a strong committee which shall be charged with the business of providing desirable publicity for continued teaching of mathematics in the schools and maintenance of mathematics as a desirable and necessary part of college entrance requirements," there was a general discussion of this question including a further statement by Professor Hart, who was present by invitation. As an outcome of the discussion of this and related questions, it was voted to empower the President to appoint (1) a commission to study the place of mathematics in schools and junior colleges, and (2) a commission to study the training and utilization of advanced students of mathematics. A nucleus of this committee will initially act as a committee of organization and planning and it is expected that fuller information will be given at an early date. In this connection the following resolution was adopted:

> Resolved, That this body would welcome a decision by the National Council of Teachers of Mathematics to hold a meeting at the time of the Boston and Cambridge meetings next December, in affiliation with the A.A.A.S. and the Mathematical Association, so as to come into touch with the members of these associations and with the wider scientific interests.

The trustees authorized the Secretary to issue permits to members of the Association to wear an official pin or button adopted as the official emblem of the Association.

W. D. CAIRNS, *Secretary-Treasurer*

AIR RESISTANCE TO FALLING SPHERES

By E. V. HUNTINGTON, Harvard University

Introduction

Let us consider a sphere, of weight W and radius r, falling from rest in air.

If the resistance of the air is assumed to be proportional to the square of the velocity[1] and also to the area of the cross-section of the sphere,—that is, if the law of resistance is assumed to be

$$R = kAv^2, \tag{1}$$

where R = the retarding force, $A = \pi r^2$, and v = the velocity,—then, as is well known, the distance x fallen in any time t is given by the equation

$$x = \frac{b^2}{g} \log_e \cosh\left(\frac{gt}{b}\right), \tag{2}$$

[1] According to *Encyklopädie der Mathematischen Wissenschaften*, IV, 17 (S. Finsterwalder), p. 161, the quadratic law is generally accepted for velocities from 0.2 to 240 m/sec.

MATHEMATICAL INTERPRETATIONS OF GEOMETRICAL AND PHYSICAL PHENOMENA[1]

By G. A. BLISS, University of Chicago

Mathematical theories have been of great service in many experimental sciences in correlating the results of observations and in predicting new data afterwards verified by observation. This has happened particularly in geometry, physics, and astronomy. But the relationship between a mathematical theory and the data which it is designed to relate is often misunderstood. When such a theory has been successful as a correlating agent the conviction is likely to become established that the theory has a unique relationship to nature as interpreted for us by the observations. Furthermore, it is sometimes inferred that nature behaves in precisely the way which the mathematics indicates. As a matter of fact, nature never does behave in this way, and there are always more mathematical theories than one whose results depart from a given set of data by less than the errors of observation.

The purpose of these lectures is to explain in more detail the point of view which is indicated briefly in the preceding paragraph. A description will be given of the structures of pure and applied mathematical sciences. For this purpose the most accessible mathematical domain is geometry, and the first few paragraphs below are devoted to a discussion of various geometrical theories. But examples from astronomy and physics will also be briefly described in later sections.

1. *Geometrical measurements*. If our geometrical experiences were only those of very ancient peoples our collection of observed geometric data would be a very limited one. Nowadays, however, we have many sources to draw from. We accumulate geometric data by measurements on the drawing table, by local and geodetic surveys, by observing the motions of the planets, or even by the relatively very rough measurements of stellar distances which are now in use. In every case the measurements have a percentage of inaccuracy which can be experimentally determined, and in every case the domain of observation is of finite extent.

On the drawing table, for example, we designate as points the dots which we make with our pencil, and straight lines are the marks which we make by drawing the pencil along the edge of a ruler. It is not true experimentally, as we should perhaps like to have it, that every pair of dots is joined by one and only one straight line. If we were to sharpen our pencil sufficiently, and perhaps use a magnifying glass, we should see that a pair of dots can be joined by very many delicately drawn straight lines. It is likewise not true experimentally that the sum of the angles of a triangle is always 180 degrees. If we measure our angles honestly we find that the sum is usually not exactly 180 degrees, but that the

[1] Two lectures delivered in the General Course of the Division of Physical Sciences of the University of Chicago.

maximum variation of the sum from that amount is a small number which depends upon the instruments used and the physical characteristics of the observer. Other observations which may be cited to illustrate the inaccuracy which attends all measurement are the measurements of stellar distances. The methods used are too technical to be described here. The points to be emphasized with regard to them are that even these enormous and almost incomprehensible distances are still finite, and that the percentage of inaccuracy in their determination is very great. That they can be estimated at all is a great tribute to the ingenuity of astronomers.

2. *Fundamental postulates of geometry.*[1] In view of the inaccuracy of geometrical observations it is clear that we can never hope to find a precise mathematical theory to correlate them which will be an exact fit. Until recently it has been customary to idealize points and lines. A point was something which had no dimensions, and a line had only one dimension, namely, length. We may still find such definitions in our text books. Unfortunately our observational experience does not provide us with any such geometrical entities. The dots and straight line marks of the drawing table certainly have dimensions, and it is, after all, their properties which our mathematical theory, whatever it may be, should correlate approximately.

The procedure nowadays is quite different. We do not attempt to idealize points and lines. We rather try to construct an ideal geometrical theory by idealizing the fundamental properties which we ascribe to points and lines. The statements of these fundamental properties are called postulates, and from them, by the processes of mathematical logic, are deduced the theorems which constitute the exact science of geometry. Thus in our ideal geometry one of the postulates is that two points can be joined by one and but one straight line. The dots and straight line marks on the drawing table do not satisfy this postulate, but the dots can be made so small that the eye alone cannot distinguish between the lines which join them. A theorem in our exact geometrical theory, based upon the postulate, should therefore correspond to a property of the dots and marks with a degree of approximation similar to that with which the postulate itself is applicable. Described somewhat roughly the theorem should seem true to the unaided eye for dots and marks when the dots are made sufficiently small.

The pure mathematician is of course primarily interested in exact abstract geometrical science, but he is guided in his selection of postulates by his observational experience. It is impossible here to give a complete list of the postulates for geometry. As formulated by the German mathematician, Hilbert, they are twenty in number, and are of five types which are called the postulates of connection, order, congruence, parallelism, and continuity. Examples of each of these will be given in the following paragraphs.

The postulate mentioned above, stating that through two points there passes one and but one line, is a postulate of connection. There are six others of

[1] Hilbert, *Foundations of Geometry*, Open Court Publishing Co., 1902; Veblen and Young, *Projective Geometry*, vol. I, Introduction and Chapter I.

this type. If three points lie on the same line then one and only one of them lies between the other two. This is a postulate concerning order. It seems self-evident unless we recognize, for example, that of three lines through a point each one may be regarded as lying between the other two. By this example we see that there are other types of order besides that described in the postulate concerning the order of the points on a line, and that therefore some agreement about the order of such points is necessary if our geometry is to have a sound basis. There are in all five postulates concerning order. If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other then the triangles are congruent. This postulate is the basis of that part of geometry which has to do with congruent triangles. It is a sample of the postulates of congruence which are six in number.

Now we come to the famous assumption of Euclid concerning parallel lines. If a line l and a point P are given, then there is one and but one line through P which has no point in common with l. This is not by any means a self-evident property of the straight lines of our experience, though geometers for many centuries thought that it was so or that it could be proved to be a consequence of other simple postulates. No one was able to make such a proof, and when another geometrical theory was found with this postulate changed, but all the others just the same, it became evident that no proof of the dependence of Euclid's postulate could ever be made. The possibility of changing the postulate to another is a consequence of the facts noted above that all of our geometrical experiences are in a finite portion of space, and that our geometrical measurements agree only to within a certain percentage of error. This will be clearer after we have considered in the next section an example of a geometry in which Euclid's postulate is not true, but which nevertheless correlates the facts of our geometrical experience with as high a degree of accuracy as does Euclid's geometry itself.

The final postulate which we shall consider, the so-called postulate of continuity, is usually ascribed to the famous old Greek scientist, Archimedes. If A, A_1, and B are three points in that order on a line then there is always a sequence of points $A, A_1, A_2, \cdots, A_n$ such that the segments between adjacent points are all congruent, and furthermore such that B lies between A and A_n. This is equivalent to saying that by starting at A and laying off successively equal segments we can always ultimately pass the point B.

It will be noted that all of the postulates which have been stated above are in accord with our geometrical instincts. They and the others, which for want of space could not be mentioned, are statements of simple properties of points and lines which we intuitively willingly adopt as proper bases for a geometrical theory, and which agree to a suitable degree of approximation with geometrical data on the drawing table or in any other finite region where such data are obtainable. It is clear that none of us can follow two lines out to more than a very limited distance to see if they ever meet, and none of us can check the postulate of Archimedes for points B which are very far away. We select these postulates

because we like them and think that they will help to give us a useful theory.

3. *A simple non-Euclidean geometry.* Examples of sets of geometrical data which can be correlated by two different geometrical theories are very common. The surveyor, for example, uses the formulas of plane geometry when he measures the area of a city lot or of a farm. Theoretically he should of course use spherical geometry, since his measurements are made on the spherical surface of the earth. It would be foolish of him to do so, however, since the formulas of plane geometry are much easier to work with than those of spherical geometry, and since for a small area the surface of the earth is so nearly a plane that no ordinary surveying instrument could ever distinguish between the two. Evidently in this case either of the two theories would give results in agreement with observed data to a high degree of precision. The choice between the two for practical work is made for reasons of simplicity and convenience.

Just as the surveyor uses plane instead of spherical formulas for measurements on a sphere, so one may interpret plane geometrical phenomena by a spherical theory. To see this let us consider a very large sphere resting on a plane table-top. We may associate with each point P on the plane the point on the sphere which is the intersection of the straight line joining P to the center of the sphere. The points of a segment of a straight line in the plane then correspond to those of a segment of a great circle on the sphere. If we define the length of the plane segment as the length of the corresponding great circle segment on the sphere, and make a similar definition for the measurement of angles, a new geometrical theory for the plane can be constructed.

It should be noted first of all that when the sphere is sufficiently large the new measures for segments and angles on the plane agree with those of ordinary Euclidean geometry so closely that no measuring instrument, however delicate, could ever distinguish between them. Every straight line has a finite length, in the new geometry, equal to the length of half of a great circle on the sphere. The sum of the angles of a triangle on the plane is greater than 180 degrees, since the sum of the angles of a triangle on the sphere has this property. Finally we see that through a point P outside of a line l there passes no line parallel to l. Every pair of lines in the plane have a common point at a finite distance, since every pair of great circles on the sphere has a point of intersection. Thus Euclid's postulate is not true in the new geometry. These discrepancies with Euclid's theory need not disturb us at all, however, as far as our interpretations of geometrical data on the table-top are concerned. In this limited domain the two theories agree so closely that either one offers an acceptable explanation. We should be foolish to select the spherical theory for practical purposes, however, since the formulas of spherical geometry are much more complicated than those of Euclidean geometry in the plane.

In concluding our consideration of this example it should be stated that there are many other mathematical theories for the interpretation of geometrical phenomena in the plane. The theory which has been suggested above was discovered by Riemann and is sometimes called Riemannian non-Euclidean geom-

etry. There are also geometries in which through each point P not on a line l there pass an infinity of lines not meeting l. These lines through P form a sheaf so thin that it is not distinguishable experimentally from a single line.

4. *The structure of a mathematical science.* A pure mathematical science consists of postulates, definitions, and theorems. Thus for geometry we agree first of all that the names of elements to be studied shall be points and lines, and we postulate a set of simple properties for these elements from which all other properties are to be deduced. We define in terms of points and lines what we mean by an angle, a triangle, a polygon, or a circle, and starting from the postulates we prove by processes of mathematical logic the theorems concerning the things so defined which constitute the important results of geometrical science.

The science of arithmetic has a similar structure. The postulates for positive integers are four in number.[1] For two such integers we can define what we mean by the sum, product, difference, and quotient, and we can prove for these the well-known laws of computation, such as the theorem that the product of a number of integers has the same value irrespective of the order in which the integers are taken. Furthermore, starting from the positive integers and their properties we can define and analyze the properties of the complete system of positive and negative real numbers. Thus we see again that arithmetic is a science consisting of postulates, definitions, and theorems.

An applied mathematical science has a somewhat more complicated structure. The physicist, astronomer, or geometer frequently finds himself in possession of a set of data, obtained by observation, which he would like to correlate by means of a mathematical theory of the kind described in the two preceding paragraphs. For this purpose he must first of all select a set of postulates, as simple as possible, which are in agreement with the observed data as nearly as the observations are in agreement with themselves. When the logical consequences of the postulates have been worked out, he must devise new experiments if necessary to check again with observation his theoretical results. Thus the structure of an applied mathematical science can be suggested by the following table:

Observed data

Postulates	Postulates	. . .
Definitions	Definitions	. . .
Theorems	Theorems	. . .

Check with observed data

The form of the table indicates that for the correlation of observed data it is to be expected that more than one mathematical theory may be effective, as we have seen above in the case of geometry. The selection of one among the theories possible is based on convenience, or on accuracy of fit with the data, or both.

[1] See, for example, Landau, *Grundlagen der Analysis*, Chapter I.

The danger is always when a theory has been found to be convenient and effective over a long period of time, that people begin to think that nature herself behaves precisely in the way which is indicated by the theory. This is never the case, and the belief that it is so may close our minds to other possible theories and be a serious impedence to progress in the development of our interpretations of the world around us.

Just as the same set of data may be related by more than one mathematical theory, so it is possible that the same theory may be effective in connection with more than one set of data. If we define a point to be a pair (x, y) of real numbers, and a straight line to be the totality of pairs (x, y) which satisfy an equation of the form $ax+by+c=0$, we may prove that these points and lines satisfy precisely the postulates of Euclidean plane geometry. Thus the science of geometry is applicable not only approximately to the dots and straight line marks of the drawing table, but also with exactitude to the number-pairs and linear equations of the Cartesian coordinate system. By the mechanism of such a coordinate system we thus bring to bear on geometry the powerful and familiar processes of arithmetic. The theory of geometry from this point of view is called analytic geometry.

In concluding this section it should be re-emphasized that the purposes of an applied mathematical science are two-fold, first to correlate and systematize data which may otherwise appear heterogeneous and unrelated in character, and second to predict by logical processes new results which might be difficult or impossible to discover by experimental methods alone. Many examples of the effectiveness of mathematical theories in both of these respects could be given if space permitted.

5. *Mathematical theories in astronomy.* The Ptolemaic and Copernican theories of the solar system are excellent examples of theories widely different in character but which with suitable modifications of the former describe with equal accuracy the motions of the planets. In the Ptolemaic theory the earth is regarded as the center of the solar system. The stars and the sun move on circles with the earth very near the centers, and in the modification suggested by Tycho Brahe the planets move on circles whose centers are at the sun and move with the sun. The curves thus described by the planets, as seen from the earth, are called epicycloids. This is the picture which we see when we stand out on the lawn and observe the heavens, and it has much to recommend it. Numerically it is not sufficiently accurate for astronomical purposes, but it could be made so by replacing the circles by ellipses, and superposing a suitable perturbation theory. In the familiar and more popular Copernican theory the sun is the central body of the solar system and the stars are fixed. The earth and the other planets move about the sun on ellipses each of which has the sun as a focus. The mathematical justification of the motions of the planets in either theory is based upon the famous law of gravitation of Newton.

There is really no advantage for either of these theories as compared with the other, as far as their adaptability to explain numerically the facts of the

solar system is concerned. The Copernican theory is, however, much the simpler geometrically and mathematically. For this reason it has been adopted and developed until astronomers can predict coming celestial events with most surprising accuracy, and it has resulted in the discovery of the two outmost planets, Neptune and Pluto.

But even the Copernican theory based upon Newtonian mechanics has failed to fit all of the observed data satisfactorily. The perihelion of a planet is the point nearest to the sun on its elliptical orbit. According to Newtonian mechanics the perihelion of the inmost planet, Mercury, should move about the sun each year a distance which differs from the observed motion by a perceptible amount. Many explanations for this discrepancy have been offered, but none of them was well received by astronomers until Einstein suggested a slight modification of Newtonian mechanics itself in accord with his so-called general theory of relativity. The new theory agrees with observed data, including the motion of the perihelion of Mercury, with discrepancies which are less than the errors of observation, but it is much less convenient for computation than the Newtonian theory. For most purposes we can still retain Newtonian mechanics since the two theories disagree by less than the errors of observation in all except a very few instances.

The theory of general relativity as applied to the solar system is already threatened, even before its usefulness has been completely established. Recently when Einstein and de Sitter were sojourning in the United States, the newspapers reported them as discussing still newer theories from which it might be concluded that the universe is finite. This conclusion should never be made. All that we could be justified in saying is that the data which we have concerning the distances and motions of the stars are in closer accord with a theory which is finite than with others which have not this property. The finiteness which can be demonstrated is a property of the theory and not of the universe. By a finite theory we mean one, such as the Riemannian geometry described in a preceding section, in which the distances of all points from any given fixed point are finite. The geometrical theory of Euclid has not this property.

6. *Mathematical theories in physics.* The physicist makes use of mathematics primarily as a correlator of heterogeneous data. One of the best examples of the possibility of using two slightly different theories for the correlation of the same physical phenomena lies in the relationship between Newtonian mechanics and the mechanics of the special theory of relativity. Every event in physical experience occurs at a particular time and place. If an observer A has a rectangular system of coordinates in space, and a mechanism for measuring time, he can locate each event by giving its time t and its three space coordinates x, y, z. Thus the location of an event requires the specification of four numbers t, x, y, z. This is all that the mathematical physicist means when he says that the world of events is a four-dimensional world.

If a second observer A' is moving with constant velocity v in the direction of the positive x-axis of the system of coordinates of A, and if A' has a clock and a

system of space coordinates parallel to those of A and moving with A', then each event will have four coordinates t', x', y', z' for A' as well as the four coordinates t, x, y, z of A. Newtonian mechanics tells us that these two sets of coordinates are related by equations of the form

$$t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z. \tag{1}$$

The special theory of relativity asserts, on the other hand, that these relations should be replaced by the equations

$$t' = \beta(t - vx/c^2), \quad x' = \beta(x - vt), \quad y' = y, \quad z' = z, \tag{2}$$

in which c is the velocity of light and β is merely a convenient symbol for the more complicated expression $\beta = 1/(1 - v^2/c^2)^{1/2}$. It is unnecessary here to explain in detail the derivation of either of these sets of equations. For both theories they enable us to calculate the coordinates t', x', y', z' of an event when the coordinates t, x, y, z are given, or vice versa. Such relations are, of course, necessary if the world of events as observed by one of A and A' is to have any meaning to the other. Since the velocity of light in kilometers per second is the very large number $c = 300{,}000$ it is evident that for velocities v not exceeding a few kilometers per second the ratio v/c is exceedingly small and the value of β is very near to unity, so that the formulas (2) are almost identical with formulas (1). Thus for all moving bodies which we ordinarily consider on the surface of the earth, or for the motions of the planets, the two sets of equations give sets of values t', x', y', z' which are experimentally indistinguishable. The velocities of electrons in a current passing through a vacuum tube may, however, be a large fraction of c. For such cases the formulas (2) have been found to give results which agree with observations much better than those given by equations (1). Here, then, are two theories whose results differ by less than the errors of observation for every-day velocities. But one of them has a much better fit with observed data than the other when the range of velocities is enlarged.

The equations (2) are fundamental in the deduction of many of the results of special relativity theory. From them we can prove as mathematical theorems the often quoted assertions that according to the special theory of relativity a clock runs slower when it is moving than when it is stationary, that a yard-stick is shortened when moving in the direction of its length, or that there exist events occurring in one time order for A which appear to A' to take place in the opposite order. We need not worry about these matters from the practical standpoint, for the retardation of the clock and the shortening of the yard-stick at any velocities which we can experimentally impose upon them are too small to be measured by any apparatus which we now possess, and events whose orders are inverted must be so near together as to be practically indistinguishable.

As a final example of applied mathematical theories in physics one should mention the quantum theory. It is perhaps the most recently developed of all mathematical-physical theories, and is still undergoing modification and improvement. When an electric current consisting of electrons is passed through rarified

hydrogen gas in a vacuum tube, the atoms of the gas are excited and give off radiation which has a very definite and well-known line spectrum. One of the fundamental problems of the quantum theory is the determination of the character of the atoms of hydrogen and the mechanism by means of which they can give off the observed radiation. In the Bohr theory the atom is a tiny solar system consisting of a nucleus and an electron moving about each other in accord with a modified Newtonian theory of mechanics. By Newtonian mechanics the positions and velocities of the particles constituting the atom are determined uniquely at all times t when their positions and velocities at one time are known. This fact is a mathematical expression of the so-called principle of cause and effect. A second theory, matrix mechanics, provides for each time t only percentages of probability that the particles will be in various positions or have various velocities. According to this theory we would conclude that among the myriad of atoms of hydrogen in the tube a certain percentage have their particles in one position at a given time t, a second percentage have their particles in another, and so on. In some cases one of the percentages may be 100, or very nearly 100, and then we know with great precision what to expect. But in no case can a probability for positions and a probability for velocities be simultaneously 100. Thus we can never expect to have accurate information about positions and velocities at the same time. This, very roughly, is a description of the now famous "principle of uncertainty" of matrix mechanics, as contrasted to the principle of cause and effect of the older Newtonian theory. The principle of uncertainty has caused much discussion among philosophers and physicists since it seems to affect fundamentally our conception of the universe of phenomena in which we find ourselves immersed. If we accept the point of view of the preceding paragraphs, however, we must agree that the universe itself, sparsely interpreted for us by disconnected data, is not affected by either of these theories. Neither the principle of cause and effect, nor the principle of uncertainty, can be precisely characteristic of the behavior of nature. They are merely most interesting theorems in two different theories by means of which we endeavor to correlate and interpret observed data. The ultimate choice between the two must be determined by convenience or by their relative accuracies of fit with observation, and not because of any supposedly precise correspondence with nature on the part of either one of them.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A RECTANGULAR HYPERBOLA THEOREM

By J. R. MUSSELMAN, Western Reserve University

By proper choice of axes and scale the equation of any rectangular hyperbola

in Cartesian coordinates can be written parametrically as

$$x = t, \qquad y = t^{-1}. \tag{1}$$

The equation of the chord joining two points P_1 and P_2, whose parameters are t_1 and t_2 respectively, is

$$x + t_1t_2y = t_1 + t_2. \tag{2}$$

Given any three connected chords of a rectangular hyperbola P_1P_2, P_2P_3, P_3P_4; *if one draws* P_4P_5 and P_5P_6 *parallel respectively to* P_1P_2 *and* P_2P_3, *then* P_6P_1 *is parallel to* P_3P_4.

Due to the assumptions we have

$$\begin{aligned} t_1t_2 &= t_4t_5 \\ t_5t_6 &= t_2t_3. \end{aligned} \tag{3}$$

Upon multiplication of these two equations we have

$$t_1t_6 = t_3t_4 \tag{4}$$

and the theorem follows. We are then led to: *given any six connected chords of a rectangular hyperbola, so drawn that the last three chords are parallel respectively in order to the first three chords, the figure closes.* An interesting case occurs if the chords form either all positive, or all negative, 60° angles with each other. We then have an inscribed equiangular hexagon with opposite sides parallel. *Theorem: Given any* n *connected chords* (n *odd*) *of a rectangular hyperbola, if one draws* $n-1$ *more chords parallel respectively to the first* $n-1$ *chords, then* $P_{2n}P_1$ *is parallel to* P_nP_{n+1}.

From the assumptions it follows that

$$\begin{aligned} t_1t_2 &= t_{n+1}t_{n+2} \\ t_{n+2}t_{n+3} &= t_2t_3 \\ \cdot \quad &\quad \cdot \\ \cdot \quad &\quad \cdot \\ \cdot \quad &\quad \cdot \\ t_{n-2}t_{n-1} &= t_{2n-2}t_{2n-1} \\ t_{2n-1}t_{2n} &= t_{n-1}t_n. \end{aligned} \tag{5}$$

Upon multiplication of the above equations we have

$$t_1t_{2n} = t_{n+1}t_n \tag{6}$$

and the theorem is proved. Hence it follows that *given* $2n$ *connected chords* (n *odd*) *of a rectangular hyperbola, so drawn that the last* n *chords are parallel respectively in order to the first* n *chords, the figure closes.*

If, in equation (1) we think of x and y as conjugate complex variables and t as a turn, i.e. $|t| = 1$; the algebra above shows that the theorems stated for chords of a rectangular hyperbola are also true for chords of a circle.

$$\left.\begin{aligned}\frac{1}{F(D)}\sinh ax &= \frac{F(-D)\sinh ax}{F(a)F(-a)}\\ \frac{1}{F(D)}\cosh ax &= \frac{F(-D)\cosh ax}{F(a)F(-a)}\end{aligned}\right\} F(a)F(-a) \neq 0.$$

Note on the Laplacian of a Vector Point Function

By J. J. Slade Jr., Rutgers University

In most of our text books on vector analysis there appears, without comment, the formula

$$\nabla^2 u = \text{grad div } u - \text{curl curl } u \tag{1}$$

where u is the vector

$$u_1(x, y, z),\ u_2(x, y, z),\ u_3(x, y, z)$$

and $\nabla^2 u$ is

$$\nabla^2 u_1,\ \nabla^2 u_2,\ \nabla^2 u_3.$$

This formula is quite misleading, for whereas the right hand member of (1) is invariant for all the transformations of classical physical space, the left hand member is not.

For let u be transformed into v by the substitution

$$T: \begin{cases} x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta) \end{cases}$$

which is supposed to be orthogonal in the sense that

$$ds^2 = \left(\frac{d\xi}{h_1}\right)^2 + \left(\frac{d\eta}{h_2}\right)^2 + \left(\frac{d\zeta}{h_3}\right)^2.$$

The divergence and curl of u are transformed into the divergence and curl of v, and the gradient of a scalar point function ϕ goes into the gradient of $T\phi$. But, if we let $\overline{\nabla}^2$ stand for the operator

$$h_1h_2h_3\left\{\frac{\partial}{\partial\xi}\frac{h_1}{h_2h_3}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\frac{h_2}{h_3h_1}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\zeta}\frac{h_3}{h_1h_2}\frac{\partial}{\partial\zeta}\right\}$$

we see that

$$\nabla^2 u \to \overline{\nabla}^2\left[\sum v_i h_i \frac{\partial x}{\partial\xi},\ \sum v_i h_i \frac{\partial y}{\partial\xi},\ \sum v_i h_i \frac{\partial z}{\partial\xi}\right] \tag{2}$$

which is never the same as $\overline{\nabla}^2 v$ except for the trivial case when T is the identity transformation.

If u satisfies Laplace's equation, $\nabla^2 u=0$, then each of the components of the right-hand member of (2) vanishes. Expanding the first of these we get

$$(3) \qquad \sum v_i \overline{\nabla}^2\left(h_i \frac{\partial x}{\partial \xi}\right)+2\sum \overline{\nabla} v_i \cdot \overline{\nabla}\left(h_i \frac{\partial x}{\partial \xi}\right)+\sum h_i \frac{\partial x}{\partial \xi} \overline{\nabla}^2 v_i=0,$$

where $\overline{\nabla}$ stands for the operator

$$h_1 \frac{\partial}{\partial \xi}, \quad h_2 \frac{\partial}{\partial \eta}, \quad h_3 \frac{\partial}{\partial \zeta}.$$

From this it is seen that, unless there is a functional relation between v and the coördinate elements, the vanishing of $\overline{\nabla}^2 v$ implies the vanishing of $\overline{\nabla}$ $(h_i \partial x/\partial \xi)$; that is, $h_i \partial x/\partial \xi=$ const. and the transformation T is a linear orthogonal transformation. Conversely, if T is linear, then $\overline{\nabla}(h_i \partial x/\partial \xi)=0$, ∇^2 $(h_i \partial x/\partial \xi)=0$ and $\sum h_i(\partial x/\partial \xi)\overline{\nabla}^2 v_i=0$ which implies that $\overline{\nabla}^2 v_i=0$, $i=1, 2, 3$; that is, $\overline{\nabla}^2 v=0$.

In other words, although, when T is not the identity transformation, $\nabla^2 u$ never transforms into $\overline{\nabla}^2 v$, $\nabla^2 u=0$ implies $\overline{\nabla}^2 v=0$ when and only when T is a linear orthogonal transformation.

Formula (1) has appeared in the literature in a rather artificial manner; it is a statement limited to a vector point function referred to a fixed rectangular cartesian frame; it is a fairly interesting relation, but should not in my opinion be included among the relations of vector analvsis.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Algebra. By Oskar Perron. Second Edition. Berlin and Leipzig, Walter de Gruyter and Co. Volume I, Die Grundlagen, 1932. viii+301 pages. Volume II, Theorie der Algebraischen Gleichungen, 1933. viii+261 pages.

The first edition of this two-volume work, which has for several years had a recognized place in our libraries, was distinguished by the prominence given to the field concept. In the new edition there is an improved presentation of this concept and its fundamental significance in many parts of the theory is further emphasized. The author retains as his objective the desire to write a book available to a beginner, but one which takes account of recent research in the subject.

In the first chapter of volume one an abstract definition of *field* is given and discussed before it is specialized to the number field definition used in the earlier edition. Here, as well as in many other places where fundamental ideas are presented, developments which are not essential but which are confusing to the

novice are excluded; only fields of characteristic zero and rings with units and no divisors of zero are discussed. The complex field is obtained by use of the Hamilton number pair definition, a treatment which finds satisfactory generalization in a later chapter where the König procedure is employed to set up a field which contains a root of a given equation in one variable irreducible in a given field. This treatment leads to a discussion of the field of decomposition of any polynomial in one variable, the results of which are applied in the proof of the so-called "Fundamental Theorem of Algebra," where the older Gauss-Gordan proof has been replaced by the Artin-Schreier proof of 1927 (given also by Dörge in 1928). This proof has the merit of greater brevity and is an elegant application of the field theory already mentioned.

Consistent with the change in the introduction of the field notion is the change in the treatment of groups in the second volume. The definition of an abstract group is given at the outset and the isomorphism of a finite abstract group with a substitution group proved. This makes possible a simpler and more satisfactory treatment of factor groups than appeared in the first edition. Throughout the second volume the author has achieved considerable simplification in the proofs of his theorems and greater clarity in their statement.

A section on number theory has been inserted before the discussion of linear groups; the subject is presented in such a way that it grows naturally out of the group notions already discussed and gives a satisfactory preparation for the linear group theory which follows.

After the subject of finite extensions of a field has been treated, there is an excellent exposition of the Galois group, co-ordinating the presentation given in the first edition with the more recent procedure based on the automorphisms of the field of decomposition of the given equation. The notion of automorphisms, however, is not used in subsequent proofs; the author retains the proofs based on his earlier definitions. But it must be said that a student who has read and understood this treatment should have no difficulty in understanding a presentation of the subject such as the one given by van der Waerden.

Chapter VII of the first volume has been enlarged to include additional material on systems of equations and the conditions for the existence of solutions, and a new and improved proof of Bézout's theorem which takes account of the multiplicity of solutions. In the second volume some sections on orthogonal transformations and characteristic equations have been added, but, as in the first edition, certain familiar topics such as elementary divisors and pairs of bilinear forms are not mentioned.

Insistence on clarity and concreteness in the presentation of details make this a good text for beginning students, though the more experienced reader may feel at times that the author has been overscrupulous in the inclusion of all details. Professor Perron has succeeded admirably in preserving the high standard of the first edition, while making revisions which should enable the student to read with greater ease current algebraic research.

MINA S. REES

Elementary Mathematical Analysis. By Mayme Irwin Logsdon. New York, McGraw-Hill Book Company. Vol. I (1932) xiv+212 pages, 5 tables. $2.25. Vol. II (1933) ix+188 pages. $1.75.

These two volumes represent an attempt to incorporate into a more or less harmonious whole the topics usually treated in the courses preparing for Calculus, namely, Trigonometry, Algebra, and Analytics. In order to achieve this harmonious integration the concept of function is introduced at the outset. This naturally leads to the study of the derivative and its application to the elementary functions. These ideas are the only ones that are foreign to the usual courses in these subjects. This, of course, in the words of the author, makes for "simplicity and continuity" although it does rearrange somewhat the order of the topics.

Roughly, the order of discussion is that mentioned above, although some topics found in the traditional course in Algebra have been placed at the end of Volume II because they do not fit into the logical treatment that is followed. Each subject receives approximately the same number of pages.

Review in the technique acquired in high school mathematics is effectively presented. For example, the theory of exponents is recalled in the chapter on logarithms and restated in terms of logarithms. This plan has been followed throughout the books, that is, a technique is reviewed at the time it is needed.

As for Trigonometry, the usual material is given. The functions are defined for the general angle in the beginning, thereby effecting a considerable saving of time and space. In the treatment of oblique triangles only the cosine law is made available for the case of three sides given. Would it not be desirable to have an adequate table of squares included so as to place this case on a more equal footing, as regards accuracy and simplicity of solution, with the other cases? A short chapter on the history of the development of Trigonometry is placed just before the solution of triangles. A feature that will find favor with the student is that the pronunciation of Greek letters is given when they are first introduced. Did the reviewer overlook the statement that τ is tau (I, p. 162)?

The discussion of the number system of elementary mathematics (II, Chap. 1) is excellent and will gratify those instructors who have been in the habit of supplying such material as a preliminary to the study of complex numbers. There is a paragraph (II, p. 149) to help the student understand when mathematical induction is needed for the proof of a formula and when not. May it remove some of the terrors of that topic!

A feature that, fortunately, one finds more and more in texts on analytics, is stressed in these volumes, namely, the close correspondence between ideas, elements, and language of geometry and those of algebra. (I, p. 9, II, p. 51). In fact, whenever facts can be expressed from two points of view, these statements are exhibited in parallel columns (I, p. 68, II, p. 122, for example). The treatment of conics varies somewhat from that to which one is accustomed. For example, it is stated that an ellipse is the graph of a certain equation. Then

the string property is stated and the equation of this locus is shown to be the same as the one first given. Many may object to the fact that there are few, if any, locus problems. The only discussion of this fundamental problem that the reviewer noticed, in this connection, is in Volume II, p. 98 where the equation of the general conic is derived from its definition as a locus. The author feels constrained to defend her short treatment of translation of axes. The writer believes that the insertion of exercises (there are none) at this point would be of assistance to the student.

Each chapter is opened by a paragraph stating what is to be accomplished in the chapter and at the end a short summary of the contents is listed. For quick reference the chapter and section numbers are printed at the head of each page. The two volumes afford a comprehensive analysis of the polynomial, rational, trigonometric, logarithmic, and exponential functions. In fact, an ϵ-definition of a continuous function is given (I, p. 24) and certain of the above functions are *proved* to be continuous!

The two volumes have been designed to give a sound preparation for a course in Calculus and require four hours per week for a year, or, three hours per week for a year and a half. The reviewer believes that they would serve for a year's work at three hours weekly, provided one is willing to postpone to a later course such chapters as those on determinants and the simultaneous solution of three linear equations, pencils of curves, etc.

We list here the few typographical errors that have come to notice:

Vol. I, p. 17, line 16, for sam read same.
Vol. II, p. 53, bottom of page, for concurrent read collinear.
Vol. II, p. 112, line 10, omit parentheses.

C. A. Nelson

Knotentheorie. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 1, No. 1. By K. Reidemeister. Berlin, Julius Springer, 1932. vi+74 pages. RM 8.75.

This is the first of a series of monographs sponsored by the editors of the Zentralblatt fuer Mathematik. In a preface to the series they express the belief that these monographs will supplement the work of the Zentralblatt by strengthening and consolidating the positions attained by individual research on a particular topic. It is expected that the series will supplement also the Encyklopädie by giving the more recent results in each special field in the form of monographs which will be largely complete in themselves. This first member of the series adheres very closely to these ideals. There is a detailed and careful exposition of the modern theory of knots and their projections, invariants and groups. At the end of the monograph there is an excellent bibliography of the literature, and there are frequent references to this in the text so that the reader may orientate himself in the literature of knots. There is also appended the table of knots due to Alexander and Briggs.

W. L. Ayres

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1932–33

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of California

The officers for the coming year are: Mr. John Oxtoby, Director; Mr. Raphael M. Robinson, Vice Director; Miss Myra Waddell, Secretary; Mr. Willis E. Lamb, Jr., Treasurer; Mr. Gabriel G. Bejarano, Librarian. These officers were elected by a majority vote of the members present at a regular meeting of the society.

There are fifty active members in the society, twelve were initiated on October 8, 1932, and seventeen on March 4, 1933.

The meetings and programs were as follows:

August 24, 1932: "Polygenic functions" by Mr. William Hutchings.
September 21, 1932: "The orbit problem" by Dr. Sophia Levy.
October 19, 1932: "Our journey through space" by Miss Phyllis Hayford.
November 16, 1932: "Hypergeometry" by Dr. Wong.
January 25, 1933: "Vector analysis" by Mr. Floyd Fisher.
February 15, 1933: "Transfinite numbers" by Mr. Raphael Robinson.
March 16, 1933: "Number ratios" by Mr. John Oxtoby.
April 16, 1933: "Coordinate systems" by Mr. Everett Matthews.

Our annual picnic was held at Cordonices Park on May 5, 1933, and despite the fact that this was the last week of finals it was well attended.

MYRA WADDELL, *Secretary*

Pi Mu Epsilon of the University of California at Los Angeles

We are very glad to be able to report a most successful year under the direction of the following officers elected in the fall of 1932: Caroline Dutton, Director; Ruth Cunningham, Vice Director; Walter Roberts, Secretary; Wendell Mason, Treasurer; Carroll Brady, Librarian.

The director, the vice director and the secretary are undergraduate students. We have at the present time an active membership of thirty-five and a faculty membership of fourteen. Ten students were admitted to membership in the fall and six in the spring.

During the eight regular monthly meetings, the following activities were enjoyed:

October 10, 1932: "Some analogies between algebraic and differential equations" by Mr. D. Hyers.
November 9, 1932: Informal introduction of pledges.
December 7, 1932: "Related difference and differential equations" by Miss Bordon; "Applications and tricks of integration" by Mr. Garver.
January 11, 1933: "The series method in differential equations" by Mr. Roberts; "Thomas Jefferson and his mathematics today" by Miss Glazier.
February 15, 1933: The presentation and discussion of revised By-Laws.

April 12, 1933: "Quadratic involutions" by Miss Cunningham; "Calculation with a mathematical computer" by Mr. Daus.
May 17, 1933: "Tensor analysis" by Mr. Richardson.
June 7, 1933: Election of officers for 1933–1934.

The Fall initiation was held November 19, 1932 at the home of Mr. and Mrs. Wendell Mason. The Spring initiation, a semi-formal banquet, took place on May 6, 1933 at the Miramar Beach Club in Santa Monica. At Christmas an informal party was held in conjunction with the University Mathematics Club.

Every year our organization offers a prize calculus examination. This year it was held on May 16, 1933. Mr. D. Kalbfell and Mr. Ralph Phillips tied for highest honors. Honorable mention was awarded to Miss Sarah Bordon. We are also pleased to announce the election of three of our members to Phi Beta Kappa. They are Elizabeth Breuer, Donald Hyers, Reginald Richardson.

WALTER ROBERTS, *Secretary*

Pi Mu Epsilon of Hunter College of the City of New York

The Hunter College Chapter of Pi Mu Epsilon devoted the four program meetings of the fall term to the topic: "Invariants and Covariants." Attention was restricted to the fundamental processes of invariant theory and to the concomitants of the binary quadratic and cubic. Nine students reported on this topic. During the second term eleven students read papers selected from "Higher plane curves" and the "Calculus of finite differences."

Twenty-seven students and two faculty members were initiated during the year, bringing the active membership up to fifty-three.

In November the fraternity held its formal initiation at a dinner party at the Hotel Barbizon. Sixty-three active and alumni members attended. The chapter was especially honored on this occasion by the presence of Professor Edward Kasner as guest speaker. The second initiation of the year followed by a tea and bridge party was held in March.

FRIEDA ZWECHER, *Corresponding Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Clubs of Carleton College

At Carleton College there are two organizations of individuals interested in mathematics. Their activities are more or less interrelated. First, there is the School of Crotona with membership limited to those who have passed a comprehensive oral examination in mathematics through integral calculus. Second, there is the Mathematics Club with unlimited membership. The School of Crotona serves as the executive council of the latter since the members of this smaller group conduct the programs and select the officers of the Mathematics Club.

At present, William Watson is President of the Mathematics Club; Philip Nason is Secretary-Treasurer. These officers were elected by the members of the School of Crotona. There are usually eight active members in the School of Crotona. The attendence at the meetings of the Mathematics Club varied from about twenty to forty.

The primary aim of the organizations is to combine a student discussion of mathematics with coffee, tea, and rolls—the Greek idea of congenial intellectual discussion.

Six meetings were held this year. They were as follows:
October 14, 1932: "The cocoanut problem" by William Watson.
November 7, 1932: "Mathematical problems and puzzles" by William Lee.
December 7, 1932: "Solutions of quadratic equations" by Edward Tomastic.
January 18, 1933: "Probability" by Philip Nason.
February 20, 1933: "The use of mathematics in objective testing" by Dr. Selmer Larson.
May 4, 1933: "Trisection of angles and other mathematical problems and puzzles" by Ray Wendland, Raymond Jurgensen, and Selma Kjontvedt.

WILLIAM WATSON, *President*

The Mathematics Club of Butler University

The officers for the academic year 1932–1933, elected by secret ballot in May 1932, were: Rhom Settles, President; Thelma Tacoma, Vice President; Douglas Ewing, Treasurer; Fletcher Rahke, Secretary.

The purpose of the club is two-fold: to consider and discuss historic and current mathematical topics and to give opportunity to all students in the department to become acquainted.

Membership in the club is granted to all students in the department of mathematics, to those who have been students in the department, and to sponsors. The active membership numbered twenty-two for the year 1932–1933.

The meetings and programs for the year were as follows:

October 6, 1932: "The organization and history of the club" by Douglas Ewing; "Mathematics and music" by Thelma Tacoma.

November 3, 1932: "Magic squares" by Anna K. Suter, Assistant in the department of mathematics.

December 8, 1932: Christmas party, mathematical games and puzzles.

January 13, 1933: The reunion and anniversary meeting. Forty-five persons attended. At this meeting, Dr. S. E. Elliott, Professor of Physics of Butler University discussed "The Deloperator."

February 2, 1933: "The Einstein theory of relativity" by Professor W. D. Carnahan, Head of the department of mathematics at Shortridge High School.

March 2, 1933: "The trisection of an angle" by Miss Florence Rathart (read by John Batcheler); "Professor Callahan's attempt at the trisection of an angle" by Miss Donnabelle Naylor.

April 6, 1933: "The science of mathematics, its origin, development and importance" by Mr. Walter Gingery, Principal of Washington High School.

May 11, 1933: "A trip through space" by Panoria Apostol (read by Miss Harriet Summers).

June 2, 1933: Annual picnic on the Butler campus.

FLETCHER T. RAHKE, *Secretary*

The Mathematics Club of Adelphi College

The officers of our mathematics club were: Dorothy Hill, President; Hazel Geis, Vice President; Clara Strein, Secretary; Doris Bettman, Treasurer; Dr. Joseph Bowden, Honorary President. The annual election of officers occurs at the May meeting. There are approximately thirty active members.

The primary aim of the club is to further interest in mathematics. Membership in the club is open to all students interested in mathematics.

The meetings and programs were as follows:

September 27, 1932: Business meeting.

October 17, 1932: Social meeting. A tea was held to welcome the new members into the club.

October 31, 1932: "Life of Sir Isaac Newton" by Miss Doris Bettman; "Life of Sir Christopher Rene" by Miss Isabel Hill.

November 21, 1932: "Generalized coordinates" by Miss Leon, a member of the faculty.

December 6, 1932: "Discussion of Teacher-In-Training Examination" by Miss Gordon, a 1932 graduate.

January 17, 1933: "Scales of notation" by Dr. Bowden, Honorary President.

February 20, 1933: "Recent methods in the teaching of mathematics" by Miss Flora D'Amato; "Primitive methods of counting" by Miss Hazel Geis.

March 16, 1933: Joint meeting of the Chemistry and Mathematics clubs.

April 3, 1933: "Relation of mathematics to astronomy" by Professor David, Professor of Physics and Astronomy.

May 1, 1933: "Scales of notation" by Dr. Bowden.

May 9, 1933: Special business meeting; nomination of officers for the year 1933–1934.

May 16, 1933: Special business meeting; election of officers for the year 1933–1934.

May 22, 1933: Special meeting; installation of officers.

CLARA STREIN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 54. *Proposed by P. R. Hill, University of Georgia.*

If the probability that an event will occur in a single trial is $1/N$, then as N increases without limit the probability that the event will occur at least once in N trials approaches the limit, $1-1/e$.

E 55. *Proposed by J. Rosenbaum, Milford, Connecticut.*

Obtain the general solution in positive integers of

$$2z^3 = x + (x^2 - 4y^3)^{1/2}.$$

E 56. *Proposed by Otto Dunkel, Washington University, Saint Louis, Missouri.*

From the base vertices A and B of an isosceles triangle ABC, segments of straight lines AL and BM of equal length are drawn to the opposite equal sides. Determine by plane geometry the locus of P, the intersection of AL and BM.

E 57. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Tom, Dick and Harry were comparing notes on their fishing experiences of the previous summer, and found that one had caught only perch, one only trout, and one only salmon. They remarked that the number of fish Tom had caught was seven more than three-fifths of the number of perch that were caught; that the number of fish Dick had caught was three more than five-sevenths of the number of salmon that were caught; and that the total number of all the fish caught was a three-place prime number. Determine how many and what kind of fish each caught, and show that the solution is unique.

E 58. *Proposed by R. M. Sutton, Haverford College, Pa.*

In the following division of a three-place number into a five-place number each digit has been replaced by a code letter. Assuming only that the remainder, Y, is not zero, reconstruct the problem and show that the solution is unique.

$$\begin{array}{rcccccl}
L\ M\ N) & R & S & T & U & N & (U\ X \\
 & R & T & Y & X & & \\
\cline{2-6}
 & & T & Y & Y & N & \\
 & & T & Y & Y & J & \\
\cline{2-6}
 & & & & & Y &
\end{array}$$

E 59. *Proposed by J. H. Butchart, Indianapolis, Indiana.*

In the angle ACB of triangle ABC circles are inscribed tangent respectively to AC at A and to BC to B. Prove that the chords intercepted on the side AB are equal.

SOLUTIONS

E 26. [1933, 175] *Proposed by H. T. R. Aude, Colgate University.*

All proper rational fractions in lowest terms can be separated into two classes:

1. Those in which the numerator and denominator are both odd.
2. Those in which the numerator and denominator are not both odd. Show that for any fraction F in one class there is just one corresponding fraction F' in the other such that $\arctan F + \arctan F' = \pi/4$.

Solution by A. E. Andersen, Wagner College, Staten Island, N. Y.

Since $(F+F')/(1-FF') = \tan \pi/4 = 1$, we have $F' = (1-F)/(1+F)$.

If F belongs to class 1, it may be written in the form $(2n+1)/(2m+1)$, where both m and n are positive integers. Substitution and reduction gives $F' = (m-n)/(m+n+1)$, and since this numerator and denominator differ by the odd number, $2n+1$, one must be odd and the other even, so that F' belongs to class 2.

If, on the other hand, F belongs to class 2, it may be written in one of the two forms, $2n/(2m+1)$ or $(2n+1)/2m$. In these cases substitution shows $F' = (2m-2n+1)/(2m+2n+1)$ or $(2m-2n-1)/(2m+2n+1)$, each of which forms is of class 1.

Solved also by L. S. Johnston, Theodore Lindquist, J. Rosenbaum, Simon Vatriquant and R. N. Walter.

E 27. [1933, 175] *Proposed by E. P. Starke, Rutgers University.*

Derive the algebraic formula for the sum of n fractions whose numerators are in arithmetic progression and whose denominators are in geometric progression.

Solution by H. E. H. Greenleaf, De Pauw University, Greencastle, Indiana.

Let the sum of the fractions be

$$S = \frac{a}{b} + \frac{a+d}{br} + \frac{a+2d}{br^2} + \frac{a+3d}{br^3} + \cdots + \frac{a+(n-1)d}{br^{n-1}}$$

$$= \frac{a[1+r+r^2+\cdots+r^{n-1}]+d[(n-1)+r(n-2)+r^2(n-3)+\cdots+2r^{n-3}+r^{n-2}]}{br^{n-1}}.$$

The coefficient of a is seen to be $(1-r^n)/(1-r)$.

The coefficient of d may be written with each power of r copied as many times as called for by its coefficient, thus:

$$\begin{array}{l} \quad 1 \\ +\,1+r \\ +\,1+r+r^2 \\ +\,1+r+r^2+r^3 \\ +\,\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \quad\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ +\,1+r+r^2+\cdots+r^{n-3}+r^{n-2} \end{array}$$

for $n-1$ rows.

If this is summed horizontally and the results added, the coefficient of d reduces to $(1-r+1-r^2+1-r^3+\cdots+1-r^{n-1})/(1-r)$, or

$$[n(1-r)-(1-r^n)]/(1-r)^2,$$

so that finally

$$S=\frac{(a-ar-d)(1-r^n)+nd(1-r)}{br^{n-1}(1-r)^2}.$$

In their solutions of this problem, F. C. Gentry and R. MacKay point out that the series of fractions converges if r exceeds unity, and that if n then increases without limit, the limit of S is

$$\frac{ar^2+r(d-a)}{b(r-1)^2}.$$

R. MacKay and R. N. Walter remark that this problem is very closely related to one on page 44 of Hall and Knight's *Higher Algebra.*

Solved also by Martha Allen, A. E. Andersen, W. E. Buker, L. S. Johnston, Theodore Lindquist, F. L. Manning, Lazerus Medveson, Jr., Charles Molloy, W. R. Ransom, C. C. Richtmeyer, J. Rosenbaum, H. D. Ruderman, Simon Vatriquant and someone who forgot to sign his solution.

E 29. [1933, 175] *Proposed by J. Rosenbaum, Milford, Connecticut.*

The faces of a tetrahedron are congruent triangles whose sides are a, b and c. If $2S=a^2+b^2+c^2$, show that the volume is $V=[(S-a^2)(S-b^2)(S-c^2)]^{1/2}/3$.

Solution by Simon Vatriquant, Athénée Royale d'Ixelles, Brussels, Belgium.

Denoting by A, B and C the angles opposite the sides a, b and c in each of the four triangles, the area of each triangle is $K=\frac{1}{2}ab\sin C$.

The altitude h of the tetrahedron is $c\sin H$, where H is the angle between the edge c of the trihedral angle at a vertex of the tetrahedron and the opposite

face of that trihedral angle. Cutting this trihedral angle by a sphere centered at its vertex gives a spherical triangle of sides A, B and C, in which H is the arc-altitude to the side C. In this spherical triangle, $\sin H = \sin A \sin P$, where P is the angle of the spherical triangle opposite the side B. Finally P is given in terms of the sides of the triangle by the formula

$$\sin P = 2\left[\sin \tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(B+C-A)\right.$$
$$\left.\sin\tfrac{1}{2}(C+A-B)\sin\tfrac{1}{2}(A+B-C)\right]^{1/2}/\sin A \sin C.$$

But $A+B+C=180°$, so that this formula reduces to

$$\sin P = 2[\cos A \cos B \cos C]^{1/2}/\sin A \sin C.$$

Then the altitude h of the tetrahedron is $2c[\cos A \cos B \cos C]^{1/2}/\sin C$, and the volume $V = Kh/3 = abc[\cos A \cos B \cos C]^{1/2}/3$.

Now from the cosine law for a plane triangle we may readily derive $ab \cos C = \frac{1}{2}(a^2+b^2-c^2) = S-c^2$, and the two similar relations. When we use these relations to eliminate the cosines from the formula just derived for the volume, there results the desired formula, $V = [(S-a^2)(S-b^2)(S-c^2)]^{1/2}/3$.

Solved also by W. E. Buker, H. E. H. Greenleaf, R. N. Walter, the proposer and the anonymous solver of E 27.

E 30. [1933, 175] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The arithmetic mean between NED and $SASH$ is $SHUN$. Their geometric mean is $SEND$, and their harmonic mean is $SEED$. Assuming that the seven letters here involved represent different digits, identify them.

Solution by W. E. Buker, Leetsdale, Pa.

1. Since $NED+SASH=2(SHUN)$, S must be 1.
2. $(NED)(SASH)=(SEND)^2$. Thus $(D)(H)$ must equal a number whose right hand digit could be the right hand digit of D^2. The values of (D, H) for which this is possible are (2,7), (4,9), (8,3) and (5, any odd digit).
3. But from (1), $D+H$ must be even, so that D is 5 and H is 3, 7 or 9.
4. From (1), $SHUN<1500$, so that $H=3$.
5. Since from (1), $2(SHUN)$ now ends in 8, N is 4 or 9. But if N were 4, $SHUN$ could not equal $13xx$ because of (1), so N must be 9.
6. (2) has now become $(9E5)(1A13)=(1E95)^2$, so that $(9E5)(1A13)$ must end in 25. This forces E to be 2. Then $(925)(1A13)=(1295)^2$, and A is 8.
7. (1) has now become $925+1813=2(13U9)$, so U is 6.
8. Finally, $SEED$ is 1225, which is the harmonic mean of 925 and 1813.

It is interesting to note that the complete solution was obtained without using the given information concerning the harmonic mean.

Solved also by M. L. Constable, F. C. Gentry, F. L. Manning, W. R. Ransom, Elijah Swift, Simon Vatriquant, J. H. Weaver and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3631. *Proposed by Norman Anning, University of Michigan.*

On the Argand diagram a regular pentagram is constructed with its center at the origin and one vertex on the axis of positive real numbers. Find the equation whose roots are the numbers associated with the ten points where the lines intersect. Can the scale be so chosen that the equation will have integral coefficients?

3632. *Proposed by Morgan Ward, California Institute of Technology.*

If $r_1, r_2, \cdots, r_k$ are k fixed positive numbers whose sum is greater than one, and if

$$u_n = \sum_{(m_1+m_2+\cdots+m_k=n)} m_1^{-r_1} m_2^{-r_2} \cdots m_k^{-r_k},$$

where the summation variables m have the range $1, \cdots, n$, then, after a certain value n_0 of n, u_n decreases monotonically to zero with increasing n.

3633. *Proposed by N. A. Court, University of Oklahoma.*

If the three face angles of a trihedral angle of a tetrahedron are right angles, the line joining this vertex of the tetrahedron to the centroid of the opposite face is equal to two thirds of the circumradius of the tetrahedron.

3634. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, the bisectors of the angles A and B meet the opposite sides at D and E. Prove that if DE divides the angles CDA and CEB into parts having equal ratios then the sides CA and CB are equal.

3635. *Proposed by Martin Rosenman, Brooklyn, New York.*

Given irrational numbers, x and y, and any positive quantity e, determine integers r, s, and t such that

(a)
$$0 < tx - r < e,$$
$$0 < ty - s < e.$$

(b)
$$0 < rx - t < e,$$
$$0 < sy - t < e.$$

3636. *Proposed by B. F. Finkel, Drury College.*

An equilateral triangular lamina of uniform density and of mass m is suspended in a horizontal position by three equal inextensible strings fastened to its three vertices A, B, C, and also to a point, P, above the lamina. A weight, w, is placed at any point Q inside the triangle ABC. Determine the tension in each string.

3637. *Proposed by Otto Dunkel, Washington University.*

Triangle ABC is isosceles with the vertical angle C greater than $60°$ and less than arc cos $1/3$. Through a point P in its interior the segments of straight lines $AL=p$, $BM=q$, $CN=r$ are drawn to the opposite sides. Show that there are two points on its interior such that $p=q=r$, and that each of these segments has the length $3^{1/2}c/2$, where $c=AB$.

Show also that, adjacent to each of the equal sides $AC=BC$, there is in the interior of the triangle a region bounded by a part of that side and by parts of two curves which meet in one of the above two points, such that for each interior point r is the longest of the three segments. See problem 3576 [1932, 549].

3638. *Proposed by Oystein Ore, Yale University.*

Let p be a prime and N the smallest exponent such that $p^N \equiv 1 \pmod{n}$ for a given number n. The irreducible factors of $x^n-1 \pmod{p}$ then have degrees dividing N. Find the necessary and sufficient condition, that there exist prime divisors of $x^n-1 \pmod{p}$ of degree N' for all divisors N' of N.

SOLUTIONS

3575. [1932, 549] *Proposed by Frank Morley, Johns Hopkins University.*

Given two circles in a Euclidean space, which are not interlaced and not cospherical; show that there are four circles which touch both, and that these break into two pairs, the three pairs forming a symmetrical configuration.

Solution by N. A. Court, University of Oklahoma.

If a circle is tangent to the two given circles (a), (b), the point L common to the planes of the three circles has equal powers with respect to (a) and (b), or, what is the same thing, with respect to the two spheres (A), (B), having (a), (b), for diametral circles. Thus the point L is the trace of the radical plane σ of (A), (B), on the line of intersection m of the planes of (a), (b), and may therefore, in general, be determined.

Let P, Q; R, S; be the points of contact of the two pairs of tangents LP, LQ; LR, LS; from L to (a) and (b). The circle (PR) passing through P, R, and tangent to LP at P will also be tangent to LR at R; hence (PR) is tangent to both (a) and (b). Similarly, for the circles (PS), (QR), (QS).

Each of the six circles $(PQ)\equiv(a)$, $(RS)\equiv(b)$, (PR), (PS), (QR), (QS), is tangent to four of the remaining five circles. Since

$$LP = LQ = LR = LS,$$

the four points P, Q, R, S, lie on a sphere (L) having L for center. The planes of the six circles are the faces of the complete four-edge determined by the radii LP, LQ, LR, LS, of (L). The figure may thus be obtained by taking four points P, Q, R, S, on a given sphere (L), and it is immaterial how these four points are separated into two groups to obtain the first two circles (a), (b).

Special cases. (i). The plane σ passes through the line m. Every point of m may be made to play the role of the point L. The problem has an infinite number of solutions. However, this case is excluded by the wording of the problem, for it may be shown that the circles (a), (b), are cospherical in this case.

Indeed, since σ is perpendicular to the line of centers AB, and m, by hypothesis, lies in σ, the two lines AB, m, are orthogonal, and the plane through AB perpendicular to m contains the axes c, d, of (a), (b). Let (G), (G'), be the two spheres passing through (a), (b), respectively, and having for center the point $G \equiv cd$. Since m lies in σ, any point U of m is the center of a sphere (U) orthogonal to both (A) and (B). Now U lies in the radical plane of (A) and (G); hence (U) is also orthogonal to (G). Similarly, (U) is orthogonal to (G'). But with the point G as center only one sphere may be drawn orthogonal to (U); hence (G) and (G') are identical, i.e., the two circles (a), (b) will have to be co-spherical, which is contrary to the assumption of the problem.

(ii). If the plane σ is parallel to m, the points P, Q, R, S are the ends of the two diameters of (a), (b), which are perpendicular to m, and the solution does not differ from the general case. The axes c, d, have a point in common, but (a), (b), are not cospherical.

(iii). If the planes of (a), (b), are parallel, and the line AB is perpendicular to these planes, the two circles are co-spherical. If AB is not perpendicular to the planes of (a), (b), the points P, Q, R, S are the ends of the two diameters of (a), (b), which lie in the plane through AB perpendicular to the planes of (a), (b). The solution is the same as in the general case.

(iv). If the two circles (a), (b), are concentric, the same is true about the spheres (A), (B), and σ is at infinity, so that the point L coincides with the point at infinity of m, and the points P, Q; R, S, are the ends of the two diameters of (a), (b), perpendicular to m.

Note. The problem is proposed as an exercise in Jacques Hadamard's *Leçons de Géométrie Élémentaire*, Paris, 1901 vol. II, p. 149, ex. 689.

Also solved by Rufus Crane, C. A. Rupp and the proposer.

3579. [1932, 607] *Proposed by B. F. Kimball, Schenectady, N. Y.*

Given

$$F(x, a) = \sum_{s=-\infty}^{+\infty} \frac{\sin (x + sa)}{x + sa}$$

where s takes on all integral values including zero. Evaluate $F(x, a)$ for all real values of x and a. Discuss the discontinuities of the function.

Solution by the Proposer

Given

$$F(x, a) = \sum_{s=-\infty}^{+\infty} \frac{\sin (x + sa)}{x + sa}. \tag{1}$$

Expand $\sin z(\pi - \theta)$ in sine series in θ over interval $0 < \theta < 2\pi$.

$$\sin z(\pi - \theta) = \frac{\sin \pi z}{\pi} \sum_{s=-\infty}^{+\infty} \frac{\sin s\theta}{z + s}.$$

Similarly,

$$\cos z(\pi - \theta) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{+\infty} \frac{\cos s\theta}{z + s}, \qquad 0 < \theta < 2\pi.$$

Combining we have

$$\sin (x + \pi z - \theta z) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{+\infty} \frac{\sin (x + s\theta)}{z + s}, \qquad 0 < \theta < 2\pi. \tag{2}$$

Denote the function of θ which is the sum of the series on the right by $f(\theta)$. Now $f(\theta)$ is periodic with period 2π. Denote $\theta - 2n\pi$ by $\bar{\theta}$. Then if θ lies in the interval

$$2n\pi < \theta < 2(n + 1)\pi, \tag{3}$$

n a positive integer or zero, $\bar{\theta}$ lies between 0 and 2π and accordingly

$$\sin (x + \pi z - \bar{\theta} z) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{+\infty} \frac{\sin (x + s\theta)}{z + s}, \qquad \begin{matrix} 0 < \bar{\theta} < 2\pi, \\ 2n\pi < \theta < 2(n + 1)\pi. \end{matrix}$$

Hence for θ in the interval (3),

$$\sum_{-\infty}^{+\infty} \frac{\sin (x + s\theta)}{z + s} = \frac{\pi}{\sin \pi z} \sin \left[(2n + 1)\pi z + x - \theta z\right]. \tag{4}$$

Now divide by a, set $\theta = a$ and $z = x/a$. We get

$$F(x, a) = \frac{\pi}{a} \frac{\sin\left[(2n + 1)\pi \dfrac{x}{a}\right]}{\sin \dfrac{\pi x}{a}}, \qquad 2n\pi < a < 2(n + 1)\pi, \tag{5}$$

where n is a positive integer or zero.

The series (2) being a Fourier's series, we have for $\theta = 2n\pi$,

$$(6)\quad f(2n\pi) = f(0) = \frac{\pi}{2\sin\pi z}[\sin(x+\pi z)+\sin(x-\pi z)] = \pi\sin x\cot\pi z.$$

Hence

$$(7)\qquad F(x,a) = \frac{\pi}{a}\sin x\cdot\cot\left(\pi\frac{x}{a}\right),\qquad a = 2n\pi,\qquad n\geqq 1.$$

Formulae (6) and (7) indicate that $F(x, a)$ has no value for $a=0$. Clearly from the definition of $F(x, a)$ it follows that

$$F(x, -a) = F(x, a).$$

Thus $F(x, a)$ is evaluated for all real values of x and a for which it is defined.

The behavior of the function at the discontinuities $a=2n\pi$ is best brought out by expressing formulas (5) and (7) in a different form. Using the trigonometric identity

$$\tfrac{1}{2}+\sum_{m=1}^{n}\cos m\theta = \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\frac{\theta}{2}},$$

one obtains

$$(8)\quad F(x,a) = \frac{2\pi}{a}\left(\tfrac{1}{2}+\sum_{m=1}^{n}\cos m\frac{2\pi x}{a}\right),\qquad 2n\pi < a < 2(n+1)\pi,\qquad n\geqq 0,$$

$$(9)\quad F(x,a) = \frac{2\pi}{a}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos x+\sum_{m=1}^{n-1}\cos m\frac{2\pi x}{a}\right),\qquad a = 2n\pi,\qquad n\geqq 1.$$

These last formulae bring out clearly the behavior of the function in the neighborhood of points $a=2n\pi$, $n\geqq 1$.

3580. [1932, 607]. *Proposed by J. B. Reynolds, Lehigh University.*

If a body dropping vertically from an airplane reaches a velocity of 120 mi./hr., which is 99 percent of its terminal velocity, in 40 sec. what is the value of n on the assumption that the resistance of the air is directly proportional to the weight of the body and the nth power of its velocity?

Solution by Elijah Swift, University of Vermont.

I assume that by "terminal velocity" is meant the limiting velocity approached as the time becomes infinite.

By hypothesis, the force on the body at any time is made up of gravity, downwards, and a resisting force, proportional to mass times the nth power of the velocity. Let x be the distance from the starting point, measured downwards as positive, v the velocity, t the time in seconds, k the constant of proportionality, g the acceleration due to gravity. Then the equation of motion is

$$m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mg - kmv^n.$$

The terminal velocity, v_T, is given by the equation $g - kv_T^n = 0$. Separating the variables and integrating, we have

$$\int_0^{.99v_T} \frac{dv}{g - kv^n} = \int_0^{40} dt.$$

The substitution $v = v_T z$ reduces this to the equation

$$\int_0^{.99} \frac{dz}{1 - z^n} = \frac{40g}{v_T},$$

since $kv_T^n = g$; or, reducing 120 mi./hr. to ft./sec. and substituting in the given values ($g = 32$), we have to solve for n the equation

$$I_n = \int_0^{.99} \frac{dz}{1 - z^n} = 7.2.$$

I_n clearly decreases as n increases. We compute its value for several values of n, obtaining the following table

$n =$	1,	$\frac{1}{2}$,	$\frac{2}{3}$
$I_n =$	4.605,	8.606,	6.603
$I_n - 7.2 =$	$-$ 2.595,	$+$ 1.406,	$-$ 0.597.

Interpolation or a rough graph shows that $n = 0.6$ is close to the correct value of n. Computing $I_{0.6}$, we find $I_{0.6} = 7.275$ (the last figure inaccurate) and interpolating we find $n = 0.607$ as a more accurate value. The value 0.6 is, however, accurate to within 1 percent and gives a value to I_n accurate to the same percent. Considering the nature of the data, the value $n = 0.6$ may be taken as the correct value of n and as verified by direct computation.

3581. [1932, 607]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let a, b, c, be the edges of a trihedral angle S-abc, and a', b', c', the perpendiculars at S to the planes bc, ca, ab. If u is the axis of a cone of revolution passing through a, b, c, the line u is also the axis of a cone of revolution tangent to the planes $a'b'$, $b'c'$, $c'a'$.

Solution by S. Vatriquant, Athênée Royale d'Ixelles, Brussels (Belgium).

Cutting the figure by a sphere of centre S, we obtain two spherical triangles ABC and $A'B'C'$, which are reciprocally polar, i.e. such that each vertex of one of them is the pole of the opposite side in the other. The cone passing through a, b, c, cuts the sphere along the circumscribed circle of ABC, and the line u meets the sphere at the pole U of this circle. If we join on the sphere (by arcs of

great circles) AU, BU, CU, these lines are equal and cut the sides of $A'B'C'$, respectively, at M, N, P, so that we have $AM=BN=CP=90°$. Thus $MU=NU=PU$, and since arcs issued from a pole are perpendicular to the polar circles, U is equidistant from the sides of $A'B'C'$. Hence U is the pole of the inscribed circle, cut on the sphere by the cone tangent to the planes $a'b'$, $b'c'$, $c'a'$, and the line u is the axis of this cone.

Solved also by A. D. Bradley, A. Pelletier, and W. H. Rasche.

3583. [1932, 608]. *Proposed by H. Grossman, New York.*

If from any point on a circle, chords be drawn to all vertices of a regular inscribed $3n$-gon, the sum of the n longest chords is equal to the sum of the $2n$ shortest ones.

Solution by Laurence Hadley, Purdue University.

Ptolemy's Theorem may be stated thus: The product of the diagonals of a convex quadrilateral inscribed in a circle equals the sum of the products of the opposite sides. Let ABC be an equilateral triangle, and let P be any point on the arc CA of the circumscribed circle. Applying Ptolemy's theorem to $ABCP$, we have $PB\cdot AC=PA\cdot BC+PC\cdot AB$. Since $AC=BC=AB$, we have at once $PB=PA+PC$. Therefore the following statement may be made as a corollary under Ptolemy's Theorem:

If chords be drawn from any point on a circle to the vertices of an inscribed equilateral triangle, the longest chord, so determined, equals the sum of the other two chords.

To solve the problem proposed, let us assume as given a regular inscribed $3n$-gon with P as any point on the circumscribing circle. Letter the vertices

$$A_1, A_2, \cdots A_n; \qquad B_1, B_2, \cdots, B_n; \qquad C_1, C_2, \cdots, C_n$$

in counter clock-wise direction beginning with $A_1=A$, where P lies on the arc C_nA_1. Denote the chords from P to the vertices of the polygon as follows:

$$PA_i = a_i; \qquad PB_i = b_i; \qquad PC_i = c_i; \qquad i = 1, 2, \cdots, n.$$

It is clear that $A_iB_iC_i$ is an equilateral triangle. If P is, for example, at the end A_1 of arc C_nA_1 then we may take the b's as the longest chords and the a's and c's as the shortest chords. In this case the shortest of the first set is equal in length to the longest of the second set, i.e., $b_1=c_1=$ a side of equilateral triangle $A_1B_1C_1$, and $a_1=0$. If P is not at an end of C_nA_1, the b chords are each longer than a side of the inscribed equilateral triangle while the a and c chords are each shorter than a side of such a triangle.

Applying the above corollary to the equilateral triangles $A_iB_iC_i$, we have

$$b_i = a_i + c_i, \qquad \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} c_i.$$

The last equation gives the desired result.

Solved also by M. G. Boyce, A. D. Bradley, W. B. Campbell, Mannis Charosh, D. C. Duncan, H. G. Green, A. S. Householder, Frank Irwin, A. Pelletier, and F. Underwood.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

AN OFFICIAL EMBLEM OF THE MATHEMATICAL ASSOCIATION

In response to numerous requests at various times by members of the Association, the Trustees have authorized the manufacture and sale of an official emblem to be worn either as a pin or as a button. A description of the emblem may be found on the inserted sheet at the end of the August-September issue of the Monthly. The design of the emblem was taken from the monogram which has been used on official publications of the Association for many years, but this emblem is much more beautiful because the central figure of an icosahedron is shown in bold relief. Of course the purchase and use of the emblem is entirely optional with the member.

The two hundredth anniversary of the discovery of the normal probability curve falls this year. The discovery was made by DeMoivre, who published his results on November 12, 1733, in a brief paper of seven pages which he presented privately to a few friends. This paper was entitled "Approximatio ad summam terminarum Binomii $(a-b)^n$ in Seriem Expansi." Only two copies of this rare paper are reported extant.

The Econometric Society, an international society for the advancement of economic theory in its relation to statistics and mathematics, has established a quarterly journal, entitled *Econometrica*, under the editorship of Ragnar Frisch, with H. T. Davis, A. H. Hansen, and F. C. Mills as associate editors. The editorial advisory board includes Irving Fisher, Harold Hotelling, Oystein Ore, and C. F. Roos. The business address of the journal is care of Alfred Cowles, Mining Exchange Building, Colorado Springs, Colo.

The degree of Doctor of Science was conferred upon Lipót Fejér, professor of mathematics at the University of Budapest, by Brown University at its commencement exercises on June 16.

The degree of Doctor of Science was conferred upon E. W. Brown, Josiah Willard Gibbs professor of mathematics, emeritus, of Yale University, by that institution on June 16. The following are some statements concerning Professor Brown made by Professor William Lyon Phelps at the conferring of the degree: Professor Brown was born at Hull, England, and received his B. A. degree at Christ College, Cambridge, in 1887. In 1891 he became professor of mathematics at Haverford College where he remained until he was called to Yale in

1907. He became professor emeritus in 1932. His publications on lunar theory and celestial mechanics have given him an international reputation. His tables on the motion of the moon is a monumental work. In youth he expected to be concert pianist, but later took up the music of the spheres.

At the commencement exercises at Harvard University the degree of Doctor of Science was conferred on Professor G. D. Birkhoff. Professor Birkhoff has also been elected Perkins professor of mathematics to succeed Professor W. F. Osgood, who on September 1 retired with the title of professor emeritus.

Oglethorpe University has conferred an honorary doctorate on Professor Archibald Henderson of the University of North Carolina.

Professor J. F. Reilly of the State University of Iowa has been elected chairman of the mathematics section of the Iowa Academy of Science, year 1933–34.

D. E. Smith, professor emeritus of Teachers College, Columbia University, has been decorated by Persia for his study of the Mathematical Works and Philosophy of Omar Khayyam.

The following will work at the School of Mathematics in the Institute for Advanced Study at Princeton: A. A. Albert, University of Chicago; Meyer Salkover, University of Cincinnati; T. Y. Thomas, Princeton University; E. R. vanKampen, Johns Hopkins University; W. E. Bleick, Johns Hopkins University.

Dr. J. V. Atanasoff of Iowa State College, has been promoted to an assistant professorship.

Associate Professor E. T. Browne, of the University of North Carolina has been promoted to a professorship of mathematics.

Dr. L. P. Eisenhart, professor of mathematics and dean of the faculty at Princeton University, has been elected dean of the graduate school.

Assistant Professor W. H. Gage, of Victoria College, has been appointed assistant professor of mathematics at the University of British Columbia.

Alan Hazeltine has returned to the Stevens Institute of Technology as professor of physical mathematics.

Dr. I. O. Horsfall, of Cornell University, has been appointed president of Snow College, Ephraim, Utah.

Assistant Professor K. W. Lamson of Lehigh University has been promoted to an associate professorship of mathematics.

Assistant Professor A. E. Meder of the New Jersey College for Women has been promoted to an associate professorship of mathematics.

Professor Helen Merrill of Wellesley College has retired as professor emeritus.

Dr. F. H. Miller of Columbia University has been appointed to an assistant professorship at Cooper Union.

Professor B. P. Reinsch of the Southern Methodist University, Dallas, Texas, has been appointed professor and head of the department of mathematics at Southern College, Lakeland, Florida.

Assistant Professor C. N. Shuster of the State Teachers College, Trenton, New Jersey, has been promoted to a professorship.

Professor Mary C. Spencer of Newcomb College has retired as professor emeritus.

Professor W. W. Weber of Columbia College, South Carolina, has been appointed professor of mathematics at Lander College.

G. A. Yanosik has been promoted to an assistant professorship at New York University.

Dr. A. L. Foster has been appointed instructor at the University of California at Berkeley.

Miss Nancy Cole has been appointed instructor at Sweet Briar College.

Miss Vevia Blair, Head of the department of mathematics at Horace Mann School for Girls, an observation school of Teachers College, Columbia University, died July 9, 1933, at the age of 62.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Annual Meeting of the Association, Cambridge, Mass., Dec. 27-30, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merges with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 2.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

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EDITED BY

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PROBABILITY FUNCTIONS AND STATISTICAL PARAMETERS

By L. S. HILL, Hunter College of the City of New York

1. *Preliminary Notes*

It is the purpose of this article to present a connected account of the more important concepts used in the modern theory of statistical distributions of sets of n continuous quantitative attributes. For the effective accomplishment of this purpose, it has seemed wise to avoid, so far as possible, any discussion of the philosophical groundwork, and to devote the space thus saved to a more complete exposition of the working ideas, about which little, if any, controversy exists.

Some form of definition of the fundamental terms is, however, necessary. That given in Sections 2 and 3 has the merit of being brief and explicit, and of affording a geometrical background against which the interrelations of the statistical ideas can be conveniently displayed. The circumstance that its soundness may be questioned in some quarters is, from our present standpoint, of only limited interest; for, as already stated, we are not attempting to deal with "foundations."

Certain notions of the theory of sets of points in an n-dimensional Euclidean space E_n will be required. If ϵ is an arbitrary positive real number, and $a=(a_1, a_2, \cdots, a_n)$ is a point of E_n, we denote by $V_\epsilon(a)$ the set of all those points of E_n of which the distances from a are less than ϵ; thus $V_\epsilon(a)$ consists of all points $x=(x_1, x_2, \cdots, x_n)$ of E_n such that $\sum_{j=1}^{n}(x_j-a_j)^2<\epsilon^2$.

A set J of points of E_n constitutes a *region* in E_n if, for each point a of J, there exists an $\epsilon>0$ such that every point of $V_\epsilon(a)$ is a point of J. The *frontier* of a region J consists of all those points of E_n which, while not points of J, are limit points of J; and the frontier is a closed set (a set containing all its limit points). The term *range* will be applied to any set in E_n which is either (1) a region, or (2) a region supplemented by the adjunction of some, or all, points of its frontier. The space E_n is a region.

If, for some point a of E_n, there exists a positive real number c (sufficiently great) such that every point of the range R is a point of the region $V_c(a)$, then R is called a *bounded* range. Let R be a bounded range in E_n, and let a, b denote points of R. Then the least upper bound of the distance (a, b), as a and b vary independently over R, is called the *diameter* of R.

Every bounded range R in E_n has a definite (positive) Lebesgue measure, which we shall call the *extent* of R. Extent coincides with length, area, volume, $\cdots$, in the cases $n=1, 2, 3, \cdots$.

2. *Statistical Populations, and Typical Samples*

Let P denote any set for each element (individual) of which n characteristics (attributes) are distinguished. Let each of these attributes be measurable

by a real number, so that there corresponds, to each individual of P, an ordered set $(x_1, x_2, \cdots, x_n)$ of n real numbers, and consequently a point of the space E_n. Let the set of all those points of E_n each of which corresponds to one or more individuals of P constitute the range T of E_n (T may or may not embrace all of E_n). Under these conditions, P will hereinafter be designated as a *statistical population* with the *domain* T.

The idealization of an infinite (and, indeed, non-denumerably infinite) population is made purely for mathematical convenience, and leads to results which are not seriously in error when, as is usually the case, the statistical material is very extensive. Since only quantitative and continuous attributes are under consideration, we employ the term "statistical population" in a somewhat restricted sense.

Every finite subset of the statistical population P will be called a *sample* of P. Some or all samples belong to a class known as *typical samples*. This class is supposed to be capable of exact definition, but we shall leave it *undefined* except in so far as the postulates of Section 3 imply restrictions upon it. The notation S_q will be used for a typical sample consisting of q individuals of P. Also, if R is any range every point of which is in the domain T, and if M is the number of individuals of a typical sample S_q which correspond to points of R, then M will be called the *frequency* and M/q the *relative frequency* of S_q in R.

3. *Probability Functions*

We make the following postulates regarding typical samples.

(A). *If k is any positive integer, P contains typical samples S_q with $q>k$.*

(B). *If $S_{q_1}, S_{q_2}, S_{q_3}, \cdots$ is any infinite sequence of typical samples of P such that $q_1<q_2<q_3<, \cdots$, and if F_i is the relative frequency of S_{q_i} in R, then the sequence $F_1, F_2, F_3, \cdots$, converges to a positive real number F which depends only upon R.*

The number F thus determined as a function of the range R may be called the "expected relative frequency" of a typical sample of P in R, or simply the *typical relative frequency in R*. Denoting by L the extent of a *bounded* range R, we designate the number $w=F/L$ as the *weight* of R. Our third postulate is

(C). *Let $R_1, R_2, R_3, \cdots$, be any infinite sequence of bounded ranges in T, each of which is contained in the preceding one, and all of which contain the point $x=(x_1, x_2, \cdots, x_n)$. Let the diameter of R_i be d_i, and let the sequence $d_1, d_2, d_3, \cdots$, converge to zero. Then if w_i is the weight of R_i, the sequence $w_1, w_2, w_3, \cdots$, converges to a non-negative real number, $f(x_1, x_2, \cdots, x_n)$ which depends only upon the point $x=(x_1, x_2, \cdots, x_n)$.*

The function $f(x_1, x_2, \cdots, x_n)$ thus determined from T is defined to be the *probability function* for the distribution, in the population P, of the set $(x_1, x_2, \cdots, x_n)$ of n attributes. It is evidently a density function; it specifies the density of the typical relative frequency at the point $x=(x_1, x_2, \cdots, x_n)$.

We finally adjoin two more postulates which will simplify the remainder of our discussions.

(D). *The probability function* $f(x_1, x_2, \cdots, x_n)$ *is continuous at every point* $x=(x_1, x_2, \cdots, x_n)$ *of the domain* T *of* P.

(E). *If* R *is any range of the domain* T *of* P *(the case in which* R *coincides with* T *not being excluded), the n-fold Lebesgue integral of* $f(x_1, x_2, \cdots, x_n)$ *exists over* R.[1]

While these postulates are not suggested as the best approach to a rigorous treatment of the problems of statistical probability, they serve to outline the general concept of a probability function in a way which leads very directly to the developments of the following sections. They therefore meet the specific requirements of the present paper.

4. *Fundamental Properties of the Probability Function*

Let $f(x_1, x_2, \cdots, x_n)$ be the probability function for a distribution of the set $(x_1, x_2, \cdots, x_n)$ of n attributes in a statistical population P, of domain T.

Any point $a\equiv(a_1, a_2, \cdots, a_n)$ of E_n is a vertex of an elementary rectangular region $(x_i, x_i+\delta x_i)$ consisting of all points of E_n of which the coordinates satisfy the inequalities $a_i<x_i<a_i+\delta x_i$, where the δx_i are positive and $i=1, 2, \cdots, n$. This region will be called a *cell* of E_n, and will be denoted by ΔX.

Let x be any point (except a frontier point) of T, and let the cell ΔX lie entirely within T. From the postulates laid down, we conclude at once that the typical relative frequency in ΔX is given, more and more closely as the n numbers δx_i all approach zero, by the product:

$$f(x_1, x_2, \cdots, x_n)\delta x_1\delta x_2 \cdots \delta x_n. \tag{1}$$

Moreover, if H denotes T, or any subrange of T, it is readily seen that the typical relative frequency in H is given by the n-fold integral, extended over H:

$$\int\int \cdots \int f(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n. \tag{2}$$

Since the typical relative frequency in T itself is manifestly unity, this integral, when extended over all of T, clearly has the value 1.

5. *Elementary Distribution Parameters*

Under the conditions of the first paragraph of the preceding section, let $h(x_1, x_2, \cdots, x_n)$ be any assigned function which is real-valued, single-valued, and continuous in T—and therewith also in P. Let S_q be a typical sample of P, consisting of q individuals. Let $x\equiv(x_1, x_2, \cdots, x_n)$ be any point (except a frontier point) of T, and let ΔX be a cell of E_n with a vertex at x.

If q is supposed to increase indefinitely, and the diameter of ΔX to approach zero (so that all of the δx_i approach zero), we may write, more and more accurately:

---

[1] So far as *bounded* and *closed* ranges in T are concerned, postulate E may be deduced, as a theorem, from postulate D.

$$qf(X)\Delta X = qf(x_1, x_2, \cdots, x_n)\delta x_1 \delta x_2 \cdots \delta x_n$$
$$= \text{the expected actual (not relative) frequency of } S_q \text{ in } \Delta X.$$
$$qh(X)f(X)\Delta X = qh(x_1, x_2, \cdots, x_n)f(x_1, x_2, \cdots, x_n)\delta x_1 \delta x_2 \cdots \delta x_n$$
$$= \text{the expected sum of the values assumed by } h \text{ for individuals of } S_q \text{ in } \Delta X.$$

Taking the n-fold integral of this expression over T, we obtain, more and more nearly as q increases, the expected sum of the values of $h(x_1, x_2, \cdots, x_n)$ for all individuals of S_q. Division by q then yields the typical arithmetic mean, $\bar{h}$, of the values of the function h:

$$\bar{h} = \int\int \cdots \int h(x_1, x_2, \cdots, x_n)f(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n. \tag{3}$$

Two very important arithmetic means calculated from a typical sample S_q are $\bar{x}_i$, the mean of the values of x_i; and $\lambda_t^{(i)}$, the mean of the values of $(x_i - \bar{x}_i)^t$, with $t = 2, 3, 4, \cdots$. These are:

$$\bar{x}_i = (\textstyle\sum x_i)/q, \tag{4}$$

and

$$\lambda_t^{(i)} = \text{the } t\text{th centroidal moment of } x_i = \textstyle\sum (x_i - \bar{x}_i)^t/q, \tag{5}$$

the summations being extended over all q individuals of the sample. In (4) and (5), i is not a summation index. The positive square root[1] of $\lambda_2^{(i)}$ is called the *standard deviation* of x_i, and is denoted by σ_i.

The *typical* values of (4) and (5)—the values which they may be expected to approach more and more closely as q is increased indefinitely—are found from formula (3) upon setting $h(x_1, x_2, \cdots, x_n)$ equal to x_i and to $(x_i - \bar{x}_i)^t$ respectively.[2]

A third basic parameter, denoted by r_{ij}, and called the *coefficient of correlation*[3] of the attributes measured by x_i and x_j is calculated from a typical sample S_q by means of the formula

$$q\sigma_i\sigma_j r_{ij} = \textstyle\sum (x_i - \bar{x}_i)(x_j - \bar{x}_j), \tag{6}$$

the summation being extended over all q individuals of the sample. In (6), i and j are not summation indices. The number of the parameters r_{ij} is n^2; but $r_{ji} = r_{ij}$, and $r_{ii} = 1$.

Since r_{ij} is plainly the arithmetic mean of the values, in S_q, of the function $(x_i - \bar{x}_i)(x_j - \bar{x}_j)/\sigma_i\sigma_j$, the *typical* value of r_{ij} is found from (3) upon substituting

---

[1] Clearly, $\lambda_2^{(i)} \geqq 0$. But, under our postulates, $\lambda_2^{(i)} \neq 0$. See the first footnote under the lemma, Section 7.

[2] The typical value of $\bar{x}_i$ is, of course, used in finding that of $\lambda_t^{(i)}$.

[3] This designation is, in certain respects, misleading. See Section 8.

this function for $h(x_1, x_2, \cdots, x_n)$, and using the typical values of σ_i and σ_j.

Let the variables be changed by a linear transformation of the special type $u_i = a_i x_i + b_i$, with non-vanishing determinant. It is readily seen that the means and centroidal moments of x_i and u_i are related by the formulas $\bar{u}_i = a_i \bar{x}_i + b_i$ and[1] $\lambda_t^{(u_i)} = a_i^t \lambda_t^{(x_i)}$. Moreover, the coefficient of correlation of x_i and x_j is equal to that of u_i and u_j, and is therefore an invariant.

Now subject the variables x_i and x_j, denoted by x and y, to the transformation $u = (x - \bar{x})/\sigma_x$, $v = (y - \bar{y})/\sigma_y$, and leave the remaining variables (if there are any) unchanged. By the remarks just made, $\bar{u} = \bar{v} = 0$, $\sigma_u = \sigma_v = 1$, and $r_{uv} = r_{xy} = r_{ij} = r$. Also, if the calculations are made from a sample S_q, we have $\sum u^2 = q\sigma_u^2 = q = \sum v^2 = q\sigma_v^2$, and $\sum uv = qr\sigma_u\sigma_v = qr$. We may therefore write the equations[2]

$$r = 1 - [\sum(u - v)^2]/2q \text{ and } r = -1 + [\sum(u + v)^2]/2q,$$

from which we conclude that $-1 \leqq r_{ij} \leqq 1$. Obviously, the *typical* r_{ij} must likewise lie on the interval $(-1, 1)$.[3]

Other basic parameters, which depend directly upon the r_{ij}, are: (1) the symmetric determinant $R = |r_{ij}|$, of nth order, having r_{ij} as the element in the ith row and the jth column; and (2) R_{ij}, the cofactor (the minor, with the proper sign) of the element r_{ij} in R.

The number of the parameters R_{ij} is n^2; but, of course, $R_{ji} = R_{ij}$. The typical values of R and of the R_{ij} are calculated from the typical values of the r_{ij}.

The probability function $f(x_1, x_2, \cdots, x_n)$ will be called *canonical* if the typical arithmetic mean value, $\bar{x}_i$, as determined by it, is equal to zero, $(i = 1, 2, \cdots, n)$. Any proposed probability function may be thrown into canonical form by a simple change of origin for the variables; that is, by a transformation of the type $u_i = x_i + b_i$, which leaves centroidal moments of order $\geqq 2$, as well as coefficients of correlation, invariant.

Example 1. Let the probability function be $f(x_1) = f(x) = Cx^2$, where C denotes a constant to be determined; and let the domain T be the open interval $0 < x < 4$, or the closed interval $0 \leqq x \leqq 4$, of E_1.

By Section 4, we have $\int_0^4 f(x)dx = 1$, whence $C = 3/64$. Therefore the typical values of $\bar{x}$ and $\sigma^2 (= \sigma_1^2)$ are:

$$\bar{x} = (3/64) \int_0^4 x^3 dx = 3,$$

$$\sigma^2 = (3/64) \int_0^4 (x - 3)^2 x^2 dx = 3/5.$$

Example 2. Let the probability function be $f(x_1, x_2) = Cx_1^2 e^{-(x_1^2 + x_2^2 + x_1 x_2)}$,

---

[1] $q\lambda_t^{(u_i)} = \sum(u_i - \bar{u}_i)^t$, in accordance with formula (5).

[2] These equations are taken from H. L. Rietz, *Mathematical Statistics*, p. 84.

[3] Under our postulates, the numerical value of the typical r_{ij} is less than 1 if $i \neq j$. See Section 8.

where C denotes a constant; and let the domain be the entire space E_2. From the formula $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_1, x_2)dx_1dx_2=1$, we find[1] that $C=(3\sqrt{3})/(4\pi)$. Using this value of C, and noting[2] that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_if(x_1, x_2)dx_1dx_2=0$ for $i=1, 2$, so that $f(x_1, x_2)$ is canonical in the domain E_2, we calculate the typical value of σ_1^2 as follows:

$$\sigma_1^2 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^4 e^{-(x_1^2+x_2^2+x_1x_2)}dx_1dx_2$$

$$= C\int_{-\infty}^{\infty}\left[x_1^4 e^{-x_1^2}\int_{-\infty}^{\infty} e^{-x_2(x_1+x_2)}dx_2\right]dx_1.$$

Substituting $x_2=t-x_1/2$, $dx_2=dt$, we find that the x_2-integral is equal to

$$e^{x_1^2/4}\int_{-\infty}^{\infty} e^{-t^2}dt = (\sqrt{\pi})e^{x_1^2/4}.$$

Hence

$$\sigma_1^2 = (C\sqrt{\pi})\int_{-\infty}^{\infty} x_1^4 e^{-3x_1^2/4}dx_1.$$

Integration by parts reduces this integral to that of Laplace (see the first footnote under the present example), and thus yields $\sigma_1^2=(8\pi C)/(3\sqrt{3})=2$.

In like manner, we find the typical

$$\sigma_2^2 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 x_2^2 e^{-(x_1^2+x_2^2+x_1x_2)}dx_1dx_2 = C\int_{-\infty}^{\infty} x_1^2 e^{-x_1^2}J dx_1,$$

where

$$J = \int_{-\infty}^{\infty} x_2^2 e^{-x_2(x_1+x_2)}dx_2.$$

---

[1] The n-fold integral ($n=1, 2, 3, \cdots$):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}e^{-Q}dx_1dx_2\cdots dx_n,$$

extended throughout the space E_n, is readily evaluated by elementary methods whenever:

(1) Q is a positive-definite quadratic form in the n variables $x_1, x_2, \cdots, x_n$—see Section 10; and

(2) each of the exponents $m_1, m_2, \cdots, m_n$ is a positive integer or zero. If, in addition, all of the exponents $m_1, m_2, \cdots, m_n$ are equal to zero, the value of the integral is $(\pi^n/D)^{1/2}$, where D is the value of the discriminant (determinant) of Q. Thus, in particular, for $n=1$, we have the important formula of Laplace:

$$\int_{-\infty}^{\infty} e^{-h^2x^2}dx = \sqrt{\pi}/h.$$

[2] The procedure suggested in the determination of σ_1^2 may be followed in evaluating the other integrals of the present example, and of later examples.

The substitution $x_2=t-x_1/2$, $dx_2=dt$, and reduction, yield

$$J = e^{x_1^2/4}[\sqrt{\pi}/2 + x_1^2\sqrt{\pi}/4].$$

Therefore we have

$$\sigma_2{}^2 = (C\sqrt{\pi}/2)\int_{-\infty}^{\infty} x_1^2 e^{-3x_1^2/4}dx_1 + (C\sqrt{\pi}/4)\int_{-\infty}^{\infty} x_1^4 e^{-3x_1^2/4}dx_1 = 1.$$

Finally, we calculate the typical

$$r_{12} = \frac{C}{\sigma_1\sigma_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^3 x_2 e^{-(x_1^2+x_2^2+x_1x_2)}dx_1dx_2 = -1/\sqrt{2}.$$

Example 3. Let the probability function be $f(x_1, x_2, x_3)=Ce^{-Q}$, where C is a constant, and

$$Q = 99x_1^2 + 75x_2^2 + 36x_3^2 - 170x_1x_2 - 116x_1x_3 + 100x_2x_3.$$

Let the domain be the entire space E_3.

The quadratic form Q, with discriminant

$$D = \begin{vmatrix} 99 & -85 & -58 \\ -85 & 75 & 50 \\ -58 & 50 & 36 \end{vmatrix},$$

is positive-definite,[1] and $D=400$. Hence, applying the formula

$$C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-Q}dx_1dx_2dx_3 = 1,$$

and the footnote under Example 2, we find that $C=20/(\pi^{3/2})$. Since $f(x_1, x_2, x_3)$ is obviously canonical in the domain E_3, the typical mean squared-deviations are

$$\sigma_i^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^2 f(x_1, x_2, x_3)dx_1dx_2dx_3 \qquad (i = 1, 2, 3);$$

and we find that $\sigma_1=\sigma_2=\sigma_3=1/2$. Similarly, the typical coefficients of correlation are found from the formulas

$$\sigma_i\sigma_j r_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i x_j f(x_1, x_2, x_3)dx_1dx_2dx_3 \qquad (i, j = 1, 2, 3),$$

in which the typical values of σ_1, σ_2, σ_3 are employed. These formulas yield the typical values $r_{12}=4/5$, $r_{13}=1/2$, $r_{23}=-1/10$; and, of course, $r_{11}=r_{22}=r_{33}=1$. Thus the typical basic determinant is

[1] See Section 10.

$$R = \begin{vmatrix} 1 & 4/5 & 1/2 \\ 4/5 & 1 & -1/10 \\ 1/2 & -1/10 & 1 \end{vmatrix}.$$

6. *Derived Probability Functions, and Array Distributions*[1]

Returning to the conditions stated in the first paragraph of Section 4, we now discuss certain probability functions expressing important aspects of the distribution there defined (for P, and in the domain T) by the function $f(x_1, x_2, \cdots, x_n)$.

Let that subset of P consisting of all individuals whose first k coordinates $(x_1, x_2, \cdots, x_k)$, $k<n$, correspond to points of the k-dimensional cell $a_i<x_i<a_i+\delta x_i$, $i=1, 2, \cdots, k$, with a vertex at the assigned point $(a_1, a_2, \cdots, a_k)$, be called an *array*, $P(12 \cdots k)$, of P at the *pole* $(a_1, a_2, \cdots, a_k)$; and let the greatest of the k numbers δx_i, all of which are supposed positive, be called the *norm* of the array. That subset, $S_q(12 \cdots k)$, of a typical sample S_q, consisting of all those individuals of S_q which belong to an array $P(12 \cdots k)$, will be called a *sample array* of the same pole and norm.

Since the attributes measured by the n coordinates x_i may be rearranged in any desired order, no loss of generality is involved, for the above definitions, in the selection of the *first* k attributes.

It is readily seen that the probability function for the distribution of the set $(x_1, x_2, \cdots, x_k)$ of k attributes, when the other $n-k$ attributes are ignored, is

$$g(x_1, x_2, \cdots, x_k) = \int\int \cdots \int f(x_1, x_2, \cdots, x_n)dx_{k+1}dx_{k+2} \cdots dx_n,$$

the limits of the $(n-k)$-fold integral being determined, in the familiar manner, by the frontier of the range (domain) T. That is to say, the relative frequency, in a typical sample S_q, of those individuals which belong to an array $S_q(12 \cdots k)$, of norm ζ and pole $(x_1, x_2, \cdots, x_k)$, may be expected to be given, more and more nearly as q is increased indefinitely and ζ approaches zero, by $g(x_1, x_2, \cdots, x_k)\ \delta x_1\ \delta x_2, \cdots, \delta x_k$. Thus, if an individual is selected at random from S_q, the *probability* that the individual will belong to the array $S_q(12 \cdots k)$ becomes more and more nearly equal to $g(x_1, x_2, \cdots, x_k)\ \delta x_1\ \delta x_2, \cdots, \delta x_k$.

Now let an individual N be selected at random from the sample array $S_q(12 \cdots k)$ of norm ζ and pole $(x_1, x_2, \cdots, x_k)$. Let the probability function for the distribution, *within the population array* $P(12 \cdots k)$, of the set $(x_{k+1}, x_{k+2}, \cdots, x_n)$ of $n-k$ attributes be, more and more nearly as the norm ζ of that array approaches zero[2], $r(x_1, x_2, \cdots, x_n)$. Then the probability that the last $n-k$ coordinates of N will be interior to specified intervals $(x_i, x_i+\delta x_i)$,

[1] See footnote to the heading of Section 8.

[2] The variables fixing the array have assigned values; hence r is defined, *throughout the array*, as a function of the $n-k$ variables $x_{k+1}, x_{k+2}, \cdots, x_n$.

$i = k+1, k+2, \cdots, n$, is given, more and more nearly as ζ and the $n-k$ numbers δx_i all approach zero, by the product $r(x_1, x_2, \cdots, x_n)\, \delta x_{k+1}\, \delta x_{k+2} \cdots \delta x_n$.

It follows that if an individual is selected at random from S_q, the probability that its coordinates will be interior to the intervals $(x_i, x_i + \delta x_i)$, $i = 1, 2, \cdots, n$, is given, more and more nearly as q is increased indefinitely and all of the n numbers δx_i approach zero, by the product $g(x_1, x_2, \cdots, x_k)\, r(x_1, x_2, \cdots, x_n)\, \delta x_1\, \delta x_2 \cdots \delta x_n$. But, by hypothesis, $f(x_1, x_2, \cdots, x_n)$ is the probability function for the distribution of the set $(x_1, x_2, \cdots, x_n)$ of all n attributes in P; and therefore the last-mentioned probability is also given, more and more nearly, by the product $f(x_1, x_2, \cdots, x_n)\, \delta x_1\, \delta x_2 \cdots \delta x_n$. We thus find that

$$r(x_1, x_2, \cdots, x_n) = \frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_k)}$$

is the distribution function for the set $(x_{k+1}, x_{k+2}, \cdots, x_n)$ of attributes within the array $P(12 \cdots k)$, where $x_1, x_2, \cdots, x_k$ have assigned values.

Example 4. Returning to the distribution defined, in Example 3, by the function $f(x_1, x_2, x_3)$ in the domain E_3, we derive several probability functions:

(1) $g(x_1) =$ the distribution function for x_1 when x_2 and x_3 are ignored

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_2dx_3 = (2/\sqrt{2\pi})e^{-2x_1^2}$$

(2) $r(x_1, x_2, x_3) =$ the distribution function for the pair (x_2, x_3) in an array $P(1)$ where x_1 is assigned

$$= \frac{f(x_1, x_2, x_3)}{g(x_1)}$$

$$= (10\sqrt{2}/\pi)e^{-(97x_1^2+75x_2^2+36x_3^2-170x_1x_2-116x_1x_3+100x_2x_3)}$$

(3) $t(x_2, x_3) =$ the distribution function for (x_2, x_3) when x_1 is ignored

$$= \int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1$$

$$= (20\sqrt{99}/99\pi)e^{-(40/99)(5x_2^2+x_2x_3+5x_3^2)}$$

(4) $u(x_1, x_2, x_3) =$ the distribution function for x_1 in an array $P(23)$, where x_2 and x_3 have assigned values

$$= \frac{f(x_1, x_2, x_3)}{t(x_2, x_3)}$$

$$= (\sqrt{99}/\sqrt{\pi})e^{-Z},$$

with $Z = 99x_1^2 + 7225x_2^2/99 + 3364x_3^2/99 - 170x_1x_2 - 116x_1x_3 + 9860x_2x_3/99$.

(5) $y(x_2, x_3) =$ the distribution function for x_2 when x_3 has an assigned value and x_1 is ignored

$$= \frac{t(x_2, x_3)}{\int_{-\infty}^{\infty} t(x_2, x_3)dx_2} = (\sqrt{\pi}/\sqrt{2})e^{2x_3^2}t(x_2, x_3)$$

$$= (20/\sqrt{198\pi})e^{-(1/99)(200x_2^2+40x_2x_3+2x_3^2)}.$$

Since these are all *probability* functions, their integrals, extended through the respective domains, must be equal to unity. The reader may verify that:

$$\int_{-\infty}^{\infty} g(x_1)dx_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(x_1, x_2, x_3)dx_2dx_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t(x_2, x_3)dx_2dx_3$$
$$= \int_{-\infty}^{\infty} u(x_1, x_2, x_3)dx_1 = \int_{-\infty}^{\infty} y(x_2, x_3)dx_2 = 1.$$

A number of other probability functions may obviously be derived, in the same manner, from the distribution defined in Example 3.

7. *Regression Equations, and Correlation*

The conditions of the first paragraph of Section 4 are again assumed. Thus the probability function for the distribution of x_1 in an array $P(23 \cdots n)$ is

$$H(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) \Big/ \int f(x_1, x_2, \cdots, x_n)dx_1,$$

the limits of the integral being determined, of course, by the frontier of T. The *typical* arithmetic mean, $\bar{x}_1(23 \cdots n)$, of the values assumed by x_1 *within the array* $P(23 \cdots n)$—that is, the expected mean of x_1 in the array $S_q(23 \cdots n)$ of a typical sample S_q with sufficiently large q—is therefore given, more and more nearly as the norm of the array approaches zero, by

$$\bar{x}_1(23 \cdots n) = \int x_1H(x_1, x_2, \cdots, x_n)dx_1$$
$$= \int x_1f(x_1, x_2, \cdots, x_n)dx_1 \Big/ \int f(x_1, x_2, \cdots, x_n)dx_1.$$

In the language of statistics, $x_1 = \bar{x}_1(23 \cdots n) = p(x_2, x_3, \cdots, x_n)$ is called the *regression equation* of x_1 on the set $(x_2, x_3, \cdots, x_n)$ of $n-1$ attributes; and the locus of this equation, in E_n, is called the *regression hypersurface* of x_1 on $(x_2, x_3, \cdots, x_n)$. When p is a first-degree polynomial in its $n-1$ variables, the regression is *linear*, and the regression hypersurface is a regression *hyperplane*.[1]

[1] Hypersurfaces (hyperplanes) are ordinary surfaces (planes) for $n=3$, and curves (straight lines) for $n=2$.

The formula $x_1=\bar{x}_1(23 \cdots n)$ is one of *typical* regression; if q increases indefinitely, the corresponding regression, as found from S_q by elementary methods, may be expected to be given more and more closely by this formula.

The degree of *correlation* of the attribute x_1 with the set of $n-1$ attributes $(x_2, x_3, \cdots, x_n)$ is defined, when $n>2$, to be the degree of concentration of the individuals of P about the regression hypersurface $x_1=\bar{x}_1(23 \cdots n)$. When $n=2$, there is a certain concentration about each of the regression curves $x_1=\bar{x}_1(2)$, $x_2=\bar{x}_2(1)$; and then, by definition, the greater of these two concentrations fixes the correlation of x_1 with x_2.

Let $w(x_2, x_3, \cdots, x_n)$ be any single-valued and continuous function of the $n-1$ arguments indicated. The typical mean squared-deviation of x_1 away from its corresponding values on the hypersurface $x_1=w$ is given by the n-fold integral, extended over T:

$$(7) \qquad \int\int \cdots \int [x_1 - w(x_2, x_3, \cdots, x_n)]^2 f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n.$$

Lemma. This integral assumes its minimum (and smallest possible) value, necessarily positive,[1] when $w=\bar{x}_1(23 \cdots n)=p(x_2, x_3, \cdots, x_n)$.

For, integrating first with respect to x_1, the values of the other variables being fixed, we obtain $K(w)=\int(x_1-w)^2 f dx_1$, where the parameter w, and the limits of integration,[2] are constants depending upon the values of $x_2, x_3, \cdots, x_n$. It is evident that the function $K(w)$ assumes a minimum (which is also the smallest possible) value. This value occurs when $K'(w)=0$; that is, when $\int x_1 f dx_1 = w\int f dx_1$, or $w=\bar{x}_1(23 \cdots n)$. The lemma follows immediately.

In particular, the integral (7) yields: (1) the typical σ_1^2, when w is equal to the constant $\bar{x}_1$ (typical); and (2) the typical mean squared-deviation, $\sigma^2_{1,23\cdots n}$, of x_1 away from the regression hypersurface $x_1=\bar{x}_1(23 \cdots n)$, when $w=\bar{x}_1(23 \cdots n)$.

The positive square root, $\sigma_{1,23\cdots n}$, of $\sigma^2_{1,23\cdots n}$, is a convenient distribution parameter. In fact, the ratio $(\sigma_{1,23\cdots n})/\sigma_1$ furnishes a measure of the degree to which the statistics (individuals of P) are *scattered away from the regression hypersurface* $x_1=\bar{x}_1(23 \cdots n)$. This ratio is >0 and $\leqq 1$, as shown by the lemma.

A commonly employed direct measure of the *concentration of the statistics* about the regression hypersurface $x_1=\bar{x}_1(23 \cdots n)$, and accordingly a *measure of the correlation* existing between the attribute x_1 and the set $(x_2, x_3, \cdots, x_n)$ of $n-1$ attributes, is provided by the non-negative square root, $V_{1,23\cdots n}$, of the non-negative number $V^2_{1,23\cdots n}=1-(\sigma^2_{1,23\cdots n})/\sigma_1^2$. The parameter $V_{1,23\cdots n}$ is called the *correlation ratio* of x_1 on the set of attributes $(x_2, x_3, \cdots, x_n)$. Ob-

[1] f, being a probability function, is non-negative in its domain T, so that (7) is non-negative. And if (7) were equal to zero, T would not be a *range* in E_n, as required by our postulates, but would lie on a hypersurface.

[2] The integral may be extended over more than one interval of the line corresponding to the fixed values of $x_2, x_3, \cdots, x_n$ if this line cuts the frontier of T in more than two points.

viously, $0 \leqq V_{1,23\cdots n} < 1$; and the correlation between x_1 and $(x_2, x_3, \cdots, x_n)$ rises from non-existence toward perfection as $V_{1,23\cdots n}$ increases from 0 toward 1.

Example 5. For the distribution given in Example 3, we find that:

$$\bar{x}_1(23) = \int_{-\infty}^{\infty} x_1 f(x_1, x_2, x_3)dx_1 \Big/ \int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1 = (85x_2 + 58x_3)/99,$$

$$\bar{x}_2(13) = \int_{-\infty}^{\infty} x_2 f dx_2 \Big/ \int_{-\infty}^{\infty} f dx_2 = (17x_1 - 10x_3)/15,$$

$$\bar{x}_3(12) = \int_{-\infty}^{\infty} x_3 f dx_3 \Big/ \int_{-\infty}^{\infty} f dx_3 = (29x_1 - 25x_2)/18.$$

Hence the regression of each attribute on the other two is linear, and the regression planes are $x_1 = (85x_2 + 58x_3)/99$, $x_2 = (17x_1 - 10x_3)/15$, and $x_3 = (29x_1 - 25x_2)/18$. We next find that:

$$\sigma_{1,23}^2 = \int\int\int [x_1 - \bar{x}_1(23)]^2 f(x_1, x_2, x_3)dx_1 dx_2 dx_3 = 1/198 = 0.00505,$$

$$\sigma_{2,13}^2 = \int\int\int [x_1 - \bar{x}_2(13)]^2 f dx_1 dx_2 dx_3 = 1/150 = 0.00667,$$

$$\sigma_{3,12}^2 = \int\int\int [x_1 - \bar{x}_3(12)]^2 f dx_1 dx_2 dx_3 = 1/72 = 0.01389,$$

the integration being extended from $-\infty$ to $+\infty$ with respect to each variable. Very high correlation exists between each attribute and the remaining pair of attributes. In fact, recalling (from Example 3) that $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$, we find that: $V_{1,23} = 0.9898$, $V_{2,13} = 0.9864$, $V_{3,12} = 0.9718$.

Example 6. For the distribution given in Example 2, only one of the regression functions $\bar{x}_1(2)$, $\bar{x}_2(1)$ is linear. It will be found, in fact, that the regression equation of x_2 on x_1 is:

$$x_2 = \bar{x}_2(1) = \int_{-\infty}^{\infty} x_1^2 x_2 e^{-(x_1^2 + x_2^2 + x_1 x_2)} dx_2 \Big/ \int_{-\infty}^{\infty} x_1^2 e^{-(x_1^2 + x_2^2 + x_1 x_2)} dx_2$$

$$= \int_{-\infty}^{\infty} x_2 e^{-x_2(x_1 + x_2)} dx_2 \Big/ \int_{-\infty}^{\infty} e^{-x_2(x_1 + x_2)} dx_2 = - x_1/2 ;$$

while that of x_1 on x_2 is:

$$x_1 = \bar{x}_1(2) = \int_{-\infty}^{\infty} x_1^3 e^{-(x_1^2 + x_2^2 + x_1 x_2)} dx_1 \Big/ \int_{-\infty}^{\infty} x_1^2 e^{-(x_1 + x_2^2 + x_1 x_2)} dx_1$$

$$= - (6x_2 + x_2^3)/(4 + 2x_2^2).$$

The calculation of the typical values of $\sigma_{1,2}$, $\sigma_{2,1}$, $V_{1,2}$, $V_{2,1}$ is left to the reader.

8. *Best-Fitting Hyperplanes, and Coefficients of Multiple Correlation*[1]

When appreciable correlation exists between the attribute x_1 and the set of $n-1$ attributes $(x_2, x_3, \cdots, x_n)$, it may often be exhibited, and even measured with much accuracy, by a parameter which is more easily calculated (especially from a sample S_q) than the correlation ratio $V_{1,23\cdots n}$.

The conditions of the first paragraph of Section 4 are again assumed. It is also assumed, merely for convenience and with no loss of generality, that $f(x_1, x_2, \cdots, x_n)$ is in *canonical* form, so that the typical $\bar{x}_i$ equals zero, $(i=1, 2, \cdots, n)$. Finally, it is assumed that R_{11}, of the basic determinant R of the distribution, is not equal to zero.

Then, in the class of all *linear* functions of $(x_2, x_3, \cdots, x_n)$, there exists one and only one,

$$b_{12}x_2 + b_{13}x_3 + \cdots + b_{1n}x_n + c \tag{8}$$

which, when substituted for $w(x_2, x_3, \cdots, x_n)$ in (7), yields, for that integral, a smaller value than any other linear w. The coefficients of this unique linear function are

$$c = 0, \qquad b_{1j} = -(\sigma_1 R_{1j})/(\sigma_j R_{11}) ; \tag{9}$$

and

$$x_1 = -(\sigma_1/R_{11}) \sum_{j=2}^{n}(R_{1j}x_j/\sigma_j) \tag{10}$$

will be called the "best-fitting" x_1-hyperplane[2] on $(x_2, x_3, \cdots, x_n)$.

In fact, let the n-fold integral

$$L(b_{12}, b_{13}, \cdots, b_{1n}, c) = \int\int \cdots \int \left(x_1 - c - \sum_{j=2}^{n} b_{1j}x_j\right)^2 f dx_1 dx_2 \cdots dx_n,$$

in which $f=f(x_1, x_2, \cdots, x_n)$, be extended over T. Equating the partial derivative dL/dc to zero, and indicating n-fold integration by one symbol, with differentials omitted, we obtain:[3]

$$\int (x_1 - b_{12}x_2 - b_{13}x_3 - \cdots - b_{1n}x_n - c)f = 0,$$

or $\bar{x}_1 - b_{12}\bar{x}_2 - b_{13}\bar{x}_3 - \cdots - b_{1n}\bar{x}_n = c = 0$.

---

[1] Here, as in other sections, when a particular attribute, or set of attributes, is distinguished for special analysis, this is done merely for notational convenience, and no sacrifice of generality is implied.

[2] "Best fit" is here defined in the sense of the principle of least squares.

[3] Of course, $\int f = 1$; and, since f is canonical, we have the further relations:

$$\bar{x}_j = \int x_j f = 0, \quad \sigma_i \sigma_j r_{ij} = \int x_i x_j f \qquad (i, j = 1, 2, \cdots, n).$$

Setting $c=0$, we now write the $n-1$ equations $dL/db_{1j}=0$:

$$\int (x_1 - b_{12}x_2 - b_{13}x_3 - \cdots - b_{1n}x_n)x_jf = 0.$$

Cancelling the factor σ_j, we thus have:

$$b_{12}r_{2j}\sigma_2 + b_{13}r_{3j}\sigma_3 + \cdots + b_{1n}r_{nj}\sigma_n = r_{1j}\sigma_1 \qquad (j = 2, 3, \cdots, n). \tag{11}$$

It is recognized at once that the system (11) of linear equations uniquely specifies the numbers b_{1j}, and with the values given in (9), when the determinant of the coefficients of these numbers does not vanish.[1] But that determinant is plainly the product of $(\sigma_2\sigma_3 \cdots \sigma_n)$ and R_{11}, which was assumed to be different from zero.

The typical dispersion of the statistics away from the hyperplane (10) is the positive square root, $\bar{\sigma}_{1,23\ldots n}$, of the n-fold integral over T:

$$\begin{aligned} \bar{\sigma}^2_{1,23\cdots n} &= \int \left[x_1 + (\sigma_1/R_{11}) \sum_{j=2}^{n} (R_{1j}x_j/\sigma_j) \right]^2 f \\ &= (\sigma_1^2/R_{11}^2) \int \left[\sum_{j=1}^{n} (R_{1j}x_j/\sigma_j) \right]^2 f \end{aligned}$$

which is merely (7) with w set equal to the right-hand member of (10).[2]
We may write:

$$\begin{aligned} \bar{\sigma}^2_{1,23\cdots n} &= (\sigma_1^2/R_{11}^2)\left[\sum_{j=1}^{n} R_{1j}^2 + 2\sum_{j=2}^{n} R_{11}R_{1j}r_{1j} + 2\sum_{j=3}^{n} R_{12}R_{1j}r_{2j} \right. \\ &\qquad \left. + \cdots + 2R_{1,n-1}R_{1n}r_{n-1,n} \right] \\ &= (\sigma_1^2/R_{11}^2) \sum_{j=1}^{n} R_{1j}\left(R_{1j} + \sum_{i=1}^{n}{}' R_{1i}r_{ji} \right), \end{aligned} \tag{12}$$

the (primed) i-summation omitting the term with $i=j$. By an elementary property of determinants, the parenthesis in (12) is equal to R or to 0 according as $j=1$ or $j\neq 1$. Hence $\bar{\sigma}^2_{1,23\cdots n}=\sigma_1^2(R/R_{11})$, and

$$\bar{\sigma}_{1,23\cdots n} = \sigma_1(R/R_{11})^{1/2}. \tag{13}$$

By the lemma, and the footnote just made:

$$\sigma_{1,23\cdots n} \leqq \bar{\sigma}_{1,23\cdots n} \leqq \sigma_1. \tag{14}$$

[1] See Cramer's Rule, in elementary algebra, for the solution of a system of linear equations with non-vanishing determinant.

[2] By the determination of the minimizing linear function (10), $\bar{\sigma}^2_{1,23\cdots n} \leqq \sigma_1^2$. Moreover, since σ_1^2 is given by (7) with w equal to $\bar{x}_1(=0)$, the inequality sign prevails unless $R_{1j}=0$ for $j=2, 3, \cdots, n$.

The non-negative square root, $r_{1,23\cdots n}$, of the non-negative number

$$r_{1,23\cdots n}^2 = 1 - (\bar{\sigma}_{1,23\cdots n}^2)/\sigma_1^2 \tag{15}$$

serves as a direct measure of the degree of concentration of the statistics about the best-fitting hyperplane (10), this concentration rising from non-existence toward perfection as $r_{1,23\cdots n}$ increases[1] from 0 toward 1.

When $n=2$, we have, by (13), $\bar{\sigma}_{1,2}^2=\sigma_1^2(1-r_{12}^2)$, whence

$$r_{12}^2 = 1 - (\bar{\sigma}_{1,2}^2/\sigma_1^2) = r_{1,2}^2 \tag{15a}$$

so that the (non-negative) parameter $r_{1,2}$ is numerically equal to r_{12}. When $n>2$, the parameter $r_{1,23\cdots n}$ is called the *coefficient of multiple correlation* of x_1 with the set $(x_2, x_3, \cdots, x_n)$ of $n-1$ attributes. That the designation is a little misleading appears in the paragraph above, where the exact nature of this parameter is stated. It is evident that very high correlation might exist—very high concentration of the statistics about the regression hypersurface $x_1=\bar{x}_1(23\cdots n)$—with $r_{1,23\cdots n}$ very small; but if $r_{1,23\cdots n}$ is large (nearly equal to 1), we are assured of high correlation between x_1 and $(x_2, x_3, \cdots, x_n)$. The suitability of $r_{1,23\cdots n}$ as a measure of this correlation is fully exhibited in the formula

$$r_{1,23\cdots n} \leqq V_{1,23\cdots n} \tag{16}$$

which follows immediately from (14).

By the lemma, Section 7, the regression hypersurface $x_1=\bar{x}_1(23\cdots n)$ is the "best-fitting" surface of the type $x_1=w(x_2, x_3, \cdots, x_n)$, where w is a single-valued and continuous function in T. It therefore coincides, when the regression of x_1 on $(x_2, x_3, \cdots, x_n)$ is *linear*, with the hyperplane (10); and in that case, of course, $\bar{\sigma}_{1,23\cdots n}=\sigma_{1,23\cdots n}$, $r_{1,23\cdots n}=V_{1,23\cdots n}$.

Example 7. For the distribution of Example 3, the regression of each attribute on the remaining pair of attributes was found to be linear (see Example 5). These regressions may therefore be checked by (10), with $n=3$, and by the two corresponding equations

$$x_2 = -(\sigma_2/R_{22})\sum_{j=1,3}(R_{2j}x_j/\sigma_j), \quad x_3 = -(\sigma_3/R_{33})\sum_{j=1,2}(R_{3j}x_j/\sigma_j),$$

if we set

$$\sigma_1=\sigma_2=\sigma_3=1/2, \qquad R=0.02, \qquad R_{11}=0.99, \qquad R_{22}=0.75,$$
$$R_{33}=0.36, \quad R_{12}=R_{21}=-0.85, \quad R_{13}=R_{31}=-0.58, \quad R_{23}=R_{32}=0.50.$$

Similarly, the typical values of $\sigma_{1,23}^2$, $\sigma_{2,13}^2$, $\sigma_{3,12}^2$, as obtained in Example 5, may be checked by (13) and the two corresponding formulas: $\bar{\sigma}_{2,13}^2=\sigma_2^2(R/R_{22})$, $\bar{\sigma}_{3,12}^2=\sigma_3^2(R/R_{33})$. The coefficients of multiple correlation are equal, in this case,

[1] It is obvious that $0\leqq r_{1,23\cdots n}<1$.

to the correlation ratios calculated in Example 5. For the distribution of (x_2, x_3), with x_1 ignored, the function $t(x_2, x_3)$, given in Example 4, shows that there is linear regression of x_2 on x_3; in fact, we find that

$$\int_{-\infty}^{\infty} x_2 t(x_2, x_3)dx_2 \Big/ \int_{-\infty}^{\infty} t(x_2, x_3)dx_2 = -x_3/10.$$

By formula (10), the regression equation $x_2 = -x_3/10$ ought to be the same as $x_2 = -(\sigma_2/Q_{22})(Q_{23}x_3/\sigma_3)$, where

$$Q = \begin{vmatrix} r_{12} & r_{23} \\ r_{32} & r_{33} \end{vmatrix} = \begin{vmatrix} 1 & -1/10 \\ -1/10 & 1 \end{vmatrix}$$

is the basic determinant of the distribution considered, and Q_{ij} denotes the cofactor of the element r_{ij} in Q—so that $Q_{22} = r_{33} = 1$, $Q_{23} = -r_{32} = 1/10$. Such is found to be the case. In the same manner, if i and j denote any two of the three indices 1, 2, 3, in either order, we find that the regression of x_i on x_j, with the third attribute ignored, is linear; and that its equation, which reduces to $x_i = (r_{ij}\sigma_i x_j)/\sigma_j$, is given by the formula $x_i = -(\sigma_i/Q_{ii})\ (Q_{ij}x_j/\sigma_j)$, where

$$Q = \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix}, Q_{ii} = r_{jj} = 1, Q_{ij} = -r_{ji} = -r_{ij}.$$

Example 8. For the distribution discussed in Examples 2 and 6, we found that the regression of x_2 on x_1 was linear. Hence the regression equation $x_2 = \bar{x}_2(1) = -x_1/2$ should be the same as that of the best-fitting x_2-line on x_1, given by formula (10): $x_2 = -\sigma_2 R_{21}x_1/\sigma_1 R_{22}$. Recalling that $\sigma_1^2 = 2$, $\sigma_2^2 = 1$, $r_{12} = -1/\sqrt{2}$,

$$R = \begin{vmatrix} 1 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1 \end{vmatrix}, R_{21} = 1/\sqrt{2}, R_{22} = 1,$$

we obtain the desired check.

The best-fitting x_1-line on x_2 is $x_1 = -\sigma_1 R_{12}x_2/\sigma_2 R_{11} = -x_2$. It is not, however, the regression curve (see Example 6) of x_1 on x_2; and therefore $r_{1,2}$ (=the numerical value of $r_{12} = 1/\sqrt{2} = 0.707$) does not measure the full extent of the correlation of x_1 with x_2. In this distribution, we have $V_{2,1} = r_{2,1} = r_{1,2} = 1/\sqrt{2}$; but $V_{1,2} > r_{1,2}$, and $V_{1,2}$ may perhaps be much more nearly equal to 1 than is indicated by the value of $r_{1,2}$.

9. *Array Parameters, and Coefficients of Partial Correlation*[1]

Under the conditions of the first paragraph of Section 4, we have $g(x_1, x_2, \cdots, x_k) = \int\int \cdots \int f\ dx_{k+1}dx_{k+2} \cdots dx_n$ as the distribution function for $(x_1, x_2, \cdots, x_k)$ with $x_{k+1}, x_{k+2}, \cdots, x_n$ ignored; and $U(x_1, x_2, \cdots, x_n) = f/g$ as that for $(x_{k+1}, x_{k+2}, \cdots, x_n)$ within the array $P(12 \cdots k)$, where $x_1, x_2, \cdots, x_k$ have fixed values. It is understood that k denotes any integer $\geqq 1$ and $< n$.

[1] See the footnote to the heading of Section 8.

Employing $U(x_1, x_2, \cdots, x_n)$, in which the values of $x_1, x_2, \cdots, x_k$ are assigned, just as we have hitherto employed $f(x_1, x_2, \cdots, x_n)$, we may easily define, *for the distribution of* $(x_{k+1}, x_{k+2}, \cdots, x_n)$ *within the array* $P(12 \cdots k)$, all parameters, regression equations, best-fitting hyperplanes, etc., previously set up for the distribution of $(x_1, x_2, \cdots, x_n)$ in P. We have only to observe that if $h(x_1, x_2, \cdots, x_n)$ is any single-valued and continuous function in T, the typical arithmetic mean of the values assumed by h within $P(12 \cdots k)$ is given by the $(n-k)$-fold integral

$$\int\int \cdots \int h(x_1, x_2, \cdots, x_n)U(x_1, x_2, \cdots, x_n)dx_{k+1}dx_{k+2} \cdots dx_n, \tag{17}$$

and is thus a function[1] of the assigned values of $x_1, x_2, \cdots, x_k$.

Let a typical sample S_q be partitioned into a (finite) set of arrays of the type $S_q(12 \cdots k)$, and let the arithmetic mean of h be calculated for the individuals of each sample array. Then let the arithmetic mean of these array-means be determined for the entire partition of S_q. The *typical* value of the resulting parameter—the value which it may be expected to approach more and more closely as q increases indefinitely, and the norms of the partition arrays in S_q all approach zero—is clearly the familiar

$$\bar{h} = \int\int \cdots \int h(x_1, x_2, \cdots, x_n)f(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n$$

of Section 5. Analytical verification is immediate. The typical arithmetic mean $f(h; 12 \cdots k)$, of the typical arithmetic array-mean of h given in (17) is expressed by a k-fold integral of an $(n-k)$-fold integral, namely by

$$\int\int \cdots \int \left[\int\int \cdots \int hU dx_{k+1}dx_{k+2} \cdots dx_n\right] g dx_1 dx_2 \cdots dx_k ;$$

and, recalling that g is independent of the variables $x_{k+1}, x_{k+2}, \cdots, x_n$, while $U = f/g$, we see that

$$f(h ; 12 \cdots k) = \int\int \cdots \int hf dx_1 dx_2 \cdots dx_n. \tag{18}$$

Letting i, j denote integers of the set $k+1, k+2, \cdots, n$, we signalize three especially interesting and important cases:

(A) $\bar{x}_i(12 \cdots k)$ = the typical arithmetic array-mean found, from (17), for $h = x_i$

(B) $\sigma_i^2(12 \cdots k)$ = that for $h = [x_i - \bar{x}_i(12 \cdots k)]^2$

(C) $r_{ij}(12 \cdots k)$ = that for $h = \dfrac{[x_i - \bar{x}_i(12 \cdots k)][x_j - \bar{x}_j(12 \cdots k)]}{\sigma_i(12 \cdots k)\sigma_j(12 \cdots k)}$,

[1] See below for certain cases in which these array means are known to be independent of the values of the k variables fixing the array.

where $\sigma_i(12 \cdots k)$ is the positive square root of the array parameter (B).

Evidently, (B) is the typical square of the standard deviation of x_i within the array $P(12 \cdots k)$; and (C) is the typical coefficient of correlation of x_i and x_j within that array. By (18), the typical arithmetic means of these two typical array parameters are:

$$(19)\quad \sigma^2_{i,12\cdots k} = \int\int\cdots\int [x_i - \bar{x}_i(12 \cdots k)]^2 f dx_1 dx_2 \cdots dx_n,$$

$$(20)\quad r_{ij,12\cdots k} = \int\int\cdots\int \frac{[x_i - \bar{x}_i(12 \cdots k)][x_j - \bar{x}_j(12 \cdots k)]}{\sigma_i(12 \cdots k)\sigma_j(12 \cdots k)} f dx_1 dx_2 \cdots dx_n.$$

The typical arithmetic mean of the typical $\bar{x}_i(12 \cdots k)$ is plainly $\bar{x}_i$.

The parameter $r_{ij,12\cdots k}$ is the typical *coefficient of partial correlation* of x_i and x_j, in the distribution of $(x_1, x_2, \cdots, x_n)$, when the measures of $x_1, x_2, \cdots, x_k$ are assigned, and $n-k-2$ measures are ignored. This parameter is certainly of high statistical significance in *normal* distributions of $(x_1, x_2, \cdots, x_n)$, defined in Section 10, and in distributions approximating normality. Normal distributions possess several outstanding properties, among which are the following two:

(1) If $1 \leqq k \leqq n-1$, the regression of any attribute on any set of k other attributes, with $n-k-1$ attributes ignored, is linear.

(2) If $1 \leqq k \leqq n-1$, the typical array parameter $\sigma_i(12 \cdots k)$ is a constant, and is therefore independent of the values of $x_1, x_2, \cdots, x_k$ fixing the array; and if $1 \leqq k \leqq n-2$, the typical array parameter $r_{ij}(12 \cdots k)$ is likewise a constant. Hence the typical $\sigma_i(12 \cdots k)$ is equal to the positive square root, $\sigma_{i,12\cdots k}$, of the typical $\sigma^2_{i,12\cdots k}$; and the typical $r_{ij}(12 \cdots k)$ and $r_{ij,12\cdots k}$ are equal.

For any distribution, whether normal or not, with the properties (1) and (2), the typical $\sigma_{i,12\cdots k}$ and $r_{ij,12\cdots k}$, and therewith also the typical $\sigma_i(12 \cdots k)$ and $r_{ij}(12 \cdots k)$, may be calculated directly from the elementary parameters σ_i and R. We will now develop these important formulas.

Suppose for convenience, in accordance with the footnote to the heading of Section 8, that we seek the typical $r_{12,ab\cdots c}$, the coefficient of partial correlation of x_1 and x_2, for assigned values of k other variables of which the indices, arranged in natural order, are $a, b, \cdots, c$; and that the indices of the $n-k-2$ ignored variables, arranged in natural order, are $s, t, \cdots, u$. We must first find the typical $\sigma_{1,ab\cdots c}$ and $\sigma_{2,ab\cdots c}$.

Let $r_{\alpha\epsilon}, r_{\beta\zeta}, \cdots, r_{\delta\theta}$ be any $m<n$ elements of the basic determinant R for the distribution of $(x_1, x_2, \cdots, x_n)$; and let no two of these m elements stand in the same row or in the same column of R. Deleting, from R, the m rows and m columns in which these elements stand, we denote by $R_{\alpha\beta\cdots\delta,\epsilon\zeta\cdots\theta}$ the resulting determinant,[1] which is of order $n-m$. Then the distribution, D, of $(x_1, x_a, x_b, \cdots, x_c)$, with $x_2, x_s, x_t, \cdots, x_u$ ignored, has the basic determinant

[1] In particular, R_{ij} equals $\pm R_{i,j}$, according as $i+j$ is even or odd.

$Q=R_{2st\cdots u,2st\cdots u}$, of which the order is $k+1$. For if i and j are integers of the set $1, a, b, \cdots, c$, we see that r_{ij} has the same typical value[1] whether defined by the probability function f in the full distribution, F, of $(x_1, x_2, \cdots, x_n)$ or by the probability function $G(x_1, x_a, x_b, \cdots, x_c)=\int\int\int\cdots\int f\, dx_2dx_sdx_t\cdots dx_u$ in the distribution D.

As stated above, F is assumed to have the properties (1) and (2). It is also assumed, for convenience, that f is canonical in F—so that G is obviously canonical in D.

The arguments of Section 8, applied to F with its distribution function f and its determinant R, may now be applied verbatim to D with its distribution function G and its determinant Q. It is thus found, in D, that the best-fitting linear formula for x_1 on $(x_a, x_b, \cdots, x_c)$ is[2]

$$x_1 = -(\sigma_1/Q_{11})\sum(Q_{1j}x_j/\sigma_j), \tag{21}$$

with j assuming, in the summation, the k values $a, b, \cdots, c$. By reason of the assumed linear regression, (21) is also the regression equation of x_1 on $(x_a, x_b, \cdots, x_c)$. The typical mean squared-deviation of x_1 away from the values given by (21) is the $(k+1)$-fold integral:

$$\bar{\sigma}^2_{1,ab\cdots c} = \int\int\int\cdots\int\Big[x_1+(\sigma_1/Q_{11})\sum(Q_{1j}x_j/\sigma_j)\Big]^2 G\,dx_1dx_adx_b\cdots dx_c,$$

which may be expressed as the n-fold integral (over T):

$$\bar{\sigma}^2_{1,ab\cdots c}=\int\int\cdots\int\Big[x_1+(\sigma_1/Q_{11})\sum(Q_{1j}x_j/\sigma_j)\Big]^2 f\,dx_1dx_2\cdots dx_n. \tag{22}$$

But the linear regression identifies (22) with the integral (over T):

$$\int\int\cdots\int[x_1-\bar{x}_1(ab\cdots c)]^2 f\,dx_1dx_2\cdots dx_n,$$

and consequently, by the general formula (19), with the typical $\sigma^2_{1,ab\cdots c}$.

Hence we have only to calculate $\bar{\sigma}^2_{1,ab\cdots c}$ by (22), proceeding exactly as in Section 8 (where $k=n-1$), except that R is replaced by Q. We find:

$$\bar{\sigma}_{1,ab\cdots c} = \sigma_1(Q/Q_{11})^{1/2} = \sigma_1(R_{2st\cdots u,2st\cdots u}/R_{12st\cdots u,12st\cdots u})^{1/2}. \tag{23}$$

In like manner, the distribution of $(x_2, x_a, x_b, \cdots, x_c)$, with disregard of $x_1, x_s, x_t, \cdots, x_u$, has the determinant $W=R_{1st\cdots u,1st\cdots u}$, and the probability function $\int\int\int\cdots\int f\,dx_1dx_sdx_t\cdots dx_u$; it yields the regression equation:[3]

$$x_2 = \bar{x}_2(ab\cdots c) = -(\sigma_2/W_{22})\sum(W_{2j}x_j/\sigma_j), \tag{24}$$

[1] Similarly, σ_i has the same value in both distributions.

[2] Q_{ij} is the cofactor of the element r_{ij} in Q, $(i, j=1, a, b, \cdots, c)$; but r_{ij} does not necessarily stand in the i-th row and j-th column of Q.

[3] W_{ij} is the cofactor of the element r_{ij} in W, $(i, j=2, a, b, \cdots, c)$. In the summation, j assumes the k values $a, b, \cdots, c$.

and the typical value:

$$\bar{\sigma}_{2,ab\cdots c} = \sigma_2(R_{1st\cdots u,1st\cdots u}/R_{12st\cdots u,12st\cdots u})^{1/2}. \tag{25}$$

From the general formula (20), we finally obtain:

$$\sigma_{1,ab\cdots c}\sigma_{2,ab\cdots c}r_{12,ab\cdots c}$$

$$= \int [x_1 + (\sigma_1/Q_{11}) \sum (Q_{1j}x_j/\sigma_j)] \; [x_2 + (\sigma_2/W_{22}) \sum (W_{2j}x_j/\sigma_j)]f$$

where an n-fold integral, extended over T, is denoted on the right by a single symbol, and differentials are omitted.[1] In the summations, j assumes the k values $a, b, \cdots, c$.

The simplification of this result is straightforward. It is necessary only to recall that, since f is canonical, the expressions $x_i^2 f$ and x_ix_jf yield respectively, when integrated over T, σ_i^2 and $\sigma_i\sigma_jr_{ij}$; that $Q=R_{2st\cdots u,2st\cdots u}$ and $W=R_{1st\cdots u,\,1st\cdots u}$; and that $\sigma_{1,ab\cdots c}$ and $\sigma_{2,ab\cdots c}$ are given by (23), (25). We find the formula:

$$r_{12,ab\cdots c} = \frac{R_{1st\cdots u,2st\cdots u}}{[(R_{1st\cdots u,1st\cdots u})(R_{2st\cdots u,2st\cdots u})]^{1/2}} \tag{26}$$

which becomes, when $k=n-2$, simply

$$r_{12,34\cdots n} = \frac{R_{1,2}}{(R_{1,1}R_{2,2})^{1/2}} = \frac{-R_{12}}{(R_{11}R_{22})^{1/2}}.$$

In the derivation of (26), it was assumed that F possessed properties (1) and (2). It is evidently sufficient, however, that, *for the particular set of k variables $x_a, x_b, \cdots, x_c$ chosen*, the regressions of x_1 and x_2 on this set of variables be linear, and that the parameters $\sigma_1(ab\cdots c)$ and $\sigma_2(ab\cdots c)$ be constants. In this case, the coefficient of correlation of x_1 and x_2 has the same value, that given by (26), in every array $P(ab\cdots c)$ of these k variables. If, in addition, the regression of x_1 on x_2, and that of x_2 on x_1, are linear when the other $n-2$ variables are ignored—and thus linear also within the array $P(ab\cdots c)$—formula (26) yields the actual *correlation ratio* of x_1 on x_2, as well as that of x_2 on x_1, for every array $P(ab\cdots c)$. In such cases, the coefficient of partial correlation $r_{12,ab\cdots c}$ serves as a measure of the correlation still existing between the attributes x_1 and x_2 when the possible effect of variation of the k attributes $x_a, x_b, \cdots, x_c$ has been eliminated.

Obviously,

$$r_{12,ab\cdots c} = r_{21,ab\cdots c}. \tag{27}$$

Whether or not the regression of x_1 on the set $(x_a, x_b, \cdots, x_c)$ of k variables

[1] The assumption is here made that the parameters $\sigma_1(ab\cdots c)$, $\sigma_2(ab\cdots c)$ are *constants*, so that they are equal, respectively, to the parameters $\sigma_{1,ab\cdots c}$, $\sigma_{2,ab\cdots c}$.

is linear, the reader will understand the significance of the parameters

(28) $$V^2_{1,ab\cdots c} = 1 - (\sigma^2_{1,ab\cdots c})/\sigma^2_1,$$

and

(29) $$r^2_{1,ab\cdots c} = 1 - (\bar{\sigma}^2_{1,ab\cdots c})/\sigma^2_1,$$

and of their non-negative square roots $V_{1,ab\cdots c}$, $r_{1,ab\cdots c}$. These parameters were discussed in Sections 7 and 8, for the case $k=n-1$. When the regression mentioned is linear, we have, of course, $r_{1,ab\cdots c}=V_{1,ab\cdots c}$; and, in any case,

(30) $$r_{1,ab\cdots c} \leqq V_{1,ab\cdots c}.$$

Example 9. Returning to the distribution analyzed in Examples 3, 4, 5, 7, we find, in accordance with (4) of Example 4, and (1) of Example 5, that

$$\sigma_1^2(23) = \int_{-\infty}^{\infty}[x_1 - \bar{x}_1(23)]^2 u(x_1, x_2, x_3)dx_1 = 1/198.$$

This parameter has here a value independent of x_2 and x_3, and is therefore equal to the $\sigma_{1,23}{}^2$ calculated in Example 5. Similarly, it will be found that $\sigma_2{}^2(13)=\sigma_{2,13}{}^2$ and $\sigma_3{}^2(12)=\sigma_{3,12}{}^2$.

The typical arithmetic means of x_2 and x_3 in an array $P(1)$ are (see Example 7):

$$\bar{x}_2(1) = -(\sigma_2/Q_{22})(Q_{21}x_1/\sigma_1),$$

with

$$Q = \begin{vmatrix} r_{22} & r_{21} \\ r_{12} & r_{11} \end{vmatrix},$$

and

$$\bar{x}_3(1) = -(\sigma_3/W_{33})(W_{31}x_1/\sigma_1),$$

with

$$W = \begin{vmatrix} r_{33} & r_{31} \\ r_{13} & r_{11} \end{vmatrix}.$$

Thus we obtain the two typical values:

$$\sigma_2^2(1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_2 - \bar{x}_2(1)]^2 r(x_1, x_2, x_3)dx_2dx_3$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_2 - 4x_1/5]^2 r\, dx_2dx_3,$$

$$\sigma_3^2(1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_3 - \bar{x}_3(1)]^2 r(x_1, x_2, x_3)dx_2dx_3$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_3 - x_1/2]^2 r\, dx_2dx_3,$$

where $r(x_1, x_2, x_3)$ is the function (2) of Example 4. These integrals yield the values $\sigma_2^2(1) = 9/100$ and $\sigma_3^2(1) = 3/16$, which are independent of the value of x_1 fixing the array $P(1)$. Hence $\sigma_{2,1}^2 = 9/100$ and $\sigma_{3,1}^2 = 3/16$.

Further, the typical $r_{23}(1)$ is given by the formula:

$$\sigma_2(1)\sigma_3(1)r_{23}(1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_2 - 4x_1/5][x_3 - x_1/2]r(x_1, x_2, x_3)dx_2dx_3.$$

Substituting $\sigma_2(1) = 3/10$, $\sigma_3(1) = \sqrt{3}/4$, and evaluating the integral, we find that $r_{23}(1) = -5\sqrt{3}/9$, a result which is independent of x_1. Thus $r_{23,1} = -5\sqrt{3}/9 = -0.9623$, and a much stronger correlation than that indicated by $r_{23} = -1/10$ (see Example 3) is shown to exist between x_2 and x_3 when the influence of x_1 has been eliminated. Since conditions are here satisfied for the application of the formula[1]

$$r_{23,1} = R_{2,3}/(R_{2,2}R_{3,3})^{1/2} = -R_{23}/(R_{22}R_{33})^{1/2},$$

with

$$R = \begin{vmatrix} r_{22} & r_{23} & r_{21} \\ r_{32} & r_{33} & r_{31} \\ r_{12} & r_{13} & r_{11} \end{vmatrix},$$

the reader may check the value of $r_{23,1}$.

In the same manner, it may be shown that $r_{13}(2) = r_{13,2} = 29/3\sqrt{97} = 0.972$, and $r_{12}(3) = r_{12,3} = 17/\sqrt{297} = 0.986$.

10. *Normal Distribution*

The general quadratic n-ary form is

$$A = \sum_{i,j=1}^{n} a_{ij}x_ix_j,$$

with $a_{ji} = a_{ij}$; and it may be noted that $2a_{ij} = \partial^2 A/\partial x_i\partial x_j$. Thus the unary form is $a_{11}x_1^2$, the binary is $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$, and the ternary is $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$. The *discriminant* of the form A is the determinant of n-th order:

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[1] The arrangement of R corresponds to the order of the subscripts in $r_{23,1}$.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

For $h=1, 2, \cdots, n$, let $D(h)$ denote any determinant, of order h, obtained from D by striking out $n-h$ rows, and $n-h$ columns with the same indices. When regarded as a determinant of first order, each element a_{jj} of the principal diagonal of D is a $D(1)$; and D is itself the only $D(n)$. There are $n!/h!(n-h)!$ determinants of the type $D(h)$, $h=1, 2, \cdots, n$; and each is called a "principal minor," of order h, in D. In particular, the principal minors of order h standing in the upper-left-hand and lower-right-hand corners of D will be denoted by ${}^{(h)}D$ and $D_{(h)}$ respectively.

The quadratic form A, of discriminant D, is called *positive-definite* if (1) all the coefficients a_{ij} are real numbers, and (2) the value of A is positive whenever the n variables x_j have real values which are not all equal to zero. If a positive-definite form A is converted, by a non-singular[1] linear transformation

$$x_i = \sum_{j=1}^{n} b_{ij} y_j \qquad (i = 1, 2, \cdots, n),$$

with real coefficients b_{ij}, into $c_1 y_1^2 + c_2 y_2^2 + \cdots + c_n y_n^2$, then $c_j > 0$ for $j=1, 2, \cdots, n$.

In order that the form A, with real coefficients, be positive-definite, it is necessary and sufficient that, for $h=1, 2, \cdots, n$, every $D(h)$ be greater than zero. This condition is, however, satisfied if, for $h=1, 2, \cdots, n$, the n inequalities ${}^{(h)}D > 0$, or the equivalent n inequalities $D_{(h)} > 0$, are valid.

Let the form A, with discriminant D, be positive-definite in the set of n variables $(x_1, x_2, \cdots, x_n)$. It is evident that the terms of second degree in any k of these variables, $(1 \leqq k \leqq n-1)$, constitute a quadratic form which is positive-definite in the k variables. Moreover, let fixed (constant) values be assigned to any k variables, say to $x_1, x_2, \cdots, x_k$. Then A becomes a quadratic polynomial, Q, in the $n-k$ variables $x_{k+1}, x_{k+2}, \cdots, x_n$. It may be written $Q(x_{k+1}, x_{k+2}, \cdots, x_n) = E+F+G$, where E is the quadratic form $\sum a_{ij} x_i x_j$, $(i, j=k+1, k+2, \cdots, n)$, F is the linear form $2\sum b_j x_j$, $(j=k+1, k+2, \cdots, n)$, and $G=\sum a_{ij} x_i x_j$, with $(i, j=1, 2, \cdots k)$, is a constant. The constant b_j is related to the constants $x_1, x_2, \cdots, x_k$ by the formula

$$b_j = \sum_{i=1}^{k} a_{ji} x_i, \qquad (j = k+1, k+2, \cdots, n).$$

The form E being positive-definite in $x_{k+1}, x_{k+2}, \cdots, x_n$, its discriminant, $D_{(n-k)}$, is not equal to zero, and therefore the system of $n-k$ linear equations

[1] The n-th order determinant of the b_{ij} is not equal to zero.

$$b_j + \sum_{i=k+1}^{n} a_{ji}x_i = 0, \qquad (j = k+1, k+2, \cdots, n),$$

has a unique solution: $x_{k+1}=c_1$, $x_{k+2}=c_2$, $\cdots$, $x_n=c_n$. Thus A may be represented, when $x_1, x_2, \cdots, x_k$ have constant values, as the sum of a constant, J, dependent upon the values of these k variables, and a positive-definite quadratic form in the variables $u_h=x_h-c_h$, $(h=k+1, k+2, \cdots, n)$:

$$A = J + \sum_{i,j=k+1}^{n} a_{ij}(x_i - c_i)(x_j - c_j).$$

For the cases $n=2$, $n=3$, this representation of A is familiar in elementary analytic geometry.

A distribution of the set $(x_1, x_2, \cdots, x_n)$ of n attributes is called "normal" if and only if:

(1) the range of each x_j, $(j=1, 2, \cdots, n)$, is from $-\infty$ to $+\infty$; and

(2) the probability function for the distribution can be written

$$f(x_1, x_2, \cdots, x_n) = C\, e^{-\Sigma a_{ij}(x_i-c_i)(x_j-c_j)} = C\, e^{-Q},$$

where C is a suitably determined constant, and Q is a positive-definite quadratic form in $(u_1, u_2, \cdots, u_n)$, with $a_{ji}=a_{ij}$ and $u_h=x_h-c_h$. In accordance with footnote to Example 2, we readily find that

$$1 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\Sigma a_{ij}u_iu_j}du_1du_2\cdots du_n = C(\pi^n/D)^{1/2},$$

whence $C=(D/\pi^n)^{1/2}$, where D is the discriminant of the form Q. For this normal distribution of $(x_1, x_2, \cdots, x_n)$, it is clear that the typical value of $\bar{x}_j$ is $\bar{x}_j=c_j$, $j=1, 2, \cdots, n$. When all the c_j are equal to zero, the distribution function is canonical.

Professor Karl Pearson has shown[1] that the probability function for a normal distribution of $(x_1, x_2, \cdots, x_n)$ in which the basic determinant of the r_{ij} is R, and the standard deviation of x_j is σ_j, may be expressed:

$$f(x_1, x_2, \cdots, x_n) = Me^{-\Sigma(R_{ij}u_iu_j/\sigma_i\sigma_j)/(2R)},$$

with $u_h=x_h-c_h$, $(h=1, 2, \cdots, n)$, and $M=1/[(2\pi)^{n/2}\sigma_1\sigma_2\cdots\sigma_n\sqrt{R}]$. The summation in the exponent of e is, of course, a positive-definite quadratic form in $u_1, u_2, \cdots, u_n$, the discriminant of which has $R_{ij}/\sigma_i\sigma_j$ as element in the i-th row and the j-th column. Denoting by K the determinant of n-th order having simply R_{ij} as element in the i-th row and the j-th column, we see that ${}^{(h)}K>0$ and $K_{(h)}>0$, $(h=1, 2, \cdots, n)$.

It follows that the coefficients of correlation r_{ij} in a normal distribution are bound by certain inequalities. In fact:[2]

[1] See Whittaker and Robinson, *The Calculus of Observations*, pages 340–342.

[2] See Bôcher, *Introduction to Higher Algebra*, pages 30–33.

$$K = R^{n-1} \text{ and } K_{(h)} = {}^{(n-h)}RR^{h-1}, \quad h = 1, 2, \cdots, n-1.$$

The last of these equations shows that $R>0$ when n is even. Setting $h=n-1$, and noting that ${}^{(1)}R=r_{11}=1$, we obtain $K_{(n-1)}=R^{n-2}$ and conclude that $R>0$ when n is odd. Thus, in any case, ${}^{(n-h)}R>0$, $h=1, 2, \cdots, n-1$. Therefore ${}^{(h)}R>0, h=1, 2, \cdots, n$; and consequently $R(h)>0, h=1, 2, \cdots, n$, where $R(h)$ denotes any principal minor, of order h, in R.

Using the canonical Pearsonian expression of the normal function,

$$f(x_1, x_2, \cdots, x_n) = Me^{-\Sigma(R_{ij}x_ix_j/\sigma_i\sigma_j)/(2R)},$$

we easily find that

$$\int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n)dx_1$$

is

$$Me^{-q\Sigma h_{ij}R_{ij}x_ix_j}\int_{-\infty}^{\infty} e^{-qx_1(h_{11}R_{11}x_1+2h_{12}R_{12}x_2+\cdots+2h_{1n}R_{1n}x_n)}dx_1,$$

where $q=1/2R$, $h_{ij}=1/\sigma_i\sigma_j$, $h_{jj}=1/\sigma_j^2$, and the summation indices i, j, in the power of e outside the integration sign, range *from* 2 *to* n. A change of the integration variable from x_1 to

$$t = x_1 + (\sigma_1/R_{11})(R_{12}x_2/\sigma_2 + \cdots + R_{1n}x_n/\sigma_n),$$

with $dx_1=dt$, leads at once to the following expression for the integral

$$\begin{aligned}\int_{-\infty}^{\infty} f\,dx_1 &= (M\sigma_1\sqrt{2\pi R}/\sqrt{R_{11}})e^{-q\Sigma h_{ij}R_{ij}x_ix_j}e^{(R_{12}x_2/\sigma_2+\cdots+R_{1n}x_n/\sigma_n)^2/(2RR_{11})}\\ &= \left\{1/\left[(2\pi)^{(n-1)/2}\sigma_2\sigma_3\cdots\sigma_n\sqrt{R_{11}}\right]\right\}e^{-\{\Sigma(R_{ij}x_ix_j/\sigma_i\sigma_j)-V^2/R_{11}\}/(2R)\}},\end{aligned}$$

where $V=R_{12}x_2/\sigma_2+\cdots+R_{1n}x_n/\sigma_n$, and the summation indices i, j range from 2 to n.

Denoting by T the determinant $R_{(n-1)}=R_{11}$, and by T_{ij} the cofactor, in T, of the element r_{ij}—which stands in row $i-1$ and column $j-1$ of T—, we quickly find that this result may be written[1]

$$\int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n)dx_1 = \left\{1/\left[(2\pi)^{(n-1)/2}\sigma_2\sigma_3\cdots\sigma_n\sqrt{T}\right]\right\}e^{-\Sigma(T_{ij}x_ix_j/\sigma_i\sigma_j)/(2T)}$$

with $i, j=2, 3, \cdots, n$.

Thus, in a normal distribution, N, of $(x_1, x_2, \cdots, x_n)$, with basic determinant R, the distribution of $(x_2, x_3, \cdots, x_n)$, when x_1 is ignored, is likewise

[1] See, again, Bôcher, pages 30–33. Note that:

$$\begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22}\end{vmatrix} = RR_{(n-2)} = RT_{22}, \qquad \begin{vmatrix} R_{11} & R_{13} \\ R_{21} & R_{23}\end{vmatrix} = RT_{23}, \text{ etc.}$$

normal, and has $R_{(n-1)}$ as basic determinant; and the standard deviation of x_j, $j=2, 3, \cdots, n$, is the same in both distributions. The conclusion is immediate that the distribution of any k attributes, $(1 \leqq k \leqq n-1)$, when the remaining $n-k$ are disregarded, is normal, and has a k-rowed minor of R as basic determinant; and that the standard deviation of each attribute in the distribution is the same as in N.

Let us now consider the distribution of $(x_{k+1}, x_{k+2}, \cdots, x_n)$ within an array of N where $x_1, x_2, \cdots, x_k$ have assigned values. According to the developments of Section 9, the probability function for this array distribution is

$$U(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)/g(x_1, x_2, \cdots, x_k)$$

where f is the probability function, expressed in Pearsonian and (for convenience) canonical form, for the normal distribution N, and g is the distribution function for $(x_1, x_2, \cdots, x_k)$ when the remaining $n-k$ attributes are ignored.

By the theorem just established, we readily see that

$$g(x_1, x_2, \cdots, x_k) = \left\{1/\left[(2\pi)^{k/2}\sigma_1\sigma_2 \cdots \sigma_k\sqrt{H}\right]\right\} e^{-\Sigma(H_{ij}x_ix_j/\sigma_i\sigma_j)/(2H)},$$

where $H={}^{(k)}R$, and the indices in the summation, which is a positive-definite quadratic form in $x_1, x_2, \cdots, x_k$, range from 1 to k. Hence $U(x_1, x_2, \cdots, x_n) = Ce^{-W}$ if

$$C = 1/\left[(2\pi)^{(n-k)/2}\sigma_{k+1}\sigma_{k+2} \cdots \sigma_n\sqrt{R/H}\right] \tag{1}$$

and

$$\begin{aligned} W &= \{1/(2R)\} \sum_{i,j=1}^{n} (R_{ij}x_ix_j/\sigma_i\sigma_j) - \{1/2H\} \sum_{i,j=1}^{k} (H_{ij}x_ix_j/\sigma_i\sigma_j) \\ &= \{1/(2R)\} \sum_{i,j=1}^{k} (R_{ij}x_ix_j/\sigma_i\sigma_j) - \{1/2H)\} \sum_{i,j=1}^{k} (H_{ij}x_ix_j/\sigma_i\sigma_j) + Q \end{aligned} \tag{2}$$

with Q expressible, as already remarked above, in the form

$$\begin{aligned} Q &= \{1/(2R)\}\left\{ \sum_{i,j=k+1}^{n} (R_{ij}x_ix_j/\sigma_i\sigma_j) \right. \\ &\qquad \left. + 2\sum_{j=k+1}^{n} (x_j/\sigma_j)(R_{j1}x_1/\sigma_1 + \cdots + R_{jk}x_k/\sigma_k)\right\} \\ &= \{1/(2R)\}\left\{ \sum_{i,j=k+1}^{n} (R_{ij}u_iu_j/\sigma_i\sigma_j) + h(x_1, x_2 \cdots, x_k)\right\}, \end{aligned}$$

where $u_r = x_r - c_r$, $r=k+1, k+2, \cdots, n$, c_r is a homogeneous linear function of $x_1, x_2, \cdots, x_k$, and h is a quadratic polynomial in these k variables.

It is obvious that the quadratic form

$$\sum_{i,j=k+1}^{n} (R_{ij}u_iu_j/\sigma_i\sigma_j)$$

is positive-definite in the $n-k$ variables $u_{k+1}, u_{k+2}, \cdots, u_n$; and it is readily found that $x_r - c_r = 0$ is the familiar best fitting hyperplane of x_r on $(x_1, x_2, \cdots, x_k)$, with the remaining $n-k-1$ variables ignored.

Treating $x_1, x_2 \cdots, x_k$ as constants, we find that

$$W = L(x_1, x_2, \cdots, x_k) + \{1/(2R)\} \sum_{i,j=k+1}^{n} (R_{ij}u_iu_j/\sigma_i\sigma_j),$$

where the value of the constant $L(x_1, x_2, \cdots, x_k)$ depends, of course, upon the (constant) values assigned to $x_1, x_2, \cdots, x_k$.

Since U is the probability function for the distribution of $(x_{k+1}, x_{k+2}, \cdots, x_n)$ when $x_1, x_2, \cdots, x_k$ have assigned values, we may write:

$$C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-W} dx_{k+1}dx_{k+2} \cdots dx_n = 1,$$

and therefore:

$$(1/C)e^{L(x_1,x_2,\cdots,x_k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\Sigma(R_{ij}u_iu_j/\sigma_i\sigma_j)/(2R)}$$

$$= \sigma_{k+1}\sigma_{k+2} \cdots \sigma_n(\pi^n/D)^{1/2},$$

where D is the determinant of order $n-k$ obtained upon dividing each element of $R_{(n-k)}$ by $1/2R$. Hence $L(x_1, x_2, \cdots, x_k)$ is independent of the values assigned to $x_1, x_2, \cdots, x_k$; and, in fact, simple calculation shows that $L=0$.

It follows that U is the function:

$$U = Ce^{-\Sigma(R_{ij}u_iu_j/\sigma_i\sigma_j)/(2R)},$$

$i, j = k+1, k+2, \cdots, n$, and therewith that *the distribution of* $(x_{k+1}, x_{k+2}, \cdots, x_n)$, *for assigned values of* $x_1, x_2, \cdots, x_k$, *is normal.*

We conclude at once that c_h, in $u_h = x_h - c_h$, is the typical mean of x_h, $h = k+1, k+2, \cdots, n$, in an array $P(12 \cdots k)$ where $x_1, x_2, \cdots, x_k$ have assigned values.

Since the distribution of $(x_{k+1}, x_{k+2}, \cdots, x_n)$ in an array $P(12 \cdots k)$ is now known to be normal, we may write its probability function, in Pearsonian form, as follows:

$$U = Ke^{-\Sigma(Z_{ij}p_iq_j/t_it_j)/(2Z)},$$

where

$$t_j = \sigma_j(12 \cdots k),\ p_j = x_j - \bar{x}_j(12 \cdots k),\ K = 1/[(2\pi)^{(n-k)/2}t_{k+1}t_{k+2} \cdots t_n\sqrt{Z}],$$

and

$$Z = \begin{vmatrix} r_{aa}(12 \cdots k) & r_{ab}(12 \cdots k) & \cdots & r_{ac}(12 \cdots k) \\ r_{ba}(12 \cdots k) & r_{bb}(12 \cdots k) & \cdots & r_{bc}(12 \cdots k) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ r_{ca}(12 \cdots k) & r_{cb}(12 \cdots k) & \cdots & r_{cc}(12 \cdots k) \end{vmatrix},$$

with $a=k+1$, $b=k+2, \cdots, c=n$, $r_{ij}(12 \cdots k)=r_{ji}(12 \cdots k)$, $r_{jj}(12 \cdots k) =1$ $(i, j=k+1, k+2, \cdots, n)$, and Z_{ij} equal to the cofactor, in Z, of the element $r_{ij}(12 \cdots k)$.

Comparison of the two expressions for U yields at once an argument that the array parameters $\sigma_j(12 \cdots k)$, $r_{ij}(12 \cdots k)$, $i, j=k+1, k+2, \cdots, n$, are *constants*, so that $\sigma_j(12 \cdots k)=\sigma_{j,12\cdots k}$ and $r_{ij}(12 \cdots k)=r_{ij,12\cdots k}$.

Moreover, since the regression of x_j on $(x_1, x_2, \cdots, x_k)$ has been shown to be linear, $j=k+1, k+2, \cdots, n$, we see that the parameter $\sigma_{j,12\cdots k}$ is identical with $\bar{\sigma}_{j,12\cdots k}$.

We conclude this article with an observation concerning normal distributions. Let C denote a constant, and let a distribution of the set $(x_1, x_2, \cdots, x_n)$ of n attributes, with all of n-space as domain, have the probability function Ce^{-T}, where T is any polynomial of second degree in $x_1, x_2, \cdots, x_n$ such that the terms of second degree constitute a positive-definite quadratic form in these n variables. Then the distribution is *normal;* and the constant C, as well as the parameters $\bar{x}_j$, σ_j, r_{ij}, $i, j=1, 2, \cdots, n$, may be determined without any integration whatever. The reader may easily outline the appropriate procedure.

SMALL OSCILLATIONS OF THE NEUTRAL HELIUM ATOM NEAR THE STRAIGHT LINE POSITIONS

By H. E. BUCHANAN, Tulane University

Introduction. The system considered in this paper consists of a central nucleus of mass m_2, carrying a positive charge of e_2, and two electrons of mass m_1 and m_3 carrying negative charges of $-e_1$ and $-e_3$ respectively. It is called the helium atom for the lack of a better name, the attention being centered solely on the mathematical discussion of the system. The gravitational attraction between the masses is neglected. We consider the special case in which $m_1=m_3$ and $e_2=2e=2e_1=2e_3$.

The author has previously discussed small oscillations of such a system near the equilateral triangle positions[1] and found that those positions are unstable. The mechanics of the problem of this earlier paper would lead one to expect that the straight line positions of the helium atom would be stable, but such is not the case. The instability is of the same nature as that of the straight line

[1] The AMERICAN MATHEMATICAL MONTHLY, vol. 39 (1931), p. 511.

positions of the three body problem discussed by the author[1] in the American Journal in 1923.

There are a number of interesting problems that occur in connection with this paper. The author takes the liberty of stating some of them.

1. The extension of this paper to the investigation of periodic orbits of the helium atom near the straight line positions when the masses are finite.

2. The discussion of the helium atom system when the gravitational attraction between the masses is considered.

3. In the three body problem for both the straight line case and the equilateral triangle case, and also in the helium atom problem for both cases, the characteristic exponents 0, 0, $\pm i\omega$, (ω being the angular velocity of the rotating axes) occur. Is this accidental? Are there certain types of U functions for which one could predict the occurrence of these roots of the characteristic equation?

The Circular Solutions. That there are circular solutions in which the nucleus and electrons remain in a straight line and revolve about their common center of gravity with angular velocity ω has been shown by Mr. W. E. Cox.[2] The angular velocity is $\omega=(e/2)\sqrt{7/m}$, where m is the mass of one electron and $2e$ is the charge on the nucleus. We assume that the units have been so chosen that the distance from each electron to the nucleus is unity and the proportionality factor in Coulomb's law, $k^2=1$.

The Differential Equations. If ξ_i, η_i and ζ_i, $i=1, 2, 3$, are the rectangular coordinates of the body m_i referred to axes rotating uniformly in the $\xi\eta$ plane with angular velocity ω, then the differential equations of motion are

$$
\begin{aligned}
&\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} = \omega^2\xi_i + \frac{1}{m_i}\frac{\partial U}{\partial \xi_i},\\
&\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} = \omega^2\eta_i + \frac{1}{m_i}\frac{\partial U}{\partial \eta_i},\\
(1)\qquad &\frac{d^2\zeta_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \zeta_i}, \quad i = 1, 2, 3,\\
&U = \frac{e_1e_2}{r_{12}} - \frac{e_1e_3}{r_{13}} + \frac{e_2e_3}{r_{23}}\cdot
\end{aligned}
$$

Let $m_1=m_3=m$ and $2e=e_2=2e_1=2e_3$, and let the coordinates in the circular solutions be

$$\xi_1^{(0)} = -1, \quad \xi_2^{(0)} = 0, \quad \xi_3^{(0)} = 1, \quad \eta_i = \zeta_i = 0, \quad i = 1, 2, 3.$$

We make the transformation

$$\xi_1 = -1 + x_1, \quad \xi_2 = x_2, \quad \xi_3 = 1 + x_3, \quad \eta_i = y_i, \quad \zeta_i = z_i, \quad i = 1, 2, 3.$$

Wherever x_2, y_2 and z_2 occur they are eliminated by the center of gravity relations $m_1x_1+m_2x_2+m_3x_3=0$, $m_1y_1+m_2y_2+m_3y_3=0$ and $m_1z_1+m_2z_2+m_3z_3=0$. We

[1] American Journal of Mathematics, vol. 45 (1923), p. 93.

[2] The AMERICAN MATHEMATICAL MONTHLY, vol. 40 (1933), p. 406.

expand the right members of the differential equations into a power series in x_1, x_3, y_1, y_3, and z_1, z_3. When only the linear terms are considered we have[1]

$$(2)\quad \left\{\begin{aligned}
\frac{d^2x_1}{dt^2} - 2\omega\frac{dy_1}{dt} &= A_{11}x_1 + A_{13}x_3 + B_{11}y_1 + B_{13}y_3,\\
\frac{d^2x_3}{dt^2} - 2\omega\frac{dy_3}{dt} &= A_{31}x_1 + A_{33}x_3 + B_{31}y_1 + B_{33}y_3,\\
\frac{d^2y_1}{dt^2} + 2\omega\frac{dx_1}{dt} &= C_{11}x_1 + C_{13}x_3 + D_{11}y_1 + D_{13}y_3,\\
\frac{d^2y_3}{dt^2} + 2\omega\frac{dx_3}{dt} &= C_{31}x_1 + C_{33}x_3 + D_{31}y_1 + D_{33}y_3
\end{aligned}\right.$$

$$(3)\quad \left\{\begin{aligned}
\frac{d^2z_1}{dt^2} &= E_{11}z_1 + E_{13}z_3,\\
\frac{d^2z_3}{dt^2} &= E_{31}z_1 + E_{33}z_3,
\end{aligned}\right.$$

where

$$A_{11} = \frac{e^2(8m + 11m_2)}{2mm_2}, \qquad A_{13} = \frac{e^2(16m + m_2)}{4mm_2}$$

$$A_{31} = \frac{e^2(16m + m_2)}{4mm_2}, \qquad A_{33} = \frac{e^2(8m + 11m_2)}{2mm_2},$$

$$B_{11} = B_{13} = B_{31} = B_{33} = C_{11} = C_{13} = C_{31} = C_{33} = 0,$$

$$D_{11} = D_{13} = D_{31} = D_{33} = E_{13} = E_{31} = -\frac{e^2(16m + m_2)}{8mm_2},$$

$$E_{11} = E_{33} = -\frac{e^2(16m + 15m_2)}{8mm_2}.$$

The Characteristic Equation for the System (2). By the usual process the characteristic equation is found to be

$$(4)\quad \begin{vmatrix}
-\lambda^2 + \frac{e^2(8m + 11m_2)}{2mm_2}, & \frac{e^2(16m + m_2)}{4mm_2}, & 2\omega\lambda, & 0\\
\frac{e^2(16m + m_2)}{4mm_2}, & -\lambda^2 + \frac{e^2(8m + 11m_2)}{2mm_2}, & 0, & 2\omega\lambda\\
-2\omega\lambda, & 0, & -\lambda^2 - \frac{e^2(16m + m_2)}{8mm_2}, & -\frac{e^2(16m + m_2)}{8mm_2}\\
0, & -2\omega\lambda, & -\frac{e^2(16m + m_2)}{8mm_2}, & -\lambda^2 - \frac{e^2(16m + m_2)}{8mm_2}
\end{vmatrix} = 0.$$

[1] The computations of these constants and the remainder of this paper have been checked by Miss Elizabeth Estorge as her master's dissertation.

Let $\lambda^2 = (e^2x^2)/(mm_2)$ and substitute $\omega = (e/2)\sqrt{7/m}$, then (4) becomes, after multiplying the last two columns by $\sqrt{7m_2}$ and removing the common factors from the last two rows,

$$\text{(5)} \quad \begin{vmatrix} -x^2 + \frac{8m+11m_2}{2}, & \frac{16m+m_2}{4}, & 7m_2x, & 0 \\ \frac{16m+m_2}{4}, & -x^2 + \frac{8m+11m_2}{2}, & 0, & 7m_2x \\ -x, & 0, & -x^2 - \frac{16m+m_2}{8}, & -\frac{16m+m_2}{8} \\ 0, & -x, & -\frac{16m+m_2}{8}, & -x^2 - \frac{16m+m_2}{8} \end{vmatrix} = 0.$$

In (5) subtraction of the last column from the third shows a common factor x and subtraction of the last row from the third shows another factor x. Hence $x=0$ is a double root. Removing the factor x^2 gives

$$\begin{vmatrix} -x^2 + \frac{8m+11m_2}{2}, & \frac{16m+m_2}{4}, & 7m_2, & 0 \\ \frac{16m+m_2}{4}, & -x^2 + \frac{8m+11m_2}{2}, & -7m_2, & 7m_2x \\ -1, & 1, & -2, & x \\ 0, & -x, & x, & -x^2 - \frac{16m+m_2}{8} \end{vmatrix} = 0.$$

All the elements of the third row except the first may be made zero by proper manipulations and the following third order determinant is found:

$$\begin{vmatrix} -x^2 + \frac{32m+23m_2}{4}, & 2x^2 - 8m - 4m_2, & -x^3 + \frac{8m+11m_2}{2}x \\ -x^2 + \frac{32m+23m_2}{4}, & -\frac{16m+15m_2}{2}, & \frac{16m+29m_2}{4}x \\ -x, & x, & -x^2 - \frac{16m+m_2}{8} \end{vmatrix} = 0.$$

Adding the second column to the first shows the common factor $x^2+7m_2/4$. Hence another pair of roots of (4) is $\lambda = \pm\iota\omega$ where $\iota = \sqrt{-1}$.

Expanding and arranging according to powers of x gives

$$x^4 + \left(-4m + \frac{3m_2}{2}\right)x^2 - \frac{512m^2 + 400mm_2 + 23m_2^2}{16} = 0.$$

Hence

$$x^2 = \frac{8m - 3m_2}{2} \pm 2\sqrt{36m^2 + 22mm_2 + 2m_2^2}.$$

Since m is very small compared to m_2, x^2 certainly has one negative and one positive value. Let these values be represented by $-\nu^2 mm_2/e^2$ and $\rho^2 mm_2/e^2$. Then (4) has the roots

$$0,\ 0,\ \pm\ \iota\omega,\ \pm\ \iota\nu,\ \pm\ \rho.$$

The Characteristic Equation for the System (3). The usual method gives the following characteristic equation for the z_i equations

$$(6)\qquad \begin{vmatrix} -\lambda^2 - \dfrac{e^2(16m + 15m_2)}{8mm_2}, & -\dfrac{e^2(16m + m_2)}{8mm_2} \\ -\dfrac{e^2(16m + m_2)}{8mm_2}, & -\lambda^2 - \dfrac{e^2(16m + 15m_2)}{8mm_2} \end{vmatrix} = 0.$$

The solution of this for λ^2 gives

$$\lambda^2 = -\frac{7e^2}{4m} = -\omega^2 \text{ and } \lambda^2 = -\frac{2e^2(2m + m_2)}{mm_2} = -\sigma^2.$$

Hence the roots of (6) are $\pm\iota\omega$, $\pm\iota\sigma$.

It is interesting that the value of σ is the same as the angular velocity in the equilateral triangle case.[1]

The Solutions of Equations (2) *and* (3). From the values of the characteristic exponents found above, we may write the solutions of equations (2) and (3)

$$(7)\qquad \begin{aligned} x_i &= K_{i1} + K_{i2}t + K_{i3}e^{\omega\iota t} + K_{i4}e^{-\omega\iota t} + K_{i5}e^{\nu\iota t} + K_{i6}e^{-\nu\iota t} + K_{i7}e^{\rho\iota t} + K_{i8}e^{-\rho\iota t}, \\ y_i &= L_{i1} + L_{i2}t + L_{i3}e^{\omega\iota t} + L_{i4}e^{-\omega\iota t} + L_{i5}e^{\nu\iota t} + L_{i6}e^{-\nu\iota t} + L_{i7}e^{\rho\iota t} + L_{i8}e^{-\rho\iota t}, \end{aligned}$$

$$(8)\quad z_i = M_{i1}e^{\omega\iota t} + M_{i2}e^{-\omega\iota t} + M_{i3}e^{\sigma\iota t} + M_{i4}e^{-\sigma\iota t}, \quad i = 1, 3,$$

and their derivatives.

We may choose six of the K_{ij}, L_{ij}, $i=1, 3, j=3 \cdots 8$ arbitrarily and the remainder are determined uniquely by well known methods. Similarly we may choose $m_{ij}, j=1 \cdots 4$ arbitrarily and M_{3j} are uniquely determined. The determination of the K_{ij} and L_{ij}, $i=1, 3, j=1, 2$ is of interest.

Substitute

$$x_i = K_{i1} + K_{i2}t, \quad y_i = L_{i1} + L_{i2}t, \quad i = 1, 3$$

in equations (2). There results:

$$\begin{aligned} -\ 2\omega L_{12} &= A_{11}(K_{11} + K_{12}t) + A_{13}(K_{31} + K_{32}t), \\ -\ 2\omega L_{32} &= A_{31}(K_{11} + K_{12}t) + A_{33}(K_{31} + K_{32}t), \\ 2\omega K_{12} &= D_{11}(L_{11} + L_{12}t) + D_{13}(L_{31} + L_{32}t). \\ 2\omega K_{32} &= D_{31}(L_{11} + L_{12}t) + D_{33}(L_{31} + L_{32}t). \end{aligned}$$

[1] Buchanan, AMERICAN MATHEMATICAL MONTHLY, vol. 39 (1931), p. 511.

These equations must be identities in t, hence

$$(9)\ \begin{cases} -2\omega L_{12} = A_{11}K_{11} + A_{13}K_{31}, \\ -2\omega L_{32} = A_{31}K_{11} + A_{33}K_{31}, \end{cases} \qquad (10)\ \begin{cases} A_{11}K_{12} + A_{13}K_{32} = 0, \\ A_{31}K_{12} + A_{33}K_{32} = 0, \end{cases}$$

$$(12)\ \begin{cases} 2\omega K_{12} = D_{11}L_{11} + D_{13}L_{31}, \\ 2\omega K_{32} = D_{31}L_{11} + D_{33}L_{31}, \end{cases} \qquad (13)\ \begin{cases} D_{11}L_{12} + D_{13}L_{32} = 0, \\ D_{31}L_{12} + D_{33}L_{32} = 0. \end{cases}$$

Since the determinant of K_{12} and K_{32} in (10) does not vanish, $K_{12}=K_{32}=0$. For the same reason (9) determines K_{11} and K_{31} uniquely in terms of L_{12} and L_{32}. Since the determinant of L_{12} and L_{32} in (12) and (13) does vanish L_{32} is uniquely determined in terms of L_{12} and L_{31} uniquely in terms of L_{11}. Therefore, it is better to choose L_{ij}, $j=1 \cdots 8$ as the arbitrary constants and all the others are uniquely determined in terms of these.

Types of Periodic Oscillations. An examination of equations (7) and (8) shows the following types of oscillations:

(1) Oscillations with period $2\pi/\omega$.
(2) Oscillations with period $2\pi/\nu$.
(3) Oscillations with period $2\pi/\sigma$.
(4) Oscillations which are combinations of any two or three of these.

This paper does not show whether these oscillations persist when all the terms in the expansions of the right members of equations (1) are taken into account. It is probable that there are periodic orbits with each of these periods for properly chosen initial conditions.

If we take $m_2=1$ and $m=1/1800$ then we have approximately

$$\omega = 56.1e, \quad \nu = e/20.9, \quad \sigma = 2e.$$

Thus by far the shortest period of oscillation is $2\pi/\omega$. The three periods are approximately proportional to the numbers, .112, 131, and 3.14.

A NEW METHOD FOR FINDING THE NUMERICAL SUM OF AN INFINITE SERIES

By C. C. CAMP, University of Nebraska

1. *Introduction.* Although the remainder R_n after taking n terms of an alternating series is numerically less than the next term it is not always so easy to get any desired accuracy. The Euler transformation is of limited application and is rather complicated. In the case of positive series the methods of Kummer and Markoff are even more laborious. The Maclaurin-Cauchy Integral Theorem gives what might be called a first approximation to upper and lower bounds for R_n in such series. The aim of the present paper is to present closer bounds by a simple formula and to give a very easy rule for interpolating so that most series can be evaluated without too much arithmetic to five or six decimals. Accuracy

to ten decimals may be secured by taking into account a certain asymptotic error.

In §2 the formula is derived from a geometrical point of view. Various types of extrapolation and interpolation are considered in §3 together with the so-called rule and its error. The other three sections treat alternating series, asymptotic forms and infinite products.

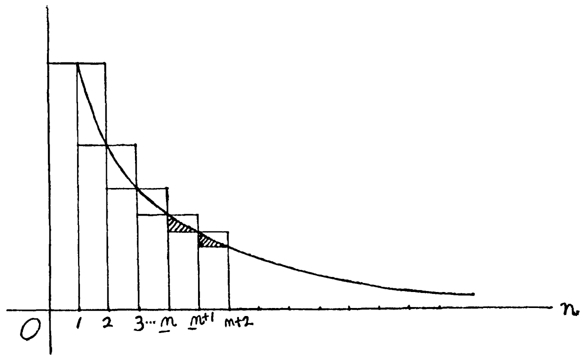


Figure 1.

2. *Derivation of the Fundamental Inequalities.* Define u_n for non-integral values of n so as to get a continuous function which tends steadily to zero. For n an integer the ordinate is the corresponding term of the series $\sum_{n=1}^{\infty} u_n$. Using exterior rectangles one sees readily that

$$\int_{n+1}^{\infty} u_n dn + \tfrac{1}{2}u_{n+1} < R_n$$

provided the curve lies below its chords. This restriction on the curve (and the series) is sufficient to insure the inequality, but not necessary. Using interior rectangles and defining k_n as the ratio of the total of the shaded areas[1] in figure 1 to the sum of the two rectangles containing them one sees that

$$R_n + k_n u_n < \int_{n}^{\infty} u_n dn,$$

[1] The use of two shaded areas rather than one is made for two reasons. First, it makes the approximate determination by (4) in section 4 much simpler for the case in which u_n is difficult to integrate. Secondly, because it leads to much greater accuracy after interpolation by the rule in section 3, than other definitions involving simple ratios of areas. The ratio k'_{n+1} of the second shaded area to its containing rectangle gives a smaller difference between the upper and lower bounds for R_{n+1}, which would be an advantage when p is unknown. However, for Σn^{-p} the error after interpolation is asymptotic to $p/\{24(n+2)^{p+2}\}+2(2p-1)(5-p)/\{240(n+2)^{p+3}\}$, instead of $p(3-p)/\{60(n+1)^{p+3}\}$, which for $p=2$, $n=10$, is over twenty times as great, namely 0.0000043.

The error resulting from (4) section 4 is close to $p(p+2)(p+3)/\{180(n+1)^3 n^p\}$. The algebra required to derive the asymptotic results in this paper is rather tedious and is omitted to save space.

provided we now restrict our series so that k_n will increase with n. This is the usual case for slowly converging series, and in particular for $u_n = c\, n^{-p}$, $p>1$, as well as for series whose general term is asymptotic to such an expression. The two inequalities may be combined in the formula

$$\int_{n+1}^{\infty} u_n dn + \tfrac{1}{2}u_{n+1} < R_n < \int_{n}^{\infty} u_n dn - k_n u_n, \tag{1}$$

where

$$k_n = \left(\int_{n}^{n+2} u_n dn - u_{n+1} - u_{n+2}\right)/(u_n - u_{n+2}). \tag{2}$$

3. It can be shown that the upper bound a_1 for the series mentioned above is nearer the true value than the lower b_1 for the sum s of the series. Consequently if one calls the difference Δ_1 the value $a_1-\Delta_1/4$ is in error less than $\Delta_1/4$ numerically. By experiment it was found that $a_1-\Delta_1/3$ was a still closer value for the case $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. Indeed for $n=10$ this gives 1.6449343 with an error of 2×10^{-7} and for $n=20$ the interpolated value was correct to 8 decimals when terms were carried to nine.

One may use (1) for two different values of n, add s_n, then extrapolate by using straight lines to get a closer value for s, namely

$$t = b_2 + (b_2 - b_1)\Delta_2/(\Delta_2 - \Delta_1), \tag{3}$$

where the subscript 2 refers to the larger value of n. For $n_1=5$, $n_2=10$ this gives $t=1.6449335$ with an error of -6×10^{-7}. This is less accurate than the simple rule for interpolation, which may be stated thus:

Rule for Interpolation: For $\sum_{n=1}^{\infty} n^{-p}$ calculate a_1, b_1 from (1), then take $a_1-\Delta_1/(p+1)$. This may be verified by using the Euler-Maclaurin sum formula. Moreover the error made in using this simple rule is asymptotic to $p(3-p)/\{60(n+1)^{p+3}\}$. The maximum numerical error for any value of p and $n=10$ is 2.3×10^{-6}. For $p=2$, $n=10$ it is 2.070×10^{-7}. By using nine decimal places for $p=3$ and taking k_{10} to seven one gets readily $\sum_{n=1}^{\infty} n^{-3} = 1.202056904$ with an accidental error of 1×10^{-9}. For $p=4$ the error in the rule is -3.42×10^{-9}. By taking this into account one may carry the terms to 12 decimals and obtain $\sum n^{-4} = 1.082323233655$ correct for $\pi^4/90$ when rounded to ten decimal places.

4. *Alternating Series.* Even slowly convergent alternating series may be evaluated easily by pairing terms. For instance $\pi/4 = 1-1/3+1/5-1/7+\cdots$ or $\pi/8 = 1/(1\cdot 3)+1/(5\cdot 7)+\cdots+1/\{(4n-3)(4n-1)\}+\cdots$. Calculation of t for $n=5$, 10 gave 0.392697 with an error of -2×10^{-6}. The use of the rule for the dominant part of this, namely $\sum 1/(16n^2)$ gave the same error. Similarly the series $\log 2 = 1-1/2+1/3-1/4+\cdots = \sum 1/\{2n(2n-1)\}$ gave $t=0.6931485$ with an error of 1.3×10^{-6} whereas $a_2-\Delta_2/3$ gave six decimals correct. It is interesting to note that for these series k_5 and k_{10} are practically the same as for the case $p=2$ above.

5. *Asymptotic Forms.* Series whose general term contains a factorial may be handled by substituting the gamma function and using the asymptotic form

$$\log \Gamma(n) = (n - \tfrac{1}{2}) \log n + \tfrac{1}{2} \log 2\pi - n$$
$$+ 1/(12n) - 1/(360n^3) + 1/(1260n^5) - \cdots$$

to simplify u_n in (1) before integration. When $n \geqq 10$ the results are accurate enough for practical use. To calculate (2) one may use Simpson's Rule and write as an approximation

$$k_n = (u_n + u_{n+1} - 2u_{n+2})/3(u_n - u_{n+2}). \tag{4}$$

If greater accuracy is required we may take $n > 10$ and use (2) and if necessary more terms of Stirling's Series. The series

$$\sum \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2^n n!(8n+1)}$$

arises in the term by term integration of

$$\int_0^1 (1 - r^8)^{1/2}\, dr = \int_0^{\pi/2} (\sin x)^{5/4}\, dx,$$

where $\sin x = r^4$. The value of the integral is $1 - s$, where s is the sum of the series. The general term may be written

$$u_n = \left(n^{-5/2} + \frac{n^{-7/2}}{4} + \frac{21n^{-9/2}}{128} + \frac{43n^{-11/2}}{512} + \cdots \right)/16\pi^{1/2}.$$

If the terms are carried to nine decimals and we take $k_5 = .45118$, $k_9 = .47067$, by Simpson's Rule then $t = .0691260$. If we take $p = 5/2$ for the dominant term and use the rule for interpolation we obtain 0.0691259 all of whose digits are correct. The true value is $s = 0.069125943$ and the original integral has the value $1 - s = \Gamma(1.5)\Gamma(1.125)/\Gamma(1.625)$, which was evaluated with Legendre's 14-figure tables. It is very clear that the rule for interpolation entails much less arithmetic. The other is useful, however, as a control.

As a crucial test consider Maclaurin's series for $\sin^{-1} x$ and put $x = 1$ obtaining

$$\sin^{-1} 1 = 1 + 1/(2\cdot 3) + 1\cdot 3/(2\cdot 4\cdot 5)$$
$$+ \cdots + 1\cdot 3 \cdots (2n-3)/\{2\cdot 4 \cdots (2n-2)(2n-1)\} + \cdots.$$

This has been characterized by Bromwich as converging "so slowly as to be quite unsuitable for numerical computation." Here

$$u_n = \Gamma(2n-1)/\{2^{2n-2}[\Gamma(n)]^2(2n-1)\}.$$

As before we get the asymptotic form

$$u_n = (n^{-3/2} + 7n^{-5/2}/8 + 81n^{-7/2}/128 + 429n^{-9/2}/1024 + \cdots)/2\pi^{1/2}.$$

$$\int_0^\infty u_n dn = (1 + 7/24n + 81/640n^2 + 429/7168n^3 + \cdots)/(\pi n)^{1/2}.$$

The terms may be exhibited as follows:

1.370 381 235	$=s_8$
1.000 000 000	
.166 666 667	
.075	
.044 642 857	
.030 381 944	
.022 372 159	
.017 352 764	
.013 964 844	$=u_8$
.011 551 801	$=u_9$
.009 761 610	
.008 390 336	
.007 312 526	
1.407 397 508	$=s_{12}$
.006 447 210	$=u_{13}$
.005 740 038	$=u_{14}$

By the approximate formula (4) one has

$k_8 = 0.4753027 \quad k_{12} = 0.483238.$
$b_1 = 1.5706242 < s < 1.5709050 = a_1$
$b_2 = 1.5707313 < s < 1.5708385 = a_2$
$t = 1.5707974$ with an error of 1.1×10^{-6}.

For $p = 3/2$ the rule gives a numerical error of about 7×10^{-7}, less than 1/9,000 of the next term u_{13}. Greater accuracy would probably be obtained with the true value of k_{12} as given in (2).

6. *Infinite Products.* By taking the logarithm of an infinite product one obtains an infinite series which may be summed by the method of this paper. For example

$$P = \prod_{n=1}^{\infty} (1 + 1/n^2) = \sinh \pi/\pi$$

is treated by using $\log P = \sum \log (1 + n^{-2})$.

$$u_n = \log (n^2 + 1) - 2 \log n.$$

$$\int u_n dn = n \log (1 + n^{-2}) + 2 \tan^{-1} n.$$

Seven-place common logarithms were used to sum the first ten terms. $k_{10}=0.477164$. Strictly $u_n=n^{-2}-n^{-4}/2+\cdots$ could be summed more accurately by taking several series with different values for p. For practical purposes, however, one takes $p=2$. The rule then gives 3.676081 as against the true value 3.676078. The sum of the series for log P was in error approximately 8×10^{-7}. It is clear that finite products for n large as well as finite series with a great number of terms can quite readily be evaluated by adaptations of the methods given above.

THE NUMBER SYSTEM OF THE MAYAS[1]

By A. W. RICHESON, University of Maryland

The Number Systems of the North American Indians have recently been discussed in detail in two papers in this monthly.[2] The system of numbers developed by the semi-civilized Maya Indians of Central America is probably the most interesting of all systems developed by the early inhabitants of this continent.

The examples of the number system of the Mayas that have been found, or at least that have been deciphered, deal with the counting of time events or periods, and many authorities are of the opinion that the recording of time series was the sole purpose of their numbers. The records of their chronicles are found as glyphs on the monuments and as written in the codices. These records present two methods of writing numerals, the normal form and the head-variant form. Both forms are essentially the same, and the Mayas were able to express a number as easily by one method as by the other. The head-variant form is found with few exceptions on the monuments, while the normal form is found exclusively in the codices.

In the head-variant form there are distinctive head forms for each of the numbers from 0 to 12 inclusive, while from 13 to 19 inclusive the numbers are written by using the head form for 10 plus the form for whatever unit is needed to make up the desired number. Each number is characterized by a distinctive type of head, by means of which it can be distinguished from any other number. In the case of three numbers, 2, 11, and 12, however, the characteristic elements have not been determined with certainty. The forms for these numerals occur

[1] It is impossible to give complete references for many of the statements, but the material has been drawn largely from the following sources, except where specific references are given: S. G. Morley, *An Introduction to the Study of Maya Hieroglyphs*, Bureau of American Ethnology, Bulletin 57; Cyrus Thomas, *Numerical Systems of Central America and Mexico*, 19th Annual Report of the Bureau of American Ethnology, 1897–98, pp. 853–955; Spinden, *Ancient Civilizations of Mexico*; W. J. McGee, *Primitive Numbers*, 19th Annual Report of the Bureau of American Ethnology, pp. 821–851; S. G. Morley, *The Inscriptions at Copan*, Carnegie Institution, Washington, D. C., 1920; John Teeple, *Maya Astronomy*, No. II, Carnegie Institution, Washington, D. C.

[2] Eels, *Number Systems of the N. A. Indians*. This MONTHLY, vol. 20 (1913), pp. 263–279; pp. 292–299.

very rarely on the inscriptions, and consequently, the data are not sufficient to justify a statement as to the characteristic elements.

Figures 1–3 illustrate the head forms for 6, 10, and 16 respectively. Figure 4 shows the head form for 16 used as a multiplier with the kin or day sign to the

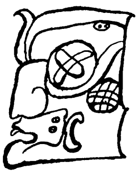


Fig 1


Fig 2


Fig 3

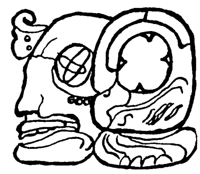
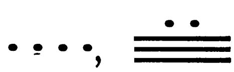


Fig 4

right. It should be noted that the head form for 16 is made up of the fleshless jaws of the character for 10 with the "hatchet" eye for 6. The characteristic elements for the numbers from 0 to 19 are given below:

Head form for	*Characteristic element*
0	Clasped hands across lower part of face.
1	Forehead ornament composed of more than one part.
2	Undetermined.
3	Banded head dress.
4	Bulging eye with square irid, snag tooth, curling fangs from back of mouth.
5	"Tun" sign for head dress.
6	Hatchet eye.
7	Large scroll passing under eye and curling under forehead.
8	Forehead ornament composed of one part.
9	Dots on lower cheek or around mouth.
10	Fleshless lower and upper jaws.
11	Undetermined.
12	Undetermined—type of head known.
13 to 19	Head for 3, 4, 5, 6, 7, 8, 9 with fleshless lower jaw for 10.

In the normal form the number combinations from 1 to 19 inclusive are formed by dots and bars. Each dot has the numerical value of 1 and each bar represents five. Generally the dots are placed horizontally over the bars or to the side of a vertical arrangement of the bars; for example, 4 and 17 were written respectively as follows:



On the inscriptions the number forms were frequently decorated to give them symmetry and a balanced form; this has often been a source of error in deciphering the inscriptions. Since the Mayas used a vigesimal system of numeration, there was no need of a symbol for twenty, since 20 units of the first order

gave one unit of the second. However, a symbol for zero was absolutely indispensable, and this symbol, which somewhat resembled the shape of a shell, is found on the inscriptions and in the codices. The symbol was first recognized by Dr. Förstemann.[1]

Methods of numeration: The Mayas developed two systems of numeration; the multiplication method and the "numeration by position." Although different in form, both methods are essentially vigesimal.

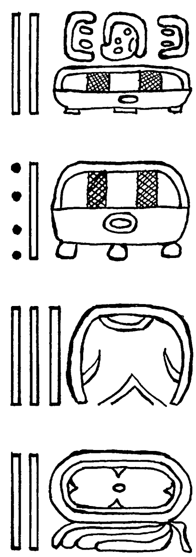


Fig 5

The first method, which is rarely found except on the inscriptions, makes use of both the normal and head-variant forms. The numbers are formed by using the bar and dot characters or the desired head form to build up the multipliers from 0 to 19 inclusive, with the time period signs as multiplicands. Until recently most authorities have stated that the Maya time count was one of days or kins and that the count was not strictly vigesimal. The following table will show the count under this assumption:

		1 kin	=	1 day
20 kins	=	1 uinal	=	20 days
18 uinals	=	1 tun	=	360 days
20 tuns	=	1 katun	=	7,200 days
20 katuns	=	1 baktun	=	144,000 days
etc.				

On the other hand, Dr. Teeple of the Carnegie Institute and Mr. Wm. E. Gates of The Johns Hopkins University have advanced the opinion that the tun is the correct unit of time used by the Mayas, and that the time count is vigesimal

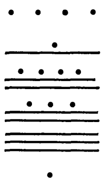
[1] Förstemann, *Zur Maya Chronologie*, Zeitschrift für Ethnologie, (Berlin) 1891.

throughout. They argue that the division of the tun or year into 18 and 20 parts is nothing more than fractional parts of the Maya time unit.[1]

Figure 5 illustrates the formation of the number 75,500 on the basis of the above table by employing the dot and bar characters for the multipliers with the kin, uinal, tun, and katun signs as multiplicands. Reading from the top down, we have 10 katuns = 72,000 kins, 9 tuns = 3,240 kins, 15 uinals = 300 kins, and 10 kins = 10 kins. The sum of the four products is 75,500 kins. This number could be expressed also by the head forms for the multipliers 10, 9, 15, and 10, in place of the dot and bar characters.

The second method of numeration, namely that by position, is very similar to the Hindu-Arabic decimal system as used today. Although the system is vigesimal and thus required 19 different combinations for the units, it was built up by the three simple characters: the dot, the bar, and the zero. With this method the Mayas were forced to fix arbitrarily a starting point and to confine themselves to one series only, or else the positional value of the nineteen digits would be useless. They accordingly adopted an ascending series which corresponds to our decimal series from right to left.

Let us illustrate the method by the number 12,489,781



= 4×	2,880,000 =	11,520,000
= 6×	144,000 =	864,000
= 14×	7,200 =	100,800
= 13×	360 =	4,680
= 15×	20 =	300
= 1×	1 =	1
		12,489,781

This is the largest number yet found in the codices.

Discussion of the numbers: The Maya numbers were no doubt written as they were spoken. The names of the numbers from 1 to 20 inclusive are given below as they appear in Beltran's *Arte del Idioma Maya.*

1 hun	11 buluc
2 ca	12 lahca
3 ox	13 oxlahun
4 can	14 canlahun
5 ho	15 lolahun
6 uac	16 uaclahun
7 uuc	17 uuclahun
8 uaxac	18 uaxaclahun
9 bolon	19 bolonlahun
10 lahun	20 hunkal or kal

Very little seems to be known concerning the origin of the number words from 1 to 5 inclusive. Dr. Brinton is of the opinion that the Maya proper and the neighboring Mayan dialects were derived from one common archaic form

[1] Teeple, *Maya Astronomy*, Carnegie Institution of Washington, D. C.

of speech and not from one another. In the case of the smaller numbers, this opinion seems to be justified, as they were no doubt formed before the beginning of their history. A number of arguments have been advanced for the derivation of the numbers from 5 to 9. Dr. Thomas believes that the hand was not used in the count until 5 was reached and that the numbers from 6 to 8 inclusive were composite. He suggests that uaxac is the answer to the whole question, that the x of ox = 3 has been combined by u with some form of 5 to give eight and that the forms for 6 and 7 are formed in a similar manner. Pio Perez on the other hand gives as a signification of the verb uac or uach "to take out one thing which is placed in another and united with it." This would seem to indicate counting on the fingers and turning them in for the first five and then opening them out while counting the second five. Bolon = 9 seems to have the meaning "on the way to 10," while lahun = 10 is lah hun; it finishes one man, i.e., counting on the fingers.

The numbers from 12 to 19 inclusive are without doubt composite numbers, i.e., $12=10+2$, $13=10+3$, etc. As we should expect in a vigesimal system, there is a definite number for 20, kal or hunkal. Henderson gives for kal "to close, to shut" or as a substantive "a fastening together," i.e., a fastening together of both hands and feet.

The count from 21 to 40 inclusive is by addition to the first 20, e.g., 21 is hun-tu-kal $=1+20$ or 1 to the 20. Forty is ca-kal or 2×20. From 41 on, the count is regular, but is different from 21 to 40. Here the count is by subtraction from the next 20.[1]

Dr. Brinton states that the Maya's use of numbers was somewhat different from ours.[2] The numbers are rarely used except with a numeral particle, which is suffixed to the numeral and indicates the character or class of the objects which are about to be enumerated. With the aid of these particles Dr. Brinton gives another method which was frequently employed to express their numbers. For eighty-one years they did not write hun tu yox kal haab, as we would expect, but can kal haab catac hunpel haab, i.e., four score years and one year.

Conclusion: It is quite evident that the Mayas had developed a number system with a place value for their characters many years before the advent of the white man. Dr. Teeple is of the opinion that the Mayan vigesimal system of numbers was a distinct part of an American civilization.[3]

The records also indicate that the Mayas were unable to handle fractions as we do today, but on the other hand they were able to and did perform long numerical computations involving multiplication and division. Just how these computations were carried out we do not know. Dr. Förstemann in his works gives an instance from the inscriptions where the calculation runs into the millions.[4]

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[1] Rosney, *Mémoire sur la numération dans la langue et dans l'écriture sacrée des anciens Mayas*, Compte-Rendu de Congrès International des Américanistes, (Paris 1875), vol. 2, p. 439.

[2] Brinton, *Maya Chronicles*, pp. 49–50.

[3] Teeple, Ibid, p. 31.

[4] Förstemann, *Zur Entzifferung der Maya-Handschriften*, No. II.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A SIMPLE CONTINUOUS FUNCIION WITH A FINITE DERIVATIVE AT NO POINT

BY T. H HILDEBRANDT, University of Michigan

Van der Waerden[1] has given a simple example of a continuous function which at no point has a finite derivative. A slight modification of .his example produces a still simpler instance.

Define $f_0(x)$ of period 1, so that

$$\begin{aligned} f_0(x) &= x \text{ for } 0 \leqq x \leqq \tfrac{1}{2} \\ &= 1 - x \text{ for } \tfrac{1}{2} \leqq x \leqq 1, \end{aligned}$$

$f_0(x)$ forms with the x-axis a set of right-angled isosceles triangles with vertices at $(\pm m, 0)$ and $(\pm m+\frac{1}{2}, \frac{1}{2})$ $m=0, 1, 2, \cdots$. Let $f_n(x) = f_0(2^n x)/2^n$. Then obviously $\phi(x) = \sum_0^\infty f_n(x)$ is a uniformly convergent series of continuous functions and therefore continuous.

To show that $\phi(x)$ has at no point a finite derivative, we make use of the well known lemma: If $\phi'(x)$ exists then

$$\lim_{(h,k)\to(0,0)} \frac{\phi(x+h) - \phi(x-k)}{h+k} = \phi'(x), \qquad h, k \geqq 0.$$

As a consequence if for every x_0 we can determine within every interval enclosing x_0 two pairs of points for which the corresponding secants on $\phi(x)$ differ by unity, $\varphi'(x_0)$ cannot exist as a finite number.

We note that $f_n(x) = 0$ for $x = p/2^n$, i.e. the value of $\phi(p/2^n)$ is determined by the functions $f_1(x), \cdots, f_k(x)$, $k < n$. Further since the slope of $f_0(x)$ is constant on $(m, m+\frac{1}{2})$, and $(m+\frac{1}{2}, m+1)$ and consequently on any interval of the form $(p/2^m, (p+1)/2^m)$, for $m > 0$, it follows that the same is true of $f_n(x)$ on any interval of the form $(p/2^{n+m}, (p+1)/2^{n+m})$ for $m > 0$. Then the slope of the line joining $[(2p+1)/2^{n+1}, \phi\{(2p+1)/2^{n+1}\}]$ and $[p/2^n, \phi\{p/2^n\}]$ differs from that of the line joining $[p/2^n, \phi\{p/2^n\}]$ and $[(p+1)/2^n, \phi\{(p+1)/2^n\}]$ by the slope of $f_n(x)$ between $x = p/2^n$ and $x = (2p+1)/2^{n+1}$, viz. 1. A similar statement holds if $2p+1$ and $p+1$ are replaced by $2p-1$ and $p-1$ respectively. These facts are also immediately evident from the graphical representation of a few of the approximating functions of $\phi(x)$. Since for any x and n, there exists an integer p such that $p/2^n \leqq x < (p+1)/2^n$, the difference quotient $\{\phi(x+h) - \phi(x-k)\}/(h+k)$, $h, k \geqq 0$, cannot approach a finite limit as $(h, k) \to (0, 0)$, for any x, i.e. if $\phi'(x)$ exists, it is infinite.

[1] Mathematische Zeitschrift, vol. 22 (1930, pp. 474–5-

Note by the Editor. This function of Professor Hildebrandt seems to be an admirably simple example of a continuous function which at no point possesses a finite derivative. The following interesting and probably not too difficult question is suggested. For what values of x (if any) does $\phi(x)$ have a definite infinite derivative, that is to say, for what values of x does

$$\frac{\phi(x + h) - \phi(x - k)}{h + k} \to + \infty \text{ or} \to - \infty$$

as $h, k \to 0, h, k \geqq 0$?

Will some reader supply the answer?

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 *Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.*

REVIEWS

Craftsmanship in the Teaching of Elementary Mathematics. By F. W. Westaway. London, Blackie and Son, 1931. xiv+665 pages. 15s net.

Mr. Westaway, a former British Inspector of Secondary Schools, sets down in this book the results of his long experience as friendly critic of young teachers of mathematics. The tone is authoritative, but not dogmatic. In many instances his judgments proceed from a balancing of one method against another in a way that appeals at once to the reader's common sense. On occasion his discussion of several alternative methods appears to be impartial when in reality it is mildy prejudiced. The reviewer usually shares Mr. Westaway's prejudice in such instances, but believes that mathematics and justice would have been better served if the author had taken more pains to exhibit the strong points of methods which he eventually rejects. A good example of this is his treatment of subtraction in arithmetic, which falls far short of Thorndike's treatment of the same topic in his Psychology of Arithmetic. On the other hand Mr. Westaway has covered a great deal of ground, interpreting Elementary Mathematics to embrace everything from the beginning of arithmetic through the elements of the calculus, including some mention of mechanics, astronomy, non-Euclidean geometry, and the philosophy of mathematics; and there is much helpful comment on all these topics. The reviewer's criticism above ought to be construed, therefore, merely as a description of the book and its scope, and not as chiding the author for giving us only six hundred pages of good counsel when he might have given us one or two hundred more. For those who wish to pursue a given matter further, there are references to other books.

The advice he gives will benefit most the relatively inexperienced teacher, and hence indirectly his earlier pupils also; but there is much to interest and challenge the teacher of experience also. Many of his comments might well be taken to heart by teachers and administrative officers in the United States. He scores the British practice of assigning teachers untrained in mathematics to classes in arithmetic and beginning algebra. The reviewer wonders if the British offend in this respect more than we do. Mr. Westaway admits that mathematics is not difficult to teach, but he wants teachers to be conversant with their subject and trained to teach it; furthermore he feels that a degree of experience is necessary, so that actually there are few good teachers of mathematics under thirty-five years of age. He regrets that there are almost no expert mathematicians in the training colleges for teachers; but he asserts that this need not be so, and that it is possible to make the training course of great value to the embryo teacher under the tutelage of men like Professor Sir Percy Nunn, to whom he acknowledges his indebtedness in many ways. He would assign the green teacher to the upper classes, insuring to the younger pupils competent instruction by teachers of experience. This has always seemed to the reviewer the wiser plan, regardless of the college entrance bugbear: older students, trained in the ways of the school, can protect themselves against the beginning teacher's earlier blunders and can be of real assistance in "breaking him in." Mr. Westaway suggests that the embryo teacher begin in a large strong school under competent guidance, rather than be thrust out into a small isolated school with responsibility from the very outset for much or all of the mathematics. This latter practice, so common with us, is fair neither to the teacher nor his pupils. In New Zealand these isolated posts are the plums given to the most experienced. Mr. Westaway is fearful lest democracy in education, like democracy in other fields, may become so interested in standardization as to stifle original methods, initiative, and intellectual independence.

The arrangement of the book is simple and obvious: a few brief chapters on teachers and methods at the start, followed by consideration of the various topics in mathematics and related fields from the kindergarten to the last years of the secondary school. Under method in mathematics he argues for that particular presentation which carries most meaning to the student, regardless of whether an expert mathematician would call it mathematically neat, or not. He would postpone all abstractions until the practical aspect of a subject had been made clear to the student. This sounds reasonable enough, and yet Mr. Westaway's exemplification of this innocent declaration seems to the reviewer to involve probably its only exception. For he asserts that he would defer all discussion of the postulates of demonstrative geometry until the very last year of school; and while his intent here is obvious and wise, he makes no provision for explaining to beginners in demonstrative geometry in some simple way that one of the advertised aims of instruction in this subject is to develop the powers of logical deduction, and that logical deductions necessarily start from certain assumptions. He has previously complimented the training colleges for

disabusing young teachers of the idea that their subject was the only one in the curriculum. Why not lead young pupils to appreciate also that their geometry is not the only geometry? It is not necessary to mention non-Euclidean geometry in this connection.

He pleads for teaching "honest thinking" through mathematics and laments the passing of Euclid. He would restore him to the schools, "frankly admitting his failings and promising reform." He admits that under the old system boys did not understand what they were doing and knew little of actual geometry; nevertheless he asserts that "they were learning to think logically and produce good authority for every assertion they made." This is reminiscent of Bertrand Russell's characterization of mathematics as the subject in which we are never sure what we are talking about, and in the same vein could lead so easily to an understanding of that very remark. It would seem as though the author ought to have taken the opportunity to discuss with the beginning teacher the whole question of demonstrative geometry in relation to "understanding" and "logical thinking," and have made some reference to the possibility of transferring this power of logical thought from situations in which it is developed to situations in which it will be applied.

The book suffers somewhat from a certain incoherence and vagueness which is a bit disconcerting. It gives the effect of being a sort of scrap book, full of important matters which are elucidated only in part. More than once one meets a detailed explanation which seems to tell only half the story, and wonders why Mr. Westaway holds back the rest. Later one finds a casual admission that he knows the sequel, and the matter is closed. He warns the reader at the outset that he will turn without notice from addressing pupils to addressing teachers. This sufficiently explains the style, without making it ideal for the young teacher. For the latter wants to know the whole story, and in detail; and is just as much interested in the scandal for adult ears only, which Mr. Westaway often glosses over, as in the expurgated edition for the innocent. But half a loaf is better than no bread, and nobody can claim to be starved who has turned to this source for help. The reviewer will indicate some of the topics where he wishes fuller treatment could have been given, and will let this suffice to indicate the general scope of the book.

In discussing mathematical reasoning (pp. 13, 14) the author says, "Mathematical reasoning is not, as commonly supposed, *deductive* reasoning; it is based upon an initial analysis of the given and, being analytical, is in essence *inductive*." And almost immediately thereafter, "In fact, the main source of fallacious reasoning almost always lies in false premises. The truth of the conclusion cannot be more true than the truth of the premises, and a scrutiny and a rigorous analysis of these is therefore always necessary. At bottom, all reasoning is much of the same kind, and it usually turns on the truth or falsehood of the premises." It seems to the reviewer that if Mr. Westaway is going to say as much as this he ought to say even more. At the very end of the book (p. 639) he mentions the differing points of view of Poincaré and Bertrand Russell with-

out relating it to his earlier pronouncements. The philosopher and psychologist can assert that all deduction is essentially induction. The mathematician eagerly pressing on to new discoveries can affirm his reliance on induction and intuition. But when the mathematician proceeds to an orderly arrangement of his results, two other points of view clamor for attention. One asserts that the validity of a conclusion depends upon the original premises; the other, that the validity of a set of postulates is determined by the conclusions to which they lead. It is not difficult to reconcile the first three of these four, showing intuition, induction, and deduction to be the same mental trait, differing only in degree. The fourth can be related to these three, but it also involves something essentially different from the others and this requires elaboration.

In commenting on rationalization versus habituation in arithmetic (pp. 16–20) the author asks whether a new rule ought to be explained and thoroughly understood before it is applied to examples, or whether it is legitimate to give the rule to the child dogmatically. "Suppose that we teach a rule 'intelligently' and the children get 50 per cent of their sums right; or suppose that we teach by rule of thumb and the children get 80 per cent of their sums right. Which plan should we adopt?" He concludes that we should test the results of our instruction as the modern psychologists have taught us to do; and further, that accuracy is the main thing, and no one long remembers the reasons behind certain operations. Still, he would give reasons to those who crave them. Now all this seems sensible and practical, but disappointing too. Has our instruction no ideal but utility, even in arithmetic? Granting that an objective attitude may help to rid us of prejudices, this problem is vital to all instruction and ought not to be so lightly dismissed. Does not our goal include appreciation of the methods we use? If so, and if our objective test probed also into the "why?," what then?

For example, Mr. Westaway contends (pp. 82–84) for a justification of the rule for placing the decimal point in multiplication which requires the pupil to write

$$72.314 \times .32 = \frac{72314}{1000} \times \frac{32}{100} = \frac{2314048}{100000} = 23.14048\ ;$$

but he winds up with the following observation. "Does not the time come when we *all* work mechanically in *all* types of calculation? Does not the *rationale* of procedure tend to fade away, until something turns up demanding revivification? Is there a more intelligent plan than teaching the boy to complete the actual multiplying before considering the decimal point at all? I doubt it." This appeals to the reviewer as good sense with respect to any mathematical technique, but not a good argument for insisting only on competent skill without requiring also some comprehension of the reasoning behind the technique. It ignores that as we advance in mathematics, while technique does not become less important, other matters involving reason and appreciation assume greater importance. Mr. Westaway's reference to "revivification" implies after all that

he too regards the rationale of a technique as having been born in the same generation with the technique, even though later "put away" for a period.

This question seems to the reviewer to depend upon the age of the pupil and to deserve different answers in different grades. Mr. Westaway actually takes this same stand himself, for in the case of signed numbers in algebra (p. 127) he insists on a justification of the algebraic rules derived from thermometer scales and insists that "boys who do not grasp the significance of directed numbers can never get to the bottom of their algebra; their work all through will inevitably be mechanical." The reviewer has every sympathy with the practical pedagogy which prompts this emphatic declaration, and would support it in general, but holds that in this special instance it deserves modification in view of the actual mathematics involved. Without the least detracting from the desirability of extended reference to the thermometer and other exemplifications of signed numbers, he would call attention to the mathematical fact that the rules for operations with negatives are not logically deducible from the rules for positives, but were purposely contrived to make the rules for positives apply without modification in the new field of negative numbers. The teacher may not wish to give the pupil an appreciation of this significance of negative numbers, but it would seem to the reviewer worth trying and better than relying solely on an explanation which is mathematically unsound. Mr. Westaway says (p. 129), "Teachers are not always quite happy about this question of directed numbers, and often ask if it is not unwise even to make the attempt to deal with it, and if a statement of just the rules ought not to suffice. Of the answer I have no doubt. Boys who do not grasp the significance of directed numbers can never get to the bottom of their algebra; their work all through will inevitably be mechanical." It has been the reviewer's experience that it was the brightest pupils who failed to follow the explanation of the text regarding the multiplication of -5 by -3, and that these brightest pupils were greatly relieved to discover that their inability to concur was to their very distinct credit and put them momentarily on a par with the best mathematicians.

It is too bad, of course, that the underlying theory seems at this point to support the uncomprehending use of a special technique, but there is a great difference between blindly following an unintelligible technique and employing this same technique—admittedly mechanical—with an appreciation of why it was so devised. Now Mr. Westaway is by no means ignorant of these matters, as appears from a brief paragraph with which he closes this topic (p. 132). He says, "It may be urged that the whole thing seems to be a little artificial. So it is. But the rule of signs is a universally accepted convention. The convention is perfectly self-consistent, and is easily justified, but by its nature it admits of no 'proof.'" Here at last is the other side of the question, but only hinted at. It would seem fairer to the teacher not to drop the matter with this vague reference, but to elaborate it and contrast it with the method which stresses the thermometric exemplifications. Then the teacher could decide whether to mention both aspects to his pupils, and just how he would reconcile one with the

other. For the two belong together mathematically and can be reconciled; the only compromise comes with regard to the pupil's appreciation and his psychology of learning.

The treatment of fractional exponents (p. 96) seems to need clarification. The author says, "We have learnt that $5^2\times 5^2=5^{2+2}=5^4$; so apparently we may assume that $5^{\frac{1}{2}}\times 5^{\frac{1}{2}}=5^{\frac{1}{2}+\frac{1}{2}}=5^1=5$. But $\sqrt{5}\times\sqrt{5}=5$; therefore $5^{\frac{1}{2}}=\sqrt{5}$." This procedure is not so obvious as it appears and requires amplification. The author practically defines $5^{\frac{1}{2}}$ as equal to $\sqrt{5}$; but he misses an excellent opportunity to set forth the real nature of algebra by his failure to emphasize that under the original definition of exponents in terms of positive integers the expression $5^{\frac{1}{2}}$ can have no meaning, and that our desire to generalize our algebraic symbolism prompts us to define $5^{\frac{1}{2}}$ so that the statement $5^m\times 5^n=5^{m+n}$ shall have wider applicability.

In solving problems by means of equations, especially those involving the partition of a sum, he prefers to let x stand for the smallest number. This is good practical advice, of course; but isn't there something to be said in favor of emphasizing the idea that we can get the answer no matter how inconvenient our choice of the number to be designated x?

There is a long and excellent section devoted to graphs including a discussion of variables and functions. Later, in considering the roots of the equation $(x+2)(x-3)=0$, no mention is made of variables, although to the reviewer it would seem as though this were an opportunity not to be missed to permit x to vary continuously from infinitely large negative values to infinitely large positive values and observe what happens to the function.

Some of Mr. Westaway's suggestions on the beginning of geometry are as follows. "No attempt should be made to develop the subject on rigorously deductive lines, from first principles, though, right from the first, precise reasons for statements made should be demanded. Young boys are never happy and are often suspicious if they feel they are being asked to prove the obvious, but they can follow a fairly long chain of reasoning if the facts are clear. . . . Proofs of propositions concerning angles at a point, parallels, and congruent triangles should not be attempted, such proofs being a matter for later treatment in the upper forms. . . . By about the age of 13, a boy ought to be able to write out a simple straightforward proof formally and to attack easy riders. . . . Solid geometry of a simple kind may with great advantage be included in the early stages of any geometry course." He goes into considerable detail on all these matters, and many others, showing how to begin the deductive treatment, continuing through similar triangles to solid geometry, orthographic projection, radial projection, plane and spherical trigonometry, complex numbers, the calculus, mechanics, astronomy, optics, map projection, statistics, time and the calendar, non-Euclidean geometry, and the philosophy of mathematics. Despite the few shortcomings noted by the reviewer, it must be evident that the book covers a wide field and is a veritable store-house of valuable information for teachers of mathematics, no matter how experienced. RALPH BEATLEY

The Nine Circles of the Triangle. By W. H. Bruce. Published by the North Texas State Teachers College, 1932. 48 pages.

The nine circles treated in this booklet are the circumcircle, the inscribed and escribed circles, the nine-point circle, and the "vertical circles," whose centers are at the vertices of the triangle and which are externally tangent to each other at the points of contact of the inscribed circle and the sides. No mention is made of these three circles except in the theorem which established their existence.

The most important and best known theorems relating to the inscribed and escribed circles and the nine-point circle are established by elementary methods. The treatment is prolix, and if the reviewer reads it correctly, not always logically irreproachable. The theorem of Feuerbach is proved in two ways; one essentially that originally used by Feuerbach, the other ingenious but depending on a construction which has not been proved possible.

There is no bibliography, and not even references either to the sources or to more extensive treatments of the same field.

R. A. Johnson

Handbook of Mathematical Tables and Formulas. Compiled by Richard S. Burington. Sandusky, Ohio, Handbook Publishers, Inc., 1933. 252 pages. $2.00; $1.25 to students and instructors.

This excellent reference work is issued in convenient form, though rather large for the pocket. The first 86 pages are devoted to a selection of the most important formulas of mathematics from elementary algebra through the calculus, with two pages of vector analysis. There is a list of 331 integrals, together with the formulas for the usual applications of the definite integral usually studied in a first course. The standard curves of analytic geometry are reasonably well drawn, though as usual even in textbooks, the cusps of such curves as the cardioid and cycloid are not drawn with sufficient sharpness.

Most of the tables are to five places, notably the table of natural trigonometric functions;—a desirable feature with the increasing use of computing machines. There is a four-place table of trigonometric functions (and their logarithms) of angles given in radian measure. There are the usual tables of powers and roots, natural logarithms, exponential and hyperbolic functions, and their logarithms, all to five places. A number of well-chosen miscellaneous tables are followed by four-place trigonometric tables. There is an index.

The book impresses the reviewer favorably, and one anticipates that it will be widely adopted.

R. A. Johnson

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB TOPICS

A HOME MADE MATHEMATICS EXHIBIT

By WILLIAM SELL, University of Alabama

Anyone who has seen the evident interest of a student in a cone cut into sections, or in a string model of a ruled surface, can not fail to appreciate the value of a permanent exhibit of mathematical objects.

The purpose of this note is to suggest a useful and interesting home made mathematics exhibit. It is hoped that its consideration may result in exhibits of this nature. While the exhibit suggested is far from complete, it may serve as a basis and we are anticipating that it may be a stimulus for ideas and suggestions in developing a really first class exhibit.

It will be found that practically all of the material suggested can be made by the students at little or no cost. Even facsimiles of the transit, level, compass, and sextant can be made in the woodshop and gilded to serve until the actual instruments are available. A perfunctory search through the literature will suggest other objects which may be included. The place for the exhibition is a local matter. The material should be open to the public, and available to the student for actual handling, under restrictions, otherwise much of the value will be lost.

No attempt has been made to include astronomy or mechanics. The theory of numbers has been omitted, although it has possibilities. Much of the exhibit will depend on the courses offered and on the particular likes and dislikes of those in charge of it.

Above all, it must be kept in mind that the exhibit is a growing thing, always open for additions and deletions. It must be interesting, and care will be required lest it become a dusty repository for dustier reliques. This requires that the individual in charge must be genuinely interested.

The objects that the exhibit will contain may be divided into the following six groups:

(1) Unusually excellent work by students such as solutions of difficult problems, 100% examination papers and the like; (2) Charts; (3) Models; (4) Instruments of construction; (5) Instruments of measure; (6) Historical objects.

The following outline gives a few suggestions for exhibits in Plane Geometry, Solid Geometry, Algebra, Trigonometry, Analytic Geometry, Calculus, History of Mathematics and Miscellaneous objects with reference to the above six groups.

I: *Plane Geometry:* (2) The nine-point circle; the twenty-one point circle; the principle of duality; (4) The straight edge and compass; (5) The graduated rule and other rules.

II: *Solid Geometry:* (3) Card-board models of the regular and other polyhedra; piles of closely piled spheres; the tesseract; wire and string models of various ruled surfaces.

III: *Algebra:* (2) Nomographs for roots of a quadratic and cubic equation; successive terms of a geometric series, i.e., the number of descendants of two insects by successive generations; the permutations of five letters; the circular permutations of five letters; (3) Stretched wires to show the relation of harmony and the harmonic progression; the ruled lines and needle used to find π by probability.

IV: *Trigonometry:* (2) The graphs of the trigonometric functions; a tide curve, analyzed; geometric proofs of the trigonometric identities; the value of π to 707 significant figures; (3) The spherical triangle, collapsible so as to show the derivation of the formulas.

V: *Analytic Geometry:* (2) The better known curves and surfaces with their equations; (3) Sections of a cone, sphere and other well known solids including the quadric surfaces; string, wire, paper and plaster models of well known curves and surfaces and sections of surfaces; (2) Various kinds of coordinate systems and coordinate paper.

VI: *Calculus:* (2) Infinite series for the elementary functions; "graph" of a differential equation; geometrical interpretation of the derivative and partial derivative; (5) planimeters and integrating machines.

VII: *Miscellaneous:* (2) Conformal mapping; the hare and the hound problem; Zeno's paradoxes of motion; the engine problem; the coin problem; how to tell a person's age; series of remarkable numbers; geometrical paradoxes and fallacies; the paradox of Tristram Shandy; magic squares, circles and cubes; the four-color map problem; Romeo and Juliet; mathematical puzzles; Cayley's color groups; derivation of mathematical terms; (3) Stereographic projection; Riemann surfaces; Moebius strip; quasiregular solids.

VIII: *Historical:* A collection of actual texts formerly used at this school; books written by graduates and teachers; portraits of eminent mathematicians; (2) Development of present notations; multiplication with Roman numerals.

Club Activities

1932–1933

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the University of Cincinnati

The purpose of the club is to promote a deeper interest in the broader, less formal, recreational, and practical aspects of mathematics; to encourage worthy mathematical undertakings; and to afford an opportunity for social relations among the members.

All graduate and honor students in the department of mathematics are eligible for membership. Also, regular undergraduates majoring in mathematics and other qualified persons including students in the department of chemistry, physics, and engineering may become members upon invitation from the club.

Election of officers takes place at the final meeting of each semester. The officers for the first semester were: Mildren Keiffer, President; Evelyn Kennedy, Vice President; Bessie Johnson, Secretary; Alta Odoms, Treasurer. For the second semester, the officers included: Paul Pepper, President; Haim Reingold, Vice President; Stanley Klein, Secretary; Jesse Epstein, Treasurer.

The club held sixteen meetings during the year 1932–1933. The program usually consisted of the presentation of a mathematical paper and the rendition of a musical number when such could be arranged. At the conclusion of the program the members customarily enjoyed refreshments.

The meetings and programs were as follows:

October 18, 1932: Reorganization meeting.

November 1, 1932: "A new method in interpolation with finite differences" by Paul Herget.

November 18, 1932: "The International Mathematical Congress" by Dr. C. N. Moore.

December 6, 1932: "A general survey of certain mathematical publications" by Violet Diller.

December 16, 1932: Christmas tea.

January 3, 1933: "Nomograms" by Guy Harris.

January 17, 1933: "Algebraic numbers" by Dr. Harris Hancock.

February 27, 1933: "The solution of simultaneous equations by the method of iteration" by Paul Pepper.

March 9, 1933: "Topology" by Dr. William L. Ayres of the University of Michigan.

March 15, 1933: Dr. I. A. Barnett told of some of the interesting personalities which he encounterd while in Europe on his sabbatical leave.

March 30, 1933: "A method of computing trigonometric functions" by Haim Reingold.

April 10, 1933: "A simple method of computing orbits of comets" by Paul Herget.

April 26, 1933: Annual dinner. After the dinner the members journeyed to the Cincinnati Observatory where Mr. Paul Herget showed slides depicting the activities of the expedition from Cincinnati to the scene of the August, 1932, eclipse.

May 10, 1933: "Magic squares and circles" by Evelyn Kennedy.

May 19, 1933: "Topology" by Dr. John H. Roberts of Duke University.

May 24, 1933: "Divergent series" by Clemmer Mitchell.

A prize consisting of an order for a book in mathematics is being offered this year to the student who has presented at a meeting of the club the paper adjudged to be the best from the standpoint of interest to the club.

Six members of the club attended the meeting of the Ohio Section of the Mathematical Association of America at Columbus, Ohio on April 6, 1933.

EVELYN M. KENNEDY, *Chairman of the Program Committee*

The Mathematics Club of the Colleges of the City of Detroit

The mathematics club of the colleges of the City of Detroit was formed in 1925 for the purpose of stimulating a greater interest in mathematics. It is a very informal organization, its membership consisting of those students who enjoy uncovering important bypaths in the realm of mathematics.

The officers for the past year were: Margaret Dumford, President; Virginia Eyre, Secretary.

The meetings and programs were as follows:

January 12, 1933: "The division of the line segment into n equal parts" by Mr. Morris Friedman.

March 7, 1933: "Some properties of integers" by Miss Emily Taisey.

March 28, 1933: "Novel proofs of well-known theorems" by Mr. W. G. Scott.

April 11, 1933: "Mis-uses of statistics" by Miss V. L. Eyre.

VIRGINIA L. EYRE, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E60. *Proposed by M. W. Aylor, University of Virginia.*

A horse is tied by a rope to one corner of a barn which is thirty feet square. Find the length of the rope such that the horse can graze over one acre of ground without grazing over any part twice.

Editor's Note: While this type of problem is not strictly new, it is being published here as an experiment in order to learn whether or not the readers of the MONTHLY care to work on such problems.

E61. *Proposed by Raphael Robinson, University of California at Berkeley.*

Given the common logarithms of the integers from 1 to 10 correct to fifteen decimal places (as in Table Ia of the MacMillan Tables), find log 2 correct to sixteen decimal places.

E62. *Proposed by W. R. Ransom, Tufts College.*

Defining a "C-angle" as the figure formed by two internally tangent circles, and its magnitude as the difference of the curvatures of those circles, show how to bisect a C-angle geometrically. (If the circles are tangent externally, the magnitude of the C-angle is the sum of their curvatures.) If the circles are tangent to the X-axis at the origin, O, and cut the circle $x^2+y^2=2x$ also at P and Q, show that the magnitude of the C-angle equals the difference between the slopes of the chords OP and OQ.

E63. *Proposed by W. B. Carver, Cornell University.*

A number of less than thirty digits begins with the two digits 15 on the left, 15 ———; and when it is multiplied by 5, the result is merely to move these two digits to the right-hand end, thus, ——— 15. Find the number, and show that the solution is unique.

E64. *Proposed by J. Rosenbaum, The Milford School, Milford, Connecticut.*

The bisectors of the interior angles of the triangle ABC meet the sides in the points P, Q and R. Prove that the ratio of the area of the triangle PQR to the area of the triangle ABC is $2abc/(a+b)(b+c)(c+a)$.

SOLUTIONS

E13. [1932, 606]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

It is required to find two factors which, together with their product, contain each of the nine digits from one to nine just once. Each of the several solutions should be found. (Zero must not appear in any solution.)

Solution by W. E. Buker, Leetsdale, Pa.

The following multiplications satisfy the requirements of the problem

$4 \times 1738 = 6952$	$4 \times 1963 = 7852$	$12 \times 483 = 5796$
$18 \times 297 = 5346$	$27 \times 198 = 5346$	$28 \times 157 = 4396$
$39 \times 186 = 7254$	$42 \times 138 = 5796$	$48 \times 159 = 7632.$

Can anyone prove that these are the only solutions, or find another?

E31. [1933, 241]. *Proposed by W. R. Ransom, Tufts College.*

Point P is on the line AB, one N-th of the way from A to B, and Q is on the line AC, one N-th of the way from A to C. PC intersects QB at R. Prove that the area of the triangle ABC is $N(N+1)/2$ times the area of the quadrilateral $APRQ$.

Solution by Roy MacKay, Albuquerque, New Mexico.

Denote the area of the triangle ABC by a and that of the quadrilateral $APRQ$ by q. Then since the triangles ABC and CRB are respectively similar to the triangles APQ and PRQ with the ratio of similitude equal to N in each case, the area of the triangle $APQ=a/N^2$ and that of triangle $CBR=N^2(q-a/N^2)$. Furthermore, the sum of the areas of the triangles APC and ABQ is $2a/N$. Hence $a=2a/N-q+N^2(q-a/N^2)$. When solved for a, this gives the desired result: $a=[N(N+1)/2]q$.

Solved also by W. E. Buker, J. H. Butchart, Wm. Douglas, F. C. Gentry, C. W. Munshower, C. T. Oergel, Simon Vatriquant, R. N. Walter and R. C. Yates.

E32. [1933, 241]. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Prove that $\int \sec x\, dx = -2\, i \tan^{-1} e^{ix} + C$, where $i=\sqrt{-1}$.

Solution by Mary Hamilton, Agnes Scott College, Decatur, Georgia.

$$\int \sec x\, dx = \int dx/\cos x = 2\int dx/(e^{ix}+e^{-ix}) = (2/i)\int ie^{ix}dx/(e^{2ix}+1)$$
$$= (2/i)\tan^{-1} e^{ix} + C = -2i \tan e^{ix} + C.$$

Solved also by Clyde Bridger, J. H. Butchart, F. C. Gentry, Roy MacKay, F. L. Manning, J. A. McLaughlin, Lazarus Medveson, Jr., W. R. Ransom, C. C. Richtmeyer, Rafael Sanchez-Diaz, Simon Vatriquant and R. C. Yates.

E33. [1933, 241]. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

The points A and B are any two points not in the plane M. Find the locus of the point X, in M, such that the lines AX and BX make equal angles with M.

Solution by J. Rosenbaum, The Milford School, Milford, Connecticut.

Let the projections of A and B on M be A' and B'. Then by the condition of the problem the right triangles $AA'X$ and $BB'X$ are similar. Hence $XA'/XB'=AA'/BB'=$a constant. From this it follows that the locus of X is an Apollonian circle. See *College Geometry* by N. A. Court, page 14, Locus 11.

Note: It is seen that the above solution holds whether A and B are on the same side or on opposite sides of M. It is also seen that when A and B are equidistant from M, the locus becomes a straight line which is the perpendicular bisector of $A'B'$.

Solved also by J. H. Butchart, F. C. Gentry, Roy MacKay, Lazarus Medveson, Jr., W. R. Ransom, Simon Vatriquant and R. N. Walter.

E35. [1933, 241]. *Proposed by W. B. Campbell, Rangoon, Burma.*

Show that, under appropriate conditions, the limit as n approaches minus one, of the integral $\int_a^b X^n dX$ is equal to the integral $\int_a^b X^{-1}dX$.

Solution by Lazarus Medveson, Jr., Albuquerque, New Mexico.

$$\lim_{n\to -1}\int_a^b X^n dX = \lim_{n\to -1}\frac{b^{n+1}-a^{n+1}}{n+1}\cdot$$

If both a and b are finite and neither is zero, the above fraction assumes the indeterminate form $0/0$ when $n=-1$. In such cases the limit of the value of the fraction as $n\to -1$ is

$$\lim_{n\to -1}\frac{D_n(b^{n+1}-a^{n+1})}{D_n(n+1)} = \lim_{n\to -1}\frac{b^{n+1}\log b - a^{n+1}\log a}{1}$$

which equals $\log b - \log a$, which is also $\int_a^b X^{-1}dX$.

Editor's Note: A further necessary condition is that both limits of integration have the same algebraic sign, as the first step in the above solution is not valid otherwise.

Solved also by E. F. Allen, W. R. Ransom, J. Rosenbaum and Simon Vatriquant.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In

general, problems in well-known textbooks or results found in readily accessible sources will not, be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3639. *Proposed by N. A. Court, University of Oklahoma.*

The two planes passing through the circumcenter of a tetrahedron and perpendicular to two bimedians (i.e., lines joining the mid-points of pairs of opposite edges) divide the third bimedian harmonically.

3640. *Proposed by V. Thébault, Le Mans, France.*

Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ be the pedal triangles of the two isogonally conjugate points P, P' with respect to the triangle ABC. The parallels through P, P' to the sides of the triangles $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$ meet the sides BC, CA, AB, in two sets of three collinear points. These two lines are respectively perpendicular to the lines joining P', P to the orthocenters of the triangles $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$.

3641. *Proposed by J. A. Bullard, University of Vermont.*

Prove, for positive integers p and q, the following summations of binomial coefficients:

$$\text{(a)} \qquad \sum_{h=0}^{p} \frac{(-1)^h}{2h+1}\, {}_pC_h = 2^{2p}(p!)^2/(2p+1)!$$

$$\text{(b)} \qquad \sum_{h=0}^{p+q} (-1)^h\, {}_{2p}C_h\, {}_{2q}C_{p+q-h} = (-1)^p(2p)!(2q)!/(p+q)!p!q!$$

If $p\neq q$ in (b) the terms which are undefined are to be omitted in the summation.

3642. *Proposed by H. Halperin, A. and M. College of Texas.*

Show that the caustic curve by reflection from a circle, of the rays issuing from a point on the circle, can be generated as follows:

Let A be the point source on the circle O; B a point on the diameter AOB, such that OB equals one third of AO; M a variable point on the circle. Then the point P of intersection of the reflection MP of the ray AM with the line BP parallel to OM generates the caustic curve.

3643. *Proposed by H. G. Green, University College, Nottingham, England.*

Show that, if a and b are positive numbers with b greater than a, and $e\xi=(b^b/a^a)^{1/(b-a)}$, then ξ lies between a and b. Given that $\log_{10} e=0.4343$ to four decimal places, deduce that $\log_{10} 99$ lies between 1.99565750 and 1.99561262, and explain theoretically the close value of the mean of these numbers to the true value of the log.

3644. *Proposed by J. Rosenbaum, the Milford School, Milford, Connecticut.*

Prove that in a tetrahedron, the three conditions:

1. The altitudes are concurrent,
2. The sums of the squares of the pairs of opposite edges are equal, and
3. The opposite edges are perpendicular to each other,

are equivalent (i.e. anyone of the conditions implies the other two).

3645. *Proposed by Paul S. Dwyer, Antioch College.*

Show that the value of the determinant formed by deleting the kth column from the array

$$\left\| \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & {}_1C_1 & {}_2C_1 & {}_3C_1 & {}_4C_1 & \cdots & {}_nC_1 & {}_{n+1}C_1 \\ 0 & 0 & {}_2C_2 & {}_3C_2 & {}_4C_2 & \cdots & {}_nC_2 & {}_{n+1}C_2 \\ 0 & 0 & 0 & {}_3C_3 & {}_4C_3 & \cdots & {}_nC_3 & {}_{n+1}C_3 \\ 0 & 0 & 0 & 0 & {}_4C_4 & \cdots & {}_nC_4 & {}_{n+1}C_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & {}_nC_n & {}_{n+1}C_n \end{array} \right\|$$

is ${}_nC_{k-1}$.

SOLUTIONS

3536. [1932, 175]. *Proposed by Martin Rosenman, Brooklyn, N. Y.*

Consider fractions of the form $1/2, 1/3, 1/4, 1/5, \cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7=41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807 \cdots$ give the desired result; in which series each denominator exceeds by 1 the product of all preceding.

Solution by D. R. Curtiss, Northwestern University.

From a communication from Professor Curtiss it has been learned by the editors that this problem is essentially the one considered by O. D. Kellogg in his paper, *On a Diophantine problem*, in this MONTHLY, vol. 28 (1921), p. 300; and that a complete solution of this problem was given by Curtiss in his paper, *On Kellogg's Diophantine problem*, in this MONTHLY, vol. 29 (1922), p. 380. The editors regret that through oversight the earlier form of the problem and its solution were overlooked at the times of the printing of problem 3536 and of the partial solution [1933, 180].

3582. [1932, 607]. *Proposed by Dewey C. Duncan, University of California.*

If α and β are positive integers and $\beta>2$, then $2^\alpha+1$ is never divisible by $2^\beta-1$.

Solution by C. H. Smiley, Brown University.

First we note that if $\beta>2$, $2^{\beta-1}(2-1)>2$, that is, $2^{\beta-1}+1<2^\beta-1$. Hence

$$\text{(1)} \qquad \text{if } \alpha<\beta, \qquad 2^\alpha+1<2^\beta-1,$$

and $2^\alpha+1$ is clearly not divisible by $2^\beta-1$.

Next, if $\alpha=\beta$, $(2^\alpha+1)/(2^\beta-1)=1+2/(2^\beta-1)$ and again, $2^\alpha+1$ is not divisible by $2^\beta-1$.

Finally suppose $\alpha>\beta$, say $\alpha=m\beta+n$ where m is a positive integer and n is zero or a positive integer less than β. Then

$$\frac{2^\alpha+1}{2^\beta-1}=\frac{2^\alpha-2^{\alpha-m\beta}}{2^\beta-1}+\frac{2^n+1}{2^\beta-1}.$$

The numerator of the first fraction on the right may be written $2^{\alpha-m\beta}(2^{m\beta}-1)$ which is clearly divisible by $2^\beta-1$; and by (1), the second fraction on the right is a proper fraction. Thus, for all cases, including $\alpha>\beta$, we have shown that $2^\alpha+1$ is not divisible by $2^\beta-1$ when $\beta>2$.

Solved also by A. G. Clark, Bernard Friedman, R. Graves, H. G. Green, Ruth G. Mason, L. C. Mathewson, A. Pelletier, J. Rosenbaum, S. Vatriquant, and the proposer.

3584. [1932, 608]. *Proposed by W. E. Buker, Leetsdale, Pa.*

Find the rational values of x and y for which

$$x^3+y^3=2^3+1^3.$$

Partial Solution by A. Pelletier, Montreal, Canada.

Set $y=kx$, then

$$x^3=\frac{27}{3+3k^3}; \tag{1}$$

and we have to make the denominator a cube. We know that it is a cube for $k=2$ or $1/2$; and in order to find other values of k, we set $k=2+m$, and then find m so that the denominator is a cube. Thus

$$\begin{aligned} 3+3k^3 &= 27+36m+18m^2+3m^3,\\ &=(3+am)^3=27+27am+9a^2m^2+a^3m^3. \end{aligned} \tag{2}$$

If we take for a the value $4/3$ we then find $m=-54/17$, and then

$$k=-\frac{20}{17},\qquad x=-\frac{17}{7},\qquad y=\frac{20}{7}.$$

Again, setting $k=-20/17+m$ and proceeding as before, we get another solution, and so on. In this manner we see that there are an infinite number solutions.

Note by the Editors. It is not proved in this solution that this process gives all rational values of x and y satisfying the given equation.

3586. [1932, 608]. *Proposed by R. E. Gaines, University of Richmond.*

If while an ellipse is turned about in its plane it remains tangent to a fixed straight line at a fixed point, its foci trace a curve whose area is $2\pi a(a-b)$.

Solution by Eugene M. Berry, Lynchburg College.

In terms of the eccentric angle, t, the equations for the ellipse are $x=a\cos t$, $y=b\sin t$, where the x-axis is the major axis and the y-axis is the minor axis of the ellipse. Let (x', y') be the coordinates of the focus referred to the fixed straight line as x'-axis and the fixed point on this line as origin. Then y' is the distance from the focus $(-c, 0)$ to the tangent line and x' the distance from $(-c, 0)$ to the normal line.

The tangent and normal lines are

$$xb\cos t + ya\sin t - ab = 0,$$
$$xa\sin t - yb\cos t - c^2\sin t\cos t = 0,$$

where $c^2=a^2-b^2$. Hence

$$y' = b(a + c\cos t)^{1/2}(a - c\cos t)^{-1/2},$$
$$x' = c\sin t(a + c\cos t)^{1/2}(a - c\cos t)^{-1/2}.$$

Since the curve is symmetrical to the y'-axis, the area K is given by

$$K = \int_{t=\pi}^{t=0} x'dy' = 2abc^2\int_0^{\pi}\sin^2 t(a - c\cos t)^{-2}dt.$$

Integrating once by parts and putting in the limits we get

$$K = 2abc\int_0^{\pi}\cos t(a - c\cos t)^{-1}dt$$
$$= -2ab\int_0^{\pi}dt + 2a^2b\int^{\pi}(a - c\cos t)^{-1}dt.$$

Hence

$$K = -2abt + 4a^2b(a^2 - c^2)^{-1/2}\arctan\left\{(a + c)^{1/2}(a - c)^{-1/2}\tan\tfrac{1}{2}t\right\}\Big]_0^{\pi}$$
$$= 2\pi a(a - b).$$

If in the above equations, we replace $-c$ by c and t by $\pi+t$, the equations are unchanged. From this we see that the curve is the same whichever focus is used.

Solved also by H. G. Green, William Hoover, A. Pelletier, and F. Underwood.

A Note by Otto Dunkel. The form of the locus may be easily seen from its equation in polar coordinates. If O is the fixed point and F and F' are the two foci, we take as the polar axis the bisector of angle $FOF'=2\theta$ cutting FF' in N. Setting $OF=\rho$, $OF'=\rho'$, we have by the Law of Cosines

$$\rho\rho' = b^2\sec^2\theta, \qquad \rho + \rho' = 2a, \qquad \rho^2 - 2a\rho + b^2\sec^2\theta = 0.$$

This last equation shows the oval form of the curve. Its area A is given by

$$A = 4a\int_0^{\cos^{-1}(b/a)} (a^2 - b^2 \sec^2 \theta)^{1/2} d\theta.$$

Setting ϕ = angle ONF', we have $\sin\theta = e\sin\phi$, and with this relation the above integral becomes

$$A = 4c^2\int_0^{\pi/2} \frac{\cos^2 \phi d\phi}{1 - e^2 \sin^2 \phi}.$$

In order to show that this gives the desired result, we consider the polar equation of an ellipse with the polar axis along the minor axis and with the pole at the center,

$$\rho^2 = \frac{b^2}{1 - e^2 \sin^2 \phi}.$$

The area between the auxiliary circle and the ellipse is then

$$B = 2c^2\int_0^{\pi/2} \frac{\cos^2 \phi d\phi}{1 - e^2 \sin^2 \phi}.$$

Hence $A = 2B$, which is the result to be proved. From the simplicity of the result one would expect that it would be possible to derive it by geometrical methods.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Contracts for the new McDonald Observatory of the University of Texas have been signed. The Observatory will have an eighty-inch reflecting telescope. The big mirror will be made of pyrex glass with a low coefficient of expansion. The Observatory will be located on Mount Locke, a seven thousand foot peak of the Davis Range in southwestern Texas. Dr. Otto Struve, director of the Yerkes Observatory of the University of Chicago, will also be director of the McDonald Observatory, under a joint agreement of the two universities. It is expected that the new telescope will be ready in two years. It is designed to gather large amounts of light and will be used for photographing faint stars in certain constellations.

In order to carry out the full intent of the National Research Council, President Roosevelt, by an executive order, has created a Science Advisory Board with authority. This board acts through the machinery and under the jurisdiction of the National Academy of Sciences and the National Research Council to appoint committees to deal with specific problems in the various departments. The Advisory Board consists of nine members appointed for a term of two years. Karl T. Compton, President of the Massachusetts Institute of Technology, and Professor R. A. Millikan, of the California Institute of Technology, have been appointed as members of the Advisory Board.

Professor W. D. Cairns of Oberlin College gave an address before the mathematics section of the Western Pennsylvania Educational Conference at Pittsburgh, October 7, 1933, on the subject "The fundamental theorem of algebra—an exposition."

Professor R. D. Carmichael, head of the department of mathematics at the University of Illinois, has been made acting dean of the Graduate School in that institution.

At the summer meeting of the North Carolina Academy of Sciences Professor E. L. Mackie of the University of North Carolina was elected chairman, and Professor E. R. C. Miles of Duke University was elected secretary of the mathematics section.

Professor E. G. Bill, who for many years has been director of admissions and dean of freshmen at Dartmouth College, has recently been made dean of the faculty at that institution. More recently, Dean Bill has been appointed by President Roosevelt to a federal commission which will effect the amalgamation of the Bureaus of Immigration and Naturalization.

Dr. L. E. Bush, instructor in mathematics at the Ohio State University, has been appointed professor of mathematics at the College of St. Thomas, St. Paul, Minnesota.

Professor L. L. Dines of the University of Saskatchewan has been appointed professor of mathematics and head of the department at the Carnegie Institute of Technology.

Mr. F. F. Middleswart has been appointed professor of mathematics at Alderson-Broaddus Junior College, Philippi, West Virginia.

Dr. A. L. O'Toole has been appointed to a professorship in mathematics at Mary Manse College, Toledo, Ohio.

Dr. J. H. Butchart has been appointed instructor of mathematics at Butler University.

Dr. J. L. Dorroh has been appointed to an instructorship in mathematics at Johns Hopkins University.

Dr. M. H. Martin, National Research Fellow at Harvard University, has been appointed instructor in mathematics at Trinity College.

National Research Fellowships in Mathematics for the year 1933–1934, were awarded to the following:

Reappointed for a second year

Doob, J. L.	Martin, M. H.	Titt, E. W.
James, R. D.	Martin, R. S.	

First year appointments

Barber, S. F.	Hestenes, M. R.	Montgomery, Deane
Blumenthal, L. M.	Hull, Ralph	Nathan, D. S.
Cameron, R. H.	Lewis, D. C., Jr.	Sullivan, M. M.
Hedlund, G. A.	Lorch, E. R.	

The Chauvenet Prize

In the year 1925, the Mathematical Association of America established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the Carus Monographs are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the Chauvenet Prize will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1932, to Professor G. H. Hardy. The next award will be in December, 1935, for the period 1931-1934.

Note that the prize is to be awarded only to a member of the Association—one more of the many good reasons for membership.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Annual Meeting of the Association, Cambridge, Mass., Dec. 27-29, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merged with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 9.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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THE AMERICAN MATHEMATICAL MONTHLY

By R. G. SANGER, The University of Chicago

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Numbers refer to pages, black-face type indicating a problem solved and solution published italics a problem solved but solution not published, ordinary type a problem proposed.

PROBLEMS—SOLUTIONS

Numbers in **black-face** type refer to problems; those in light-face to pages.

NOTES AND NEWS

PERSONAL MENTION

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CORRIGENDA

Volume XL, 1933.

P. 120, line 12, for "Bulter," read "Butler."
P. 124, line 9, for "A. L. Meder," read "A. E. Meder."
P. 174, line 3, for "W. M. Aylor," read "M. W. Aylor."
P. 178, problem E-10, the skeleton division should be as in vol. 39 (1932), p. 548.
P. 188, A correction.
P. 309, 13th line from bottom, for "American" read "America."
P. 340, A correction to vol. 40 (1933), p. 226.
P. 406, in equations (1), for "$\frac{\partial U}{\partial \xi}$, $\frac{\partial U}{\partial \eta}$, $\frac{\partial U}{\partial \zeta}$" read respectively "$\frac{\partial U}{\partial \xi_i}$, $\frac{\partial U}{\partial \eta_i}$, $\frac{\partial U}{\partial \zeta_i}$."
P. 407, line 5, in the numerator of the first fraction read "e_3" for "e_2."
P. 412, 18th line from bottom, for "4 – 224a" read "4 – 224a′."
P. 420, 5th line from bottom, for "Whalen," read "Whelan."
P. 440, A Note. An addition to a reading list in elementary theory of equations, vol. 40 (1933), pp. 77–84.
P. 445, line 15, for "Medgycy" read "Medgyesy."
P. 504, 10th line from bottom, "Columbia College, South Carolina" and "Lander College" should be interchanged.
P. 605, line 18, for "Bauer, Julia W.", read "Bower, Julia W."

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XL, 1933

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PART I

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THE APRIL MEETING OF THE WISCONSIN SECTION

The first annual meeting of the Wisconsin Section was held at Beloit College, Beloit, Wisconsin, on Saturday, April 8, 1933.

The two sessions, morning and afternoon, were presided over by the chairman, Professor G. A. Parkinson, of the University of Wisconsin Extension Division, Milwaukee.

The attendance was forty-three, including the following eighteen members of the Association; Florence E. Allen, Leon Battig, May M. Beenken, H. H. Conwell, L. A. V. DeCleene, W. W. Hart, M. L. Hartung, R. C. Huffer, M. H. Ingraham, R. E. Langer, Morris Marden, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, H. E. Slaught, I. S. Sokolnikoff, J. I. Vass.

Officers elected for the year 1933–34 were: Chairman, G. A. Parkinson; Secretary, H. P. Pettit; Program Committee, Mary B. McMillan, H. H. Conwell. The fall meeting was scheduled for the first part of November, definite arrangements to be left in the hands of the program committee. The second annual meeting was set for Saturday May 5, 1934, at Oshkosh State Teachers College.

The following five papers were read:

1. "Improving the teaching of college mathematics" by Professor May M. Beenken, Oskhosh State Teachers College.

2. "Some general remarks concerning scientific exposition" by Professor R. E. Langer, University of Wisconsin.

3. "Singularities of certain algebraic plane curves" by Professor H. P. Pettit, Marquette University.

4. "Reminiscences of Professor E. H. Moore" by Professor M. H. Ingraham, University of Wisconsin.

5. "The lag of mathematics behind literature and art in the earlier centuries" by Professor H. E. Slaught.

Abstracts of two of the papers follow:

1. Teaching is the chief function of the undergraduate college. Research is of importance as a reinforcement of teaching. The successful mathematics teacher must not only be a master of his subject with a broad knowledge of present day applications of mathematics, but he must have a genuine interest in the students he is teaching. College teaching of mathematics can be improved by giving due recognition to excellence in teaching and by providing the Ph.D. candidate with definite training for his teaching profession.

3. Professor Pettit in a paper entitled "Projective description of some higher plane curves," The Tôhoku Mathematical Journal, 1927, dealt with the construction of a curve of order $2mn$ by means of the correspondence set up between two pencils of lines with base curves of order m and n, respectively. This gives the usual projective description of the conic for $m=n=1$. The existence of m n-fold and of n m-fold points was shown. In the present paper the

author exhibits a purely projective method of determining the tangents at these sets of multiple points, and includes an analytic proof as well.

H. P. PETTIT, *Secretary*

PROJECTIVE DIFFERENTIAL GEOMETRY

By E. P. LANE, University of Chicago

I. INTRODUCTION

The Mathematical Association of America has recognized and begun to meet the need for monographs whose purpose[1] is "to make the essential features of various mathematical theories accessible and attractive to as many persons as possible who have an interest in mathematics but who may not be specialists in the particlar theory presented." The question may well be raised whether the Association could not render still further service of a kindred sort to mathematics, by encouraging the publication of short papers, not exceeding twenty pages in length, of a qualitative rather than definitive character, describing the various fields of current mathematical research. At the invitation of the Editor of the Monthly the author has attempted to write such a paper on Projective Differential Geometry. The task has not been an easy one. To what degree success has been attained will be left to the reader to decide.

The purpose of this paper then is to give to a maximum number of readers with a minimum amount of mathematical preparation as clear an idea of the nature of the special field of Projective Differential Geometry as is possible in a few pages. With this end in view it becomes necessary to make some assumptions with regard to the mathematical prerequisites for understanding what follows. The reader will be expected to know already some *differential* geometry, but not much more than is acquired in a good course in the calculus. If the reader knows something about *projective* geometry to begin with, so much the better, Finally, homogeneous coordinates, power series, and a certain amount of differential equations will be used. The author is writing especially for the student who has had a good undergraduate sequence in mathematics, hoping at the same time to make the exposition fairly intelligible to those with even more modest mathematical attainments.

For the benefit of those who may be inspired to read further, a list of books on projective differential geometry will be found at the end, to which references will occasionally be made by number. This list is complete to date. No effort will be made to refer the reader to any of the thousand or so original memoirs and papers that have been written on projective differential geometry in five languages in the last thirty years.

II. GENERALITIES

Klein in his famous *Erlanger Programm* of 1872 formulated a definition of

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[1] G. A. Bliss, *Calculus of Variations*, Carus Mathematical Monograph No. 1. See preface, p. v.

geometry which guided geometrical thinking for about fifty years. According to his definition, geometry is the study of those properties of figures which remain unchanged when the figures themselves are subjected to the transformations of a certain group. The kind of geometry under consideration depends on what group of transformations is being used. For example, the classical *metric geometry* is the study of those properties of figures which are unchanged when the figures are *moved* from one place to another. In more precise words, metric geometry is the study of the properties of figures which are invariant under the group of rigid motions. Examples of metric invariants are the distance between two points, the angle between two lines, area, shape, size, and so on.

On the other hand, the more modern *projective geometry* is the study of those properties of figures which are invariant under the group of projections. From the point of view of projective geometry, the picture on the film in the projection machine at a motion picture show is precisely the same picture as that on the screen before the audience. To be sure, the pictures are *metrically different* because for instance they have different sizes, but they are *projectively identical*, because projective geometry recognizes the existence of only those properties which are unchanged by projection. An example of such a property is the straightness of a line; if a line is straight before projection it will be straight afterward. Another example is the *united position* of point and line; if a point is on a line before projection it will be on the line afterward. Still other projective invariants are the cross ratio of four points on a line, the harmonic separation of four collinear points or of four coplanar concurrent lines, and so on.

Other classifications of geometry, besides that of Klein, are possible. For instance, *integral* geometry, or geometry *in the large*, considers a figure as a whole, whereas *differential* geometry, or *infinitesimal* geometry as it is sometimes perhaps more properly called, considers a figure only in the neighborhood of one of its elements, say a curve in the neighborhood of one of its points. The ordinary definition of the tangent line at a point of a curve, as the limit of the secant line through this point and a neighboring point on the curve as the second point approaches the first along the curve, is of a differential character. The reason why only a neighborhood of the point of tangency need be known in order to define the tangent is found in the characteristic way in which the limiting process appears in the definition. Such limiting processes are fundamental in differential geometry, and explain why the use of the differential calculus is so very convenient.

Our two classifications of geometry can be made simultaneously. Consequently we may speak,of *metric differential* geometry, meaning geometry that is both metric and differential. Some of the familiar ideas of this geometry are tangent lines, osculants, normal lines, and curvatures of curves, as well as curvatures of surfaces, and various sorts of curves on surfaces, such as minimal lines, lines of curvature, asymptotic curves, conjugate nets of curves, and geodesics. In the same way, the name *projective differential geometry* is also precisely descriptive. This kind of geometry is characterized by being both *projective* and *differential.*

Since the group of projections contains as a subgroup the group of rigid motions, and since every invariant under a group is also an invariant under a subgroup of that group, it follows that all of projective geometry may properly be included in metric geometry, and that in particular all of projective differential geometry may be logically included in metric differential geometry. As a matter of history, many projective differential theorems appeared in metric differential geometry before projective differential geometry was organized as a separate science. If the metric differential geometer wishes to study projective differential geometry too, no one will object. In fact, one fruitful program of research has been to investigate metrically configurations that were first defined and studied projectively.

However, it becomes the task of the projective differential geometer to select from metric differential geometry the material that is actually projective. So he takes into his domain tangent lines, some of the osculants, asymptotic curves, and conjugate nets. When he must leave behind something, he sometimes invents or discovers an analogue to replace it; for instance, the metric normal at a point of a surface must be relinquished, but the so-called *projective normal* is discovered to replace it. Sometimes he generalizes; for instance, the osculating circle at a point of a curve suggests the osculating conic, and the osculating sphere suggests the osculating quadric surface. In a sense, projective geometry is simpler than metric geometry because there are fewer cases and less detail to be considered; as an illustration, instead of three metrically distinct types of conics, namely, the parabola, ellipse, and hyperbola, we have only one non-singular conic in projective geometry.

Geometry can be still further classified into *synthetic* geometry, which is pure geometry, and *analytic* geometry which is geometry studied by means of coordinate systems so that the powerful methods of analysis and algebra become available. Differential geometry can be studied by synthetic methods. Indeed the analytic geometer's work is not finished until his theorems are stated in a form that is independent of the coordinate system used. There are those that would insist that when a geometric theorem and its environment are fully understood the theorem must become intuitive and appear almost self-evident. Synthetic proofs can often eventually be given for theorems originally discovered by analytic methods.

Projective differential geometers have employed three closely related analytic methods, which may be referred to as the method of power series, the method of differential equations, and the method of differential forms. We shall discuss each of these in turn.

III. Power Series

Projective differential geometry may be said to have begun with the work of G. H. Halphen (1844–1889), who studied plane curves in 1878 and space curves in 1880. He based his work upon the consideration of certain differential invariants which appear as coefficients in certain canonical power series expan-

sions. Shifting the emphasis from the differential invariants to the power series themselves, we shall now explain in some detail how plane curves may be investigated, and shall indicate some other applications of the method.

In the plane a point P_x has three projective homogeneous coordinates x_1, x_2, x_3, which may be visualized as proportional to the distances of the point P_x from the sides of an arbitrarily chosen triangle, called *the triangle of reference*, each distance being measured in such a unit and given such a sign that an arbitrarily chosen point, called *the unit point*, not on a side of the triangle of reference, may have coordinates 1, 1, 1. When $x_1 \neq 0$, non-homogeneous projective coordinates x, y of the point P_x may be introduced by the definitions

$$x = x_2/x_1, \qquad y = x_3/x_1. \tag{1}$$

The notation will make it clear in any case which kind of coordinates is being used.

Let us consider a curve C in a plane, and write its equation in the form

$$y = f(x).$$

Let us suppose that C is *analytic*, so that in a neighborhood of an ordinary point P with coordinates 0, a_0 on C, the curve C can be represented by a converging Taylor's series, of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots . \tag{2}$$

This series could be used as it stands to study the projective differential geometry of the curve C at the point P. However, it is convenient to simplify it. A well-known principle of analytic geometry states that the more intimately the coordinate system is connected with the configuration under consideration, the simpler will be the equations of the configuration. This principle is illustrated in elementary analytics when the general equation of the second degree is reduced to a simple form by translation of the origin and rotation of the coordinate axes. We shall immediately illustrate it again by showing that the triangle of reference and unit point of our coordinate system can be chosen so that the series (2) will assume the simple form

$$y = x^2 + ax^5 + bx^7 + (2a^2 + b)x^8 + a_9x^9 + \cdots . \tag{3}$$

We begin the demonstration by choosing the vertex (1, 0, 0), or in non-homogeneous coordinates (0, 0), to be the point P on the curve C under consideration. Then in the series (2) we have $a_0 = 0$. Further, we make the side $x_3 = 0$ (or $y = 0$) coincide with the tangent line of C at P. Then $a_1 = 0$, since the line $y = 0$ must cut the curve C in two *consecutive* (coincident) points at P. Since P is supposed to be an ordinary point, we have $a_2 \neq 0$; in fact, the vanishing of a_2 would imply that P is an inflexion point or point of higher singularity.

Since a conic is determined by five points, *the osculating conic* at the point P of the curve C is defined to be the conic which has five consecutive points in common with C at P, just as the tangent line has two consecutive points in common with C at P. It is known that the equation of a conic can be reduced to

the form

$$y = x^2 \tag{4}$$

by suitable choice of coordinate system. In particular, the equation of the osculating conic can be reduced to this form. Since the osculating conic already passes through the vertex $P(1, 0, 0)$ tangent to the side $x_3 = 0$, it is sufficient to suppose that the side $x_1 = 0$ is tangent to the conic at the point $(0, 0, 1)$ distinct from P where the side $x_2 = 0$ intersects the conic, and to suppose that the unit point lies on the conic (see 7, p. 17). Then the equation of the osculating conic in non-homogeneous coordinates takes the form (4). Since this conic and the curve C have five consecutive points in common it follows that the series (2), with $a_0 = a_1 = 0$, when substituted in equation (4) must satisfy (4) identically in x as far as terms of degree four. Consequently we must have $a_2 = 1$, $a_3 = a_4 = 0$. It is now obvious on inspection that the series (2) and the equation (4) agree precisely to the term in x^5 in (2), but not including this term if $a_5 \neq 0$. In fact, if $a_5 = 0$, then the osculating conic intersects the curve C in at least *six* consecutive points at P, and P is a singular point of the kind called *sextactic point* or is some higher type of singularity. Since P is supposed to be an ordinary point we have $a_5 \neq 0$. For simplicity we drop the subscript 5 hereafter, and write the reduced form of (2) which we have now obtained,

$$y = x^2 + ax^5 + a_6x^6 + a_7x^7 + a_8x^8 + \cdots . \tag{5}$$

The unit point can still be any point distinct from P on the osculating conic, and the side $x_2 = 0$ (or $x = 0$) can be any line distinct from the tangent line $x_3 = 0$ (or $y = 0$) through P. In order to choose the line $x_2 = 0$, we proceed as follows. A plane cubic curve is determined by nine points. There is a single infinity (one-parameter family) of cubics that intersect the curve C in eight consecutive points at P. Such a cubic may be referred to as an *eight-point cubic*, since it has eight-point (or seventh order) contact with C at P, just as the osculating conic has five-point (or fourth-order) contact, and the tangent line has two-point (or first-order) contact. The reader may verify that the equation of the most general eight-point cubic is

$$\begin{aligned} &h[a(x^3 + ay^3 - xy) + a_6y(y - x^2)] \\ &\qquad + k[a(y - x^2 - axy^2 - a_6y^3) - a_7y(y - x^2)] = 0, \end{aligned} \tag{6}$$

where h, k are arbitrary parameters not both zero; it is sufficient to show that equation (6) is satisfied by the power series (5) for y identically in h, k and in x as far as terms of degree seven. The one of the eight-point cubics (6) for which $k = 0$ is the only one that has a node (or double point) at the point P, as can be verified by demanding that the derivative dy/dx calculated from (6) be indeterminate when $x = 0$, $y = 0$. This cubic is called *the eight-point nodal cubic* at the point P of the curve C. The equations of its nodal tangents are found, by setting equal to zero the terms of lowest degree in its equation, to be

$$y = 0, \qquad a_6y - ax = 0. \tag{7}$$

We now choose the side $x_2=0$ (or $x=0$) of the triangle of reference to be the second of these nodal tangents, i.e., the one distinct from the tangent line $y=0$. Thus we obtain $a_6=0$. The second nodal tangent, whose equation is now $x=0$, has been called *the projective normal* at the point P of the curve C, and used to replace the metric normal. We remark that ordinarily $a_7\neq 0$, and replace a_7 by b for simplicity.

Finally, in order to choose the unit point, we proceed as follows. Among the eight-point cubics (6) there is just one osculating (or nine-point) cubic, which may be shown by the method now familiar to be the one for which

$$hb + k(2a^2 - a_8) = 0. \tag{8}$$

The osculating cubic and the osculating conic (4) intersect in six points, of course, but five of them are coincident at P. The coordinates of the intersection point distinct from P are found, by solving two equations simultaneously, to be

$$b/(a_8 - 2a^2), \qquad b^2/(a_8 - 2a^2)^2.$$

We now choose the unit point to be this point. Thus we obtain

$$a_8 = 2a^2 + b,$$

and so the reduction of the series (2) to the form (3) is complete. No further reduction is possible. The study of the projective differential geometry of the plane curve C in the neighborhood of the ordinary point P may be based on the series (3), in which all the coefficients are absolute invariants. Any further equation connecting them expresses a projective geometrical theorem about the curve C in the neighborhood of the point P (see 7, §4).

Without entering into details we merely state that the two equations of a curve in ordinary space of three dimensions can be reduced by a method similar to that just used in the case of a plane curve to the canonical form (see 7, §5, §6)

$$\begin{aligned} y &= x^2 + ax^7 + bx^8 + \cdots, \\ z &= x^3 + x^6 + cx^7 + dx^8 + \cdots. \end{aligned} \tag{8}$$

Moreover, the equation of a non-ruled surface in ordinary space can similarly be reduced to

$$z = xy + (x^3 + y^3)/6 + (Ix^4 + Jy^4)/24 + \cdots, \tag{9}$$

or if we please to

$$z = xy + (x^3 + y^3)/6 + (Jx^3y + Ixy^3)/12 + \cdots, \tag{10}$$

or to still other forms.

IV. Differential Equations

The first mathematician the major part of whose scientific career was devoted to projective differential geometry was E. J. Wilczynski (1876–1932). Beginning about 1901 he developed a new method in this field which may be

called the method of differential equations. We shall not need to formulate this method here in its most general form (see 7, p. 9, Ex. 2, p. 28; p. 64; p. 286), but will at once explain how it is used in studying plane curves, and will then see what extensions are necessary for studying surfaces in ordinary space.

Let the projective homogeneous coordinates x_1, x_2, x_3 of a point P_x be given as analytic functions of an independent variable t by equations of the form

$$x_i = x_i(t) \qquad (i = 1, 2, 3). \tag{11}$$

When t varies the locus of the point P_x is a curve C_x, of which equations (11) are the parametric equations. We suppose that the curve C_x does not degenerate into a single fixed point, and further that C_x is not a straight line.

It is possible to determine the coefficients p_i of an ordinary third-order linear homogeneous differential equation, of the form

$$x''' + 3p_1x'' + 3p_2x' + p_3x = 0, \tag{12}$$

as functions of t so that the coordinates x_i will be a fundamental system of solutions of the equation. For this purpose let us substitute the coordinates x_i one at a time in (12) with the coefficients p_i regarded as unknown. The resulting three linear algebraic equations can be solved for the p_i since the determinant of their coefficients is not zero, in virtue of the fact that the curve C_x is a proper curve not a straight line. Then C_x is called *an integral curve* of the equation (12), and (12) is spoken of as *a differential equation* of C_x.

But the differential equation (12) has many integral curves, which are all obtained by transforming C_x by a non-singular projective transformation whose equations are of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \tag{13}$$

in which the coefficients a_{ij} are constants. To convince oneself of this, one first of all observes that the point P_y whose coordinates are y_1, y_2, y_3 generates a curve C_y when t varies. This curve C_y is by definition a projective transform of C_x. That C_y is an integral curve of the differential equation (12) may be inferred at once from the theory of linear differential equations, or may be verified directly by proving that each of y_1, y_2, y_3 is a solution of (12) and that these solutions are linearly independent. So all projective transforms of C_x are integral curves of (12). Conversely, the theory of linear differential equations tells us that the most general integral curve or (12) may be obtained by transforming projectively any particular one.

The reason why a theory of the integral curves of the differential equation (12) is a projective theory is now apparent. It must be a projective theory because it is impossible by means of the equation to distinguish between the integral curves. Only such properties of the curves can be studied

as are common to them all, i.e., are invariant under projective transformation.

It should be observed that the differential equation (12) depends on more than just the curve C_x. In the first place, the equation depends on the proportionality factor of the homogeneous coordinates. The transformation

$$\lambda x_i = \xi_i \qquad (i = 1, 2, 3)$$

certainly would not change the curve C_x, since the curve C_ξ is the same curve merely denoted differently; but this transformation ordinarily would change the equation (12). Moreover, the equation depends on the parameter, or independent variable, used. The transformation of parameter from t to u,

$$t = t(u) \qquad (dt/du \neq 0)$$

leaves the curve C_x unchanged but ordinarily would change the equation. So we make the total transformation

$$\lambda x = \xi, \qquad t = t(u) \qquad (\lambda \neq 0,\ dt/du \neq 0) \tag{14}$$

on equation (12) and get another equation of the same form.

The *invariants* of the equation (12) under the transformation (14) are defined to be those functions of the coefficients p_i and their derivatives which are unchanged, except possibly for a factor depending only on the transformation. Any absolutely invariant equation connecting these invariants expresses a projective property of an integral curve C_x, and conversely every such property can be so expressed. The *covariants* of the equation (12) under the transformation (14) are defined to be those functions of the coefficients p_i and their derivatives and also of x and the derivatives of x which are similarly invariant. When the three coordinates x_1, x_2, x_3 are substituted one at a time in a covariant, the three values thus obtained for the covariant are the coordinates of a covariant point which is geometrically definable in terms of the curve C_x and the point P_x. When the parameter t varies, the covariant point generates a covariant curve which has its points in one-to-one correspondence with the points of the curve C_x, and which may be geometrically defined in terms of C_x. Conversely, every such curve can be so represented. The method of Wilczynski as applied to plane curves (see 1, Chap. III) consists in studying the curve C_x by means of the invariants and covariants of the differential equation (12) under the transformation (14). The same method can be employed for studying curves in ordinary space (see 1, Chap. XIII) and curves in n-space (see 1, Chap. II). Moreover, this method can easily be connected with the method of power series (see 1, p. 62).

Ruled surfaces, i.e., surfaces consisting of one-parameter families of straight lines, are of considerable interest in projective differential geometry. Such a surface is called *developable* if its generators (straight lines) are tangent to a curve. A non-developable ruled surface in ordinary space is studied (see 1, Chap. IV et seq., or else 7, §§11–14) according to the method under consideration by means of the invariants and covariants of a system of ordinary equations of

the form

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \qquad (15)$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0,$$

under a transformation of the form

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad t = t(u). \qquad (16)$$

When we come to study non-ruled analytic surfaces in ordinary space a new complication arises. The fundamental equations are partial differential equations, and consequently *integrability conditions* appear, which we proceed to explain.

Let us consider in ordinary space a proper surface S, whose parametric equations are

$$x_i = x_i(u, v) \qquad (i = 1, 2, 3, 4). \qquad (17)$$

Let it be supposed that the parametric curves, i.e., the u-curves for which $v=$const. and the v-curves for which $u=$const., are not too specially chosen; specifically, let the parametric curves not form a conjugate net. Then it can be shown that the four functions x_i satisfy a system of equations of the form

$$x_{uu} = px + \alpha x_u + \beta x_v + Lx_{uv}, \qquad (18)$$
$$x_{vv} = qx + \gamma x_u + \delta x_v + Nx_{uv},$$

in which the coefficients are functions of u, v and subscripts indicate partial differentiation. In fact, the parametric curves form a conjugate net if, and only if, the coordinates x_i are solutions of a Laplace equation of the form (see 7, Chap. IV; see also 2, especially, p. 51)

$$x_{uv} = ax_u + bx_v + cx.$$

The theory of this and associated equations constitutes the very significant chapter on conjugate nets in projective differential geometry, which we dismiss with this mention. Since we are now supposing that no such equation is satisfied by the coordinates x_i, it follows that substituting the coordinates x_i one at a time in the first of equations (18) with the coefficients p, α, β, L regarded as unknown leads to four linear algebraic equations which can be solved for these coefficients. Similarly the coefficients of the second of equations (18) can be determined.

The differential equations (18) can be simplified in the following way. Let us recall that an asymptotic curve on a surface is a curve such that its osculating (or three-point) plane at each of its points is the tangent plane of the surface at the point. It is known that there is a net (two one-parameter families) of asymptotic curves on our surface. Let us take them for the parametric curves. Then it can be shown, by a very simple argument which will be omitted, that $L=N=0$.

Let us now consider the reduced system of equations

$$x_{uu} = px + \alpha x_u + \beta x_v,\qquad x_{vv} = qx + \gamma x_u + \delta x_v. \tag{19}$$

It is easy to see that each of the third partial derivatives of x can be calculated from (19) and expressed in just one way as a linear combination of x, x_u, x_v, x_{uv}. But some of the fourth derivatives can be calculated in more than one way. Since, however, we are dealing with analytic functions, the two methods of calculation must give rise to identical results. So from the equation

$$(x_{uu})_{vv} = (x_{vv})_{uu}$$

we obtain an equation of the form

$$A x_{uv} + B x_u + C x_v + D x = 0, \tag{20}$$

which must be satisfied by the four functions x_i. But since we are considering a proper surface S on which the parametric curves do not form a conjugate net, it follows that we must have (see 7, Chap. III)

$$A = B = C = D = 0. \tag{21}$$

These are the integrability conditions of system (19). We shall not calculate them explicitly here. They are partial differential equations which must be satisfied by the coefficients of system (19). Conversely, any system of the form (19) for which the integrability conditions (21) are satisfied is said to be *completely integrable.* Such a system possesses four linearly independent solutions, and its most general solution is a linear combination of these with constant coefficients. Consequently such a system defines a surface in ordinary space except for a projective transformation.

The reader who is familiar with the metric differential geometry of surfaces will observe that system (19) is closely related to the system composed of the three differential equations of Gauss and the two of Weingarten satisfied by the cartesian coordinates of a point on a surface and the direction cosines of the normal to the surface at the point. The latter system has three integrability conditions, namely, the one condition of Gauss and the two of Codazzi.

The method of differential equations as explained above can easily be connected with the method of power series, but we shall not make the connection here (see 7, §17), nor shall we attempt even to mention all the situations in which the method may be employed.

V. Differential Forms

The analytic approach to projective differential geometry by means of differential forms (homogeneous polynomials in differentials) is due to G. Fubini (1879–), whose investigations in this field began about 1914. This method finds perhaps its best illustration in the theory of non-ruled surfaces referred to their asymptotic curves in ordinary space, but is in fact very widely applicable (see 5).

Let us again think of the situation in metric differential geometry. There one could organize the theory of surfaces by starting with the five differential equations of Gauss and Weingarten, subject to the three integrability conditions of Gauss and Codazzi. These determine a surface except for its position in space. Eventually there would appear two quadratic differential forms which can be written

$$Edu^2 + 2Fdudv + Gdv^2, \qquad Ldu^2 + 2Mdudv + Ndv^2. \tag{22}$$

The first of these vanishes for the minimal curves and the second for the asymptotic curves on the surface. This would be the method of differential equations.

However, Gauss did not develop the metric theory this way, and this is not the usual way. Usually the differential forms (22) appear before the differential equations, and are called the *fundamental quadratic differential forms.* The *fundamental* theorem then asserts that when these forms are given with coefficients that are functions of u, v satisfying the Gauss-Codazzi conditions, they determine a surface *except for position* in ordinary space. This in brief is the method of differential forms as applied in the metric theory.

Fubini considered the problem of defining a surface by differential forms, *except for a projective transformation* in ordinary space, and reached in particular the following result. Let us consider three differential forms, two quadratic and one cubic, which can be written

$$2\beta\gamma dudv, \qquad 2\beta\gamma(\beta du^3 + \gamma dv^3), \qquad pdu^2 - qdv^2. \tag{23}$$

Let the coefficients β, γ, p, q be functions of u, v satisfying the three following *integrability conditions* (see 7, p. 69 and p. 289; see also 3, vol. I, p. 94; or see 5, p. 83):

$$\begin{aligned} \theta_{uvv} &= (\gamma\phi)_u + 2q_u + \theta_v\theta_{uv} - \beta\gamma\psi, \\ \theta_{uuv} &= (\beta\psi)_v + 2p_v + \theta_u\theta_{uv} - \beta\gamma\phi, \\ p_{vv} - \theta_v p_v + \beta q_v + 2q\beta_v &= q_{uu} - \theta_u q_u + \gamma p_u + 2p\gamma_u, \end{aligned} \tag{24}$$

where

$$\theta = \log \beta\gamma, \qquad \phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v. \tag{25}$$

Then the forms (23) determine a non-ruled surface referred to its asymptotic curves in ordinary space, except for a projective transformation. To connect this result with the method of differential equations we may observe that it can be shown that the surface is an integral surface of the following system of differential equations, of which the equations (24) are the integrability conditions:

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v. \end{aligned} \tag{26}$$

In conclusion we may point out a few analogies between the metric and pro-

jective theories of surfaces. The normal line at a point of a surface, which plays such an important rôle in the metric theory, has no place in the projective theory, and the problem arose to find a substitute which had as many of the desirable properties of the metric normal as possible. After an interesting period of research G. M. Green of Harvard and G. Fubini in Italy almost simultaneously found what is now called the *projective normal* (see 7, §20). The geodesic curves of the metric theory are not of a projective character, but they have suggested various investigations and generalizations in the projective theory. Further, in the metric theory the parametric net is said to be *isothermally orthogonal* in case the coefficients of the first of the forms (22) satisfy the conditions

$$F = 0, \qquad (\log E/G)_{uv} = 0.$$

The analogous definition of an *isothermally conjugate net* is, in terms of the coefficients of the second of the forms (22),

$$M = 0, \qquad (\log L/N)_{uv} = 0.$$

The problem of finding the geometric significance of isothermal conjugacy led to extensive investigations. Again, in the metric theory two surfaces are said to be *applicable* under certain conditions, which can be reduced to saying that they have the same E, F, G. Analogously, two surfaces are said to be *projectively applicable* (see 5, Chap. VI) in case they satisfy certain conditions, which can be reduced to saying that they have the same β, γ in the notation of (23). Finally, the well-known theorem of Meusnier, to the effect that the osculating circles of all plane sections at a point of a surface, made by planes through a tangent line at the point, lie on a sphere has as a projective analogue the theorem of Moutard to the effect that the osculating conics of all plane sections at a point of a surface, made by planes through a tangent line at the point, lie on a quadric surface.

This is not the place to describe other projective differential investigations, such as the extensive studies of rectilinear congruences and complexes, the study of surfaces and hypersurfaces in hyperspace, and so on. There is already a voluminous literature on these and other topics, and it is still growing.

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THERMODYNAMICS—AN EXPOSITION

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I. *The Laws of Thermodynamics*[1]

1. *Introductory.* This expository paper is intended to outline the chain of reasoning by which the general principles of thermodynamics are established, and to outline a systematic mode of application of these principles to an important class of physical phenomena. Thermodynamics has an especial interest because of the fact that the hypotheses underlying it are susceptible of experimental verification, and this interest makes it desirable that the logical structure of the theory be carefully studied. A notable departure from the usual plan of development of the general theory has been proposed by Carathéodory in a paper[2] in which he undertakes to proceed without presupposing the existence of measureable quantities of "heat," and without utilizing the "temperature" readings of an arbitrary thermometric scale. Instead of proceeding in this way I have chosen to follow the traditional course of development, taking temperatures and quantities of heat as given. An effort has been made to distinguish reasoning on physical grounds from mathematical deductions; to emphasize the postulates employed; and to bridge apparent gaps in the argument as usually presented. It will be well to begin with definitions of some terms that might otherwise be misunderstood.

2. *Definitions.* A *body* is a definite material object. A given pound of water and half-ounce of table salt together constitute a body; a pound of a mixture of salt and water, in which the masses of the components are in a varying ratio, is not a body.

The *temperature* of a given body, when uniform, is the scale-reading of a given thermometer in thermal equilibrium with the body. The calibration of the thermometer is arbitrary, except that the readings must rise continuously with increasing hotness. At a given instant the temperature of a body may be non-uniform.

The *quantity of heat* added to a body in a given thermal operation is defined, in the familiar way, with reference to the use of a calorimeter. If this quantity is negative, the heat is abstracted from the body.

The set of *thermodynamic states* of a body is the set of physical states that can be reached, one from another, by transfers of work and heat to the body. A *change of thermodynamic state* of a body is a transformation of one thermodynamic state into another. It is determined by the end states. The *path* of a change of state is the set of intermediate states.

When a body has a uniform constant temperature and all its parts are at rest, it is in a state of *thermodynamic equilibrium*. For it is then in thermal and in dynamic equilibrium with the contiguous bodies.

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[1] Part II, *An Application of the Laws of Thermodynamics*, will appear in the next issue of the MONTHLY.

[2] Mathematische Annalen, vol. 67 (1909), p. 355.

The states of thermodynamic equilibrium of a given body form in general distinct continuous *regions* of states. Thus a region of states of coexistent liquid and vapor in evaporation equilibrium is distinct from the region of states of homogeneous fluid, from a region of states of coexistent liquid and solid in fusion equilibrium, and from a region in which solid, liquid, and vapor coexist.

3. *The Energy Law.* It is a familiar experimental fact that any expenditure of work against friction or viscosity produces a quantity of heat proportional to the work expended. It follows that every absorption or development of heat by a body has a mechanical aspect, in that an absorption can be imagined effected by a mechanical operation, and a development can be imagined replaced by one. Hence quantities of work and of heat can be expressed in the same unit. Henceforth they will be so expressed.

The "First Law of Thermodynamics" is the statement that the algebraic sum of the work and heat absorbed by a body in any change of its thermodynamic state is independent of the path of the change of state. Since this principle is justified by extended and exact agreement of its consequences with experience, it is to be regarded as an experimentally established fact. It follows that the sum of the work W_{ab} and the heat Q_{ab} absorbed by a body in the change of its state from any state a to any other state b, on any possible path, is equal to the corresponding change of value $E_b - E_a$ of a single-valued function E of any independent variables that can be employed to determine the states,

$$W_{ab} + Q_{ab} = E_b - E_a. \tag{1}$$

The function E is termed the *energy* of the body. Since sufficiently small changes of the independent variables (of the state) effect arbitrarily small changes of E, this function may be considered to be continuous. The equation (1) formulates the *energy law.* The energy E, since it contains an arbitrary additive constant, is set equal to zero at an arbitrarily selected state of reference.

4. *Sequences.* When a continuous one-dimensional set of states of thermodynamic equilibrium connecting states a and b of a body is the limit approached by a family of paths of the change of state ab, and also by a family of paths of the reversed change of state ba, when these changes of state are conducted with increasing slowness under the influence of the independently controlled imposed forces and temperature, the set is termed a *reversible path* of either change of state. And the limits of the quantities of work and heat absorbed on the paths of the change ab, which are equal to the negatives of the limits of the work and heat absorbed on the paths of the change ba, as the limiting path is approached, are termed the work and heat absorbed *on the reversible path ab*, or developed *on the reversible path ba.*

When a body B undergoes a change of state in which no heat is transferred to or from B, the path of the change is *adiabatic.* Any reversible cyclic change of state conducted on an isothermal path followed successively by an adiabatic, another isothermal, and a concluding adiabatic path, is a *Carnot cycle.* Any reversible isothermal absorption or development of heat can be one of the iso-

thermal stages of such a cycle. When a body traverses a Carnot cycle, it absorbs x units of heat at a temperature t_1, it absorbs a net amount of W units of work, and it develops Q units of heat at a temperature t_2. Since the change of energy of the body is equal to zero, these quantities are connected by the relation,

$$x + W = Q.$$

Although this is not the place to demonstrate the statement, it should be noted that it is possible to devise Carnot cycles for which x is either positive or negative or equal to zero, while t_2 is either greater than t_1 or less than t_1 or equal to t_1.

Let the ith cycle of a set of Carnot cycles absorb heat x_i at a temperature t_{i1}, operate between temperatures t_{i1} and t_{i2}, and develop heat Q_i at t_{i2}, where $i=1, 2, \cdots, n$. When $n=1$, and when $n>1$ and the conditions

$$t_{i1} = t_{i-1,2}, \qquad x_i = Q_{i-1}, \qquad i = 2, 3, \cdots, n,$$

are satisfied, the set of cycles will be termed a *sequence*. When W is the total work absorbed in the operation of the cycles of the sequence, the energy law yields the relation,

$$x_1 + W = Q_n. \tag{2}$$

5. *The Second Law.* We now come to the "Second Law of Thermodynamics," which is the statement that an absorption of heat by a body maintaining a uniform temperature cannot be effected at the sole expense of a development of heat by a body maintaining a lower uniform temperature. In the words of Clausius, "Heat cannot of itself pass from a colder body to a warmer one."

A consequence of this law is the theorem that work cannot be gained at the sole expense of a development of heat by a body maintaining a uniform temperature. For, if a mechanism M absorbed x units of work at the sole expense of a development of x units of heat by a body A, an expenditure of this quantity of work by M against friction or viscosity in a body B, whose temperature exceeds that of A, would effect an uncompensated transfer of heat from A to the warmer body B. But the possibility of this result is excluded by the second law.

6. *The Work Absorbed by a Sequence.* Let any sequence operated between the arbitrary terminal temperatures t_1 and t_2, as recorded on an arbitrary scale of temperatures, absorb x units of heat at t_1, absorb a net amount of W units of work, and develop Q units of heat at t_2. When $t_2>t_1$, and the sequence is so operated that x is positive, which is always possible when $x\neq 0$, we have by (2) that Q, W, and the arbitrary positive constant x satisfy

$$Q = x + W.$$

There are five possibilities:

$$W < -x, \quad W = -x, \quad -x < W < 0, \quad W = 0, \quad W > 0.$$

The first four of these can easily be shown to involve violation of the second law. Further application of the second law proves, also, that $W=0$ when $x=0$, and that $W=0$ when $t_2=t_1$. We thus find the theorem:

I. *The quantity* W *has the sign of* x *when* $x\neq 0$ *and* $t_2>t_1$; *but* $W=0$ *when* $x=0$, *and* $W=0$ *when* $t_2=t_1$.

By application of the second law, and use of theorem I, it can easily be shown that the work W is independent of all circumstances except the given values of the independent x, t_1, t_2, and that it is uniquely determined by these values. Hence, when we take it to be physically obvious that W is a continuous function, the *physical* reasoning employed in studying the operation of any sequence yields the *mathematical* statement:

II. *The work* W *absorbed by any sequence absorbing heat* x *at a temperature* t_1 *and operating between temperatures* t_1 *and* t_2 *is a single-valued continuous function* $W(x, t_1, t_2)$ *of the independent* x, t_1, t_2; *and* $W(x, t_1, t_2)$ *vanishes for* $x=0$, *and for* $t_2=t_1$.

The physical reasoning employed in this section, and also in the following sections 7 and 8, makes use of reversible isothermal transfers of heat to and from an auxiliary body having a constant volume. This is permissible, since we can imagine the employment of a body of such extent that the change of its temperature consequent on a transfer of a given quantity of heat is less than any assignable amount.

7. *The Form of* $W(x, t_1, t_2)$ *in* x. We now seek the form of dependence of W upon x. The operation of any two sequences absorbing the respective arbitrary quantities of heat x and Δx at t_1, and operating between t_1 and t_2, can be coupled with that of a sequence absorbing the heat $x+\Delta x$ at t_1 and operating between the same temperatures. The second law is then easily shown to require

$$W(x, t_1, t_2) + W(\Delta x, t_1, t_2) = W(x + \Delta x, t_1, t_2).$$

The solution of this familiar functional equation is $W=f(t_1, t_2)x$, wherefore $f=W(1, t_1, t_2)$. So the form of W in x is expressed by the identity

$$W(x, t_1, t_2) = W(1, t_1, t_2)x.$$

We thus observe that the ratios W/x and Q/x are single-valued continuous functions of t_1, t_2. This statement involves the *Theorem of Carnot*, that the "efficiency"

$$\frac{W}{Q} = \frac{W(1, t_1, t_2)}{1 + W(1, t_1, t_2)}$$

of a Carnot cycle (more generally, of a sequence) is a function of the temperatures between which the cycle operates.

8. *The Form of* $W(x, t_1, t_2)$ *in* t_1, t_2. It remains to investigate the dependence of $W(1, t_1, t_2)$ on t_1 and t_2. We are to give an arbitrary increment t_3-t_2 to the temperature interval t_2-t_1 through which the sequence operates, whereby the function receives an increment

$$W(1, t_1, t_3) - W(1, t_1, t_2).$$

Consider any sequence absorbing heat 1 at t_1 and operated in an initial stage from t_1 to t_2 followed by a concluding stage from t_2 to t_3, where t_1, t_2, t_3 are arbitrary. Since the heat absorbed at t_2 in the concluding stage is $1+W(1, t_1, t_2)$, it follows that the work-absorption of this stage is $W(1, t_2, t_3)[1+W(1, t_1, t_2)]$. Now the work-absorption of the sequence is equal to the work-absorption $W(1, t_1, t_3)$ of any sequence absorbing heat 1 at t_1 and operating between t_1 and t_3,

$$W(1, t_1, t_3) = W(1, t_1, t_2) + W(1, t_2, t_3)[1 + W(1, t_1, t_2)]. \tag{3}$$

This is a functional equation for W.

For convenience of reference in solving (3), let us assemble the necessary foregoing results.

(A) x and W have the same sign when $x \neq 0$ and $t_2 > t_1$.

(B) $W(x, t_1, t_2)$ is equal to zero for $t_2 = t_1$.

(C) W has the form $W(1, t_1, t_2)x$.

(D) $Q = [1+W(1, t_1, t_2)]x$.

On adding unity to each member of (3), and factoring the second member, we obtain

$$1 + W(1, t_1, t_3) = [1 + W(1, t_1, t_2)][1 + W(1, t_2, t_3)]. \tag{4}$$

This equation is in terms of the quantity $1+W(1, t_i, t_j)$, which by (D) is equal to the single-valued continuous function Q of x, t_i, t_j for $x=1$. If we write $q(t_i, t_j)$ for the function $Q(1, t_i, t_j)$, we define q by the identity

$$q(t_i, t_j) = 1 + W(1, t_i, t_j) ; \tag{5}$$

whereupon the functional equation (4) becomes

$$q(t_1, t_3) = q(t_1, t_2)q(t_2, t_3), \tag{6}$$

which is a functional equation for q.

By (B) we have $W(x, t_1, t_2) = 0$ for $t_2 = t_1$. For $x = 1$, $i = 1$, and $j = 2$, this converts (5) to

$$q(t_1, t_2) = 1, \qquad t_2 = t_1. \tag{7}$$

If now in (6) we put $t_3 = t_1$, and make use of (7), we obtain

$$q(t_1, t_2)q(t_2, t_1) = 1. \tag{8}$$

Interpreted physically, $q(t_2, t_1)$ is the heat developed at t_1 by any sequence absorbing heat 1 at t_2 and operating between t_2 and t_1.

By (8) and (5),

$$q(t_1, t_2) = \frac{1}{q(t_2, t_1)} = \frac{1}{1 + W(1, t_2, t_1)}.$$

When $t_2 < t_1$ we have by (A) that $W(1, t_2, t_1)$, having the sign of $x = 1$, is positive.

Hence

(9a) $$t_2 < t, \qquad 0 < q(t_1, t_2) < 1.$$

When $t_2 = t_1$ we have by (B) that $W(1, t_2, t_1) = 0$. Hence

(9b) $$t_2 = t_1, \qquad q(t_1, t_2) = 1.$$

Finally, when $t_2 > t_1$ we have by (A) that $W(1, t_1, t_2)$ is positive. Hence $q(t_1, t_2) = 1 + W(1, t_1, t_2)$ is greater than unity,

(9c) $$t_2 > t_1, \qquad q(t_1, t_2) > 1.$$

Putting $t_3 = a$ in (6), where a is an arbitrary constant, and then using (8),

(10) $$q(t_1, t_2) = \frac{q(t_1, a)}{q(t_2, a)} = \frac{q(a, t_2)}{q(a, t_1)}.$$

If we define a function $\theta(t)$ by

(11) $$\theta = q(a, t),$$

the equation (10) becomes

(12) $$q(t_1, t_2) = \frac{\theta(t_2)}{\theta(t_1)}.$$

By the equations (9), q is positive. Hence, by (11), θ is positive for any realizable temperature t. Again, by (9c), $q(t_1, t_2) > 1$ for $t_2 > t_1$, wherefore by (12)

$$\frac{\theta(t_2)}{\theta(t_1)} > 1, \qquad t_2 > t_1.$$

Hence θ is an increasing function. Thus θ is a positive, increasing, single-valued, continuous function of t.

The form of the function W is now determined. By (C), (5), and (12), we have

$$W = W(1, t_1, t_2)x = [q(t_1, t_2) - 1]x = \left(\frac{\theta(t_2)}{\theta(t_1)} - 1\right)x.$$

So the form of W is expressed by the identity

(13) $$W(x, t_1, t_2) = \frac{\theta(t_2) - \theta(t_1)}{\theta(t_1)} x.$$

9. *Definition of the Absolute Temperature.* Since $q(t_1, t_2)x = [1 + W(1, t_1, t_2)]x = Q(x, t_1, t_2)$, the equation (12) is

$$\frac{Q}{x} = \frac{\theta(t_2)}{\theta(t_1)}.$$

Since the ratio Q/x is a function of t_1 and t_2, it is employed to determine the

ratio $\theta(t_2)/\theta(t_1)$ of the "absolute" temperatures at which the transfers of heat Q and x occur. A definition of the absolute temperature $\theta(t)$ is now obtained if we let t_1 be a fixed temperature of reference t_r, for which the freezing temperature of water is chosen, and let t_2 be the general temperature t. Hereupon (12) becomes

$$q(t_r, t) = \frac{\theta(t)}{\theta(t_r)} \cdot \tag{14}$$

To fix the value of the arbitrary constant $\theta(t_r)$, the absolute temperature of freezing water, we assign Δ units of absolute temperature to a fixed interval of temperatures, for which the interval from t_r to the boiling temperature t_s of water is chosen. Setting

$$\theta(t_s) - \theta(t_r) = \Delta,$$

and writing (12) for this temperature interval, we obtain

$$q(t_r, t_s) = 1 + W(1, t_r, t_s) = \frac{\theta(t_s)}{\theta(t_r)} \cdot$$

On writing W_{rs} for the constant $W(1, t_r, t_s)$, and eliminating $\theta(t_s)$, we get

$$\theta(t_r) = \frac{\Delta}{W_{rs}} \cdot$$

Substituting this value in (14), and replacing $q(t_r, t)$ by its value $1 + W(1, t_r, t)$ we find

$$\theta(t) = \frac{\Delta}{W_{rs}}[1 + W(1, t_r, t)].$$

This equation defines the absolute temperature $\theta(t)$ as a linear function of the work absorbed by any sequence that absorbs one unit of heat at the temperature of reference t_r and operates from t_r to the general realizable temperature t. It is a positive, increasing, single-valued, continuous function of t.

10. *A Theorem about an Auxiliary Body of Gas.* On multiplying the first member of (12) above and below by x, and writing Q_1 for the heat x *absorbed* at t_1 and Q_2 for the heat $-Q$ *absorbed* at t_2 by any Carnot cycle operating between t_1 and t_2, we find that Q_1, Q_2 and the corresponding temperatures are connected by the relation

$$\frac{Q_1}{\theta(t_1)} + \frac{Q_2}{\theta(t_2)} = 0. \tag{15}$$

We now assume that the realizable aeriform states of equilibrium of a body of hydrogen are in continuous one-to-one correspondence with the points of a region of the V, t plane, where V and t are the volume and temperature of the

body; that the uniform and normally directed pressure p and the energy E of the body are single-valued continuous functions of V, t, and possess continuous derivatives,

$$\frac{\partial p}{\partial t}, \qquad \frac{\partial E}{\partial V}, \qquad \frac{\partial E}{\partial t}, \qquad \frac{\partial^2 E}{\partial V \partial t};$$

and that the body absorbs or develops work only under the action of the pressure p. And we assume that, within any area of the region in the V, t plane, it is always possible to describe a Carnot cycle.

The work added to the body in any reversible change of its state from any state a to any other state b is the line integral $W_{ab} = -\int p dV$, taken along the path of the change of state. From the energy law

$$E_b - E_a = W_{ab} + Q_{ab},$$

we thus find that the heat Q_{ab} absorbed on the path is

$$E_b - E_a + \int p dV,$$

which is the corresponding line integral of the inexact differential expression $dE + p dV$, or

$$\left(\frac{\partial E}{\partial V} + p\right) dV + \frac{\partial E}{\partial t} dt. \tag{16}$$

Hence a formulation of (15) for any Carnot cycle described within the given region R of states is

$$\int_{cc} \left[\frac{1}{\theta}\left(\frac{\partial E}{\partial V} + p\right) dV + \frac{1}{\theta} \frac{\partial E}{\partial t} dt\right] = 0, \tag{17}$$

where the double subscript indicates that the line integral is taken along the contour of any Carnot cycle in the region.

Now let us consider this line integral, taken along *any* closed contour C in the region R. On transforming the contour integral into a surface integral extended over the enclosed area, we obtain

$$\begin{aligned} &\int_{c} \left[\frac{1}{\theta}\left(\frac{\partial E}{\partial V} + p\right) dV + \frac{1}{\theta} \frac{\partial E}{\partial t} dt\right] \\ &= \iint \frac{1}{\theta}\left[\left(\frac{\partial E}{\partial V} + p\right)\frac{1}{\theta} \frac{d\theta}{dt} - \frac{\partial p}{\partial t}\right] dV dt. \end{aligned} \tag{18}$$

If the integrand of the surface integral were positive (or negative) at all points within any subregion R_1, however small, the surface integral extended over the area enclosed by any contour within R_1 would be positive (or negative), as therefore would be the line integral taken along the contour. But this

is impossible, since the contour may be that of a Carnot cycle, for which by (17) the line integral is equal to zero. And the integrand cannot be positive (or negative) at all points of a line segment, or at isolated points, since it is continuous.

Hence the integrand of the surface integral is equal to zero at *all* points in the specified field of aeriform states; and thus the surface integral extended over any finite area is equal to zero; wherefore the line integral taken along the contour C of the area is equal to zero. If we write DQ for the heat-element (16) for the region, we find

$$\int_C \frac{DQ}{\theta(t)} = 0. \tag{19}$$

11. *The Scale of Absolute Temperatures.* The fact that the integrand of the surface integral in (18) is everywhere equal to zero affords a differential equation for the absolute temperature $\theta(t)$. This integrand contains the quantity $\partial E/\partial V$, which we can express in terms of the experimentally measureable heat capacity $\partial E/\partial t$ and the "free expansion effect" $(\partial t/\partial V)_E$,

$$\frac{\partial E}{\partial V} = -\left(\frac{\partial E}{\partial t}\right)_V\left(\frac{\partial t}{\partial V}\right)_E .$$

But the appearance of the free expansion effect is unsatisfactory, since measurements of this quantity cannot be made with much accuracy. More accurate equivalent data can be obtained by measuring the changes of temperature of a thermometric gas on expansion against reduced opposing pressures, thus formulating the "throttled expansion effect,"

$$\left(\frac{\partial t}{\partial p}\right)_{E+pV} .$$

To introduce this quantity we repeat the calculation leading to (18), taking p, t independent instead of V, t, and finding

$$\left(\frac{\partial E}{\partial p} + p\frac{\partial V}{\partial p}\right)\frac{1}{\theta}\,\frac{d\theta}{dt} + \frac{\partial V}{\partial t} = 0. \tag{20}$$

To formulate the throttled expansion effect, we differentiate the function $E+pV$,

$$d(E + pV) = \left(\frac{\partial E}{\partial p} + p\frac{\partial V}{\partial p} + V\right)dp + \left(\frac{\partial E}{\partial t} + p\frac{\partial V}{\partial t}\right)dt.$$

Since $dE+pdV$ is the heat-element of the gas, the term in dt is the heat absorbed per degree for constant p — it is the specific heat c_p of the gas for constant pressure. It follows that

$$\frac{\partial E}{\partial p} + p\frac{\partial V}{\partial p} = -\,c_p\left(\frac{\partial t}{\partial p}\right)_{E+pV} - V,$$

which converts (20) into

$$(21)\qquad \frac{1}{\theta}\frac{d\theta}{dt} = \frac{\dfrac{\partial V}{\partial t}}{c_p\left(\dfrac{\partial t}{\partial p}\right)_{E+pV} + V}.$$

This equation can be integrated when the forms of V and c_p and the throttled expansion effect shall have been determined by experiment. For the temperature of an arbitrary scale we select the temperature T of the centigrade scale of the standard hydrogen thermometer. It is a fact of observation that, for any relatively low constant pressure p, the volume V of a body of hydrogen is closely proportional to the quantity $T+273$,

$$V = \phi(p)\cdot(T + 273).$$

If we employ this approximation, and neglect the throttled expansion effect, which is small, integration of (21) yields

$$\theta = c(T + 273).$$

Let T_r and T_s be the ice point and the boiling point. Then

$$\theta(T_s) - \theta(T_r) = c(T_s - T_r) ;$$

that is, $100 = c.100$, or $c = 1$. Hence, very approximately,

$$\theta = T + 273.$$

12. *The Entropy Law.* It is now desired to show that the quantity of heat absorbed by any body B, on any reversible path in any given region R_i of the realizable states of equilibrium of B, is equal to the corresponding line integral of a linear differential expression in the independent variables $x_{i1}, x_{i2}, \cdots, x_{in}$ that determine the states of R_i. We assume that the body is in equilibrium under an imposed uniform temperature and under a finite number of imposed forces; that the work absorbed by the body on any reversible path of change of its state can be considered to be developed by a mechanism M connected with the body; and that this work is equal to the corresponding line integral of a linear expression,

$$DW_i = p_{i1}dx_{i1} + p_{i2}dx_{i2} + \cdots + p_{in}dx_{in},$$

in which the p_{ij} are continuous functions of the x_{ij}, and which in general is inexact. It then follows, by the energy law $DQ_i = dE_i - DW_i$, that the heat-element DQ_i for the region is a similar linear expression

$$DQ_i = q_{i1}dx_{i1} + q_{i2}dx_{i2} + \cdots + q_{in}dx_{in}.$$

Let *any* body B undergo *any* possible reversible cyclic change of thermodynamic state, isothermal or not, the process being so conducted that all

work transferred to or from B is exchanged with a mechanism M, and all heat transferred is exchanged with a body G of hydrogen gas in thermal equilibrium with B; while all work transferred to or from G is exchanged with M. At the conclusion of the cycle the body B has regained its initial state and energy, while G has regained its initial temperature, and either has or has not regained its initial volume and hence its initial energy.

Suppose G has not regained its initial volume. On returning G to its initial state by reversible isothermal expansion or compression, the gas exchanges work with M and absorbs heat q from an outside body H, where q is either positive, negative, or zero. If q is positive, the change of state of M is a net absorption of work q, compensated by the isothermal development of heat by H.[3] This violates the second law. If q is negative, the change of state of M is a net development of work q, which compensates the isothermal absorption of heat by H. A reversal of the whole set of operations would cause M to absorb work q at the expense of an isothermal development of heat by H, thus again violating the second law. So $q=0$.

In general the described cyclic path of the reversible change of state of B traverses successive regions $R_a, R_b, \cdots, R_a$ of states of B. At every instant of the process the temperatures of B and G are equal, and the heat-element DQ_i absorbed by B is equal to the negative of the heat-element DQ_g absorbed by G. By integration along the entire path, therefore, we have that

$$\int \frac{DQ_a}{\theta} + \int \frac{DQ_b}{\theta} + \cdots + \int \frac{DQ_a}{\theta} = - \int_C \frac{DQ_g}{\theta},$$

where each term of the sum of line integrals is taken along the path-segment lying in the corresponding region. But, by (19), the second member of this equation is equal to zero. For G either regains its initial state when B does, or it can be restored to this state along an isothermal path on which it absorbs heat $q=0$. So we find

$$\int \frac{DQ_a}{\theta} + \int \frac{DQ_b}{\theta} + \cdots + \int \frac{DQ_a}{\theta} = 0.$$

Here θ is a function of t, which is a function of the regional variables and may in particular be one of these variables or be a constant. Selecting a state of reference, in a region R_0, specified by the set ρ of the values of the regional variables there, we define the function S_i at the general state σ of the general region R_i by integration along any reversible path of the change of state $\rho\sigma$,

$$S_i = \int_\rho \frac{DQ_0}{\theta} + \cdots + \int^\sigma \frac{DQ_i}{\theta}. \tag{22}$$

For the quantity so defined is independent of the reversible path between ρ

[3] For the energy of the body composed of B, G, and M is increased by q; wherefore the energy of M is increased by q, since the energies of B and G have their initial values.

and σ. The quantity S_i is termed the *entropy* of the body for the region R_i. On crossing the boundary separating a region R_i from a region R_j, the entropy S_i is succeeded by the entropy S_j, which in general is a function of a different set of independent variables and has a different analytical form. By (22), the heat-element DQ_i is expressed in terms of the two functions θ and S_i,

$$DQ_i = \theta dS_i.$$

Reversible adiabatic transfers of work, since they occur at constant entropy, are termed *isentropic*. Irreversible adiabatic changes of state are not isentropic.

It appears that the function $1/\theta(t)$ is an integrating factor of the heat-element DQ_i for any non-isothermal region of states of equilibrium. Now, the ratio of any two integrating factors contains all the independent variables. When t is one of the regional variables, θ^{-1} is a function of this variable, and it follows that any other integrating factor contains the other variables and may contain all the variables. Hence the absolute temperature $\theta(t)$ is determined by the circumstance that it is the only integrating divisor of DQ_i that is a function of the temperature alone.

13. *The Dissipation Law.* The *reversible* processes considered in the foregoing, being continuous sets of states of equilibrium, are not realizable changes of state. Actually occurring changes of thermodynamic state proceed on *irreversible* paths. And they proceed in definite *directions*. Thus a released stone falls to the ground; a mechanism expending work against friction effects a heating; the temperatures of contiguous bodies approach a common value; a body of gas expands abruptly into a vacuum, or is compressed by the fall of a heavy piston; sugar diffuses from an aqueous solution into a supernatant layer of water; a zinc rod dissolves in aqueous acid.

If these processes are conducted under control, they develop work against the controlling forces. The falling stone raises a weight; the mechanism compresses a spring; an air engine operates through absorbing heat from the hot body and transmitting heat to the cold one; the expanding gas develops work against a confining piston; the falling piston raises a weight; the sugar develops work against a semipermeable piston separating the solution from the water; the dissolving zinc develops an electric current that drives a motor.

To judge from these examples, it would seem that any actual thermodynamic process can be utilized for the production of work. When it is conducted under complete control it is conducted reversibly, and it develops the maximum work possible. The irreversible process then develops the least work possible, it follows the path of least resistance.

The validity of this idea may be tested by studying a diversity of irreversible processes. An extensive class of changes of state that admit reversible paths includes the destruction of work by impact or friction, and it includes heat conduction, expansion or compression of elastic bodies, diffusion, and some chemical reactions. When any change of state of this class is conducted under control it develops work against the controlling forces, wherefore the energy

lost as work must be restored as heat when the change of state is completed on a reversible path. Since this addition of heat increases the entropy of the set of bodies undergoing the process, it follows that the irreversible change proceeds in the direction permitting the greatest possible increase of the entropy of the set. This is merely a way of saying that the irreversible process follows the path of least resistance, or of the greatest possible avoidance of doing work. It is a way of saying that work which the process could be compelled to develop is dissipated in producing a heating effect. When a cooling is produced, it is less than that which occurs when the change of state is conducted reversibly.

The inductively reached conclusion, that actual thermodynamic processes are dissipative, will here be termed the *dissipation law*:

In any actual change of state of a dynamically and thermally isolated body, the change of the entropy of the body is positive.

When an irreversible process is not known to admit a reversible path, the conclusion stated cannot be drawn. For the process cannot be shown to be dissipative if work cannot be gained by conducting it under control; and nothing is known of the change of entropy, since this is defined only with reference to a reversible path. That such processes are dissipative is generally believed; but it is not clear that conclusive grounds for this belief can be given.

The term "second law of thermodynamics" is often employed to mean the dissipation law.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AN ALIGNMENT CHART FOR VARIOUS MEANS

By DAVID MOSKOVITZ, Carnegie Institute of Technology

Our purpose is to discuss a type of nomographic chart which seems to be peculiarly adapted for the determination of various means of two numbers. We desire a chart of the form described below.

Let $w=f(u, v)$ be a positive function of the two positive variables u and v, which are assumed to have the same range; and let $f(u, v)$ possess the property that

$$f(u, v) = v\cdot f(y, 1), \text{ where } y = u/v,$$

so that $f(u, v)$ is a homogeneous function of the first degree. We impose the further condition on $f(u, v)$ that

$$f(a, a) = a.$$

We desire to obtain the equation of a curve $x = \pm F(y)$, symmetric about the

y-axis such that when the points P and Q (one of which is to the left and the other to the right of the y-axis), whose ordinates are u and v are joined by a straight line, this line shall meet the y-axis at the point whose distance from the origin is given by $w=f(u, v)$.

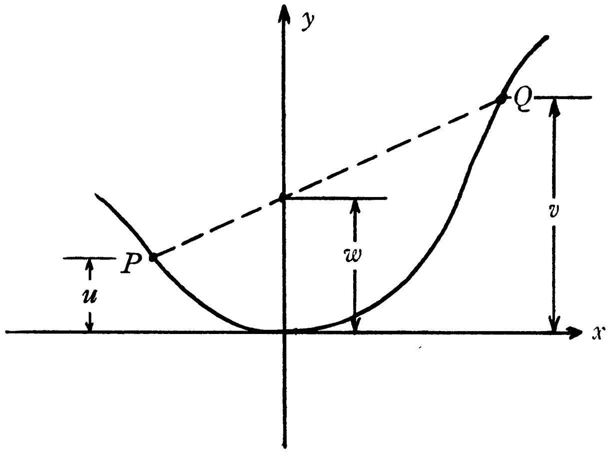


FIG. 1

Let the coordinates of P and Q be $(-x', y')$ and (x'', y''), respectively, where x', x'', y', y'' are positive. The equation of the line joining P and Q is

$$(y - y')(x'' + x') = (y'' - y')(x + x')$$

and the intercept of this line on the y-axis is given by

$$Y = (x''y' + x'y'')/(x'' + x')$$

Now let, $y'=u$, $y''=v$, $x'/x''=x$, and hence $y'/y''=u/v=y$; then

$$Y = y''\cdot\frac{(y'/y'') + (x'/x'')}{1 + (x'/x'')} = v\cdot\frac{y + x}{1 + x}.$$

But we desire Y to be equal to $w=f(u, v)=v\cdot f(y, 1)$; hence

$$v\cdot\frac{y + x}{1 + x} = v\cdot f(y, 1),$$

from which on solving for x, we find

$$x = \frac{f(y, 1) - y}{1 - f(y, 1)},$$

and this is the equation of the desired curve for our nomographic chart. Here y is a variable which ranges from 0 to 1. By assigning values to y ranging from 0 to 1 and computing the corresponding values of x, we obtain points which are to the right of the y-axis. The curve is symmetrical about the y-axis and hence points to the left of the y-axis can be located.

The choices of scales on the x and y-axis are independent of each other; any scale may be used on either axis. The chart so constructed can be used for any range of values of u and v, since the ordinate $y=1$ may be made to correspond to the largest permissible value of u or v. If the chart is to be used for a function in which the largest value of u or v is to be A, then the ordinate $y=t$ will correspond to $v=tA$. We illustrate with the following applications to various means. We have in each case taken the range of values of u and v to be 0 to 10.

(1) Arithmetic mean:

$$w = f(u, v) = (u + v)/2.$$

The required curve is given by

$$x = \frac{(y+1)/2 - y}{1-(y+1)/2} = \frac{(1-y)/2}{(1-y)/2} = 1.$$

(2) Geometric mean:

$$w = f(u, v) = \sqrt{uv}.$$

The required curve is given by

$$x = \frac{\sqrt{y} - y}{1 - \sqrt{y}} = \sqrt{y}.$$

(3) Harmonic mean:

$$w = f(u, v) = 2uv/(u + v).$$

The required curve is given by

$$x = \frac{2y/(y+1) - y}{1 - 2y/(y+1)} = \frac{y - y^2}{1 - y} = y.$$

(4) Heronian mean:

$$w = f(u, v) = (u + \sqrt{uv} + v)/3.$$

The required curve is given by

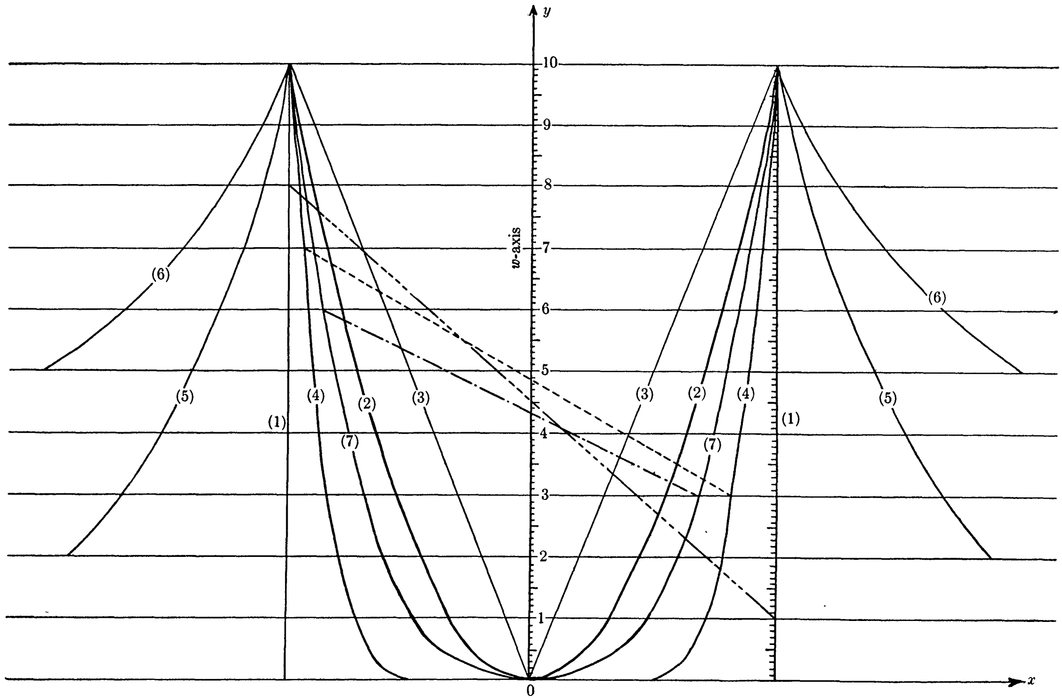
$$x = \frac{(y + \sqrt{y} + 1)/3 - y}{1 - (y + \sqrt{y} + 1)/3} = \frac{\sqrt{y} + 1 - 2y}{2 - y - \sqrt{y}} = \frac{1 + 2\sqrt{y}}{2 + \sqrt{y}}.$$

(5) Root mean-square:

$$w = f(u, v) = \sqrt{(u^2 + v^2)/2}.$$

The required curve is given by

$$x = \frac{\sqrt{(y^2+1)/2} - y}{1 - \sqrt{(y^2+1)/2}}.$$



(6) Contraharmonic mean:

$$w = f(u, v) = (u^2 + v^2)/(u + v).$$

The required curve is given by

$$x = \frac{(y^2 + 1)/(y + 1) - y}{1 - (y^2 + 1)/(y + 1)} = \frac{1 - y}{y - y^2} = 1/y.$$

(7) Logarithmic mean:

$$w = f(u, v) = \frac{u - v}{\ln u - \ln v}. \qquad [\ln u = \log_e u]$$

The required curve is given by

$$x = \frac{(y - 1)/(\ln y) - y}{1 - (y - 1)/(\ln y)} = \frac{y - 1 - y \ln y}{\ln y - y + 1}.$$

It is to be noted that in the equations of the curves of (5) and (7), the value of x is indeterminate when $y = 1$; but in each of these $x \to 1$ as $y \to 1$.

The accompanying diagram combines the above seven cases. To use the chart, we will illustrate by finding the Heronian mean of the numbers 3 and 7. The curve (4) is the one to be used for the Heronian mean of two numbers.

Join the point on the right hand branch of curve (4) whose ordinate is 3 to the point on the left hand branch of curve (4) whose ordinate is 7. The dotted line joining these points crosses the y-axis at the point for which $w=4.86$.

As has been mentioned earlier, although the accompanying chart shows the range of values of u and v to be 10, yet the same chart can be used for any other range. For example, to find the logarithmic mean of 9 and 18, we multiply the existing scale by 3 so that the chart now has a range of 30. The dotted and dashed index line on the chart between the right and left branches of curve (7) gives the required reading, crossing the y-axis at the point for which $w=3(4.33)$ $=12.99$.

A familiar fact which can be easily seen from this chart is that for given values of u and v, for which $u \neq v$, of the seven cases of means considered, the harmonic mean is the smallest mean, while the largest mean of u and v is the contraharmonic mean. The order of magnitude of the various means is the order in which the curves lie between the curve for the harmonic mean and the curve for the contraharmonic mean. We can thus see that

$$\text{harmonic} < \text{geometric} < \text{logarithmic} < \text{Heronian} < \text{arithmetic} \\ < \text{root mean-square} < \text{contraharmonic}.$$

The third index line on the chart gives the arithmetic mean of 1 and 8 to be 4.5.

AN UNUSUAL SPIRAL

By L. S. JOHNSTON, University of Detroit

We discuss in this note a spiral differing from the more familiar spirals in that its whorls all have a common point, and that all its whorls have a common tangent at this common point. The writer encountered the spiral while investigating a problem submitted to him by a member of the designing staff of a company manufacturing reflectors.[1] This problem may be stated as follows:

Using the conventional polar coordinate net, let $(a, \pi/2)$ be the coordinates of a given point P_0. Let the perpendicular to OP_0 at P_0 intersect the line $\theta=\pi/4$ at P_1, the perpendicular to OP_1 at P_1 intersect the line $\theta=\pi/8$ at P_2, etc., thus establishing the sequence $[P_n]$ of points $P_0, P_1, P_2, \cdots$. We wish to find

(a) the limit point P_∞ of $[P_n]$,

(b) a continuous function $\rho=f(\theta)$ which includes $[P_n]$ and P_∞, and

(c) an elementary geometric construction by which other points on $\rho=f(\theta)$ may be located.

We discuss these parts in order.

(a) Let $OP_i = \rho_i$ $(i = 0, 1, 2, \cdots)$ and $\theta_i =$ angle P_iOX,

OX being the line $\theta=0$. We then have

[1] See "Cochleoid", Encyclopaedia Britannica (14th edition), vol. 6, p. 894. Also solution of problem E41, this issue of the MONTHLY, p. 609.

$$\rho_0 = a, \quad \rho_1 = a \sec\frac{\pi}{4}, \quad \rho_2 = a \sec\frac{\pi}{4}\sec\frac{\pi}{8}, \cdots$$

(1)

$$\rho_n = a \sec\frac{\pi}{4}\sec\frac{\pi}{8}\sec\frac{\pi}{16}\cdots\sec\frac{\pi}{2^{n+1}}, \text{ and}$$

(2) $$\theta_0 = \frac{\pi}{2}, \quad \theta_1 = \frac{\pi}{4}, \quad \theta_2 = \frac{\pi}{8}, \cdots, \theta_n = \frac{\pi}{2^{n+1}}.$$

From the identity $\sec\theta = 2\sin\theta/\sin 2\theta$ we have

$$\rho_n = 2^n a \sin\frac{\pi}{2^{n+1}} = \frac{\pi a}{2}\left\{\frac{\sin\dfrac{\pi}{2^{n+1}}}{\dfrac{\pi}{2^{n+1}}}\right\}.$$

Hence

$$\rho_\infty = \lim_{n\to\infty}\rho_n = \frac{\pi a}{2}.$$

Also, since

$$\lim_{n\to\infty}\theta_n = 0,$$

we have $(\pi a/2, 0)$ as the coordinates of the point P_∞.

It may be remarked here that this furnishes an excellent and convenient method of finding the approximate circumference of a circle of given material radius, for OP_4 differs from the true length of the quadrant by less than one six hundredth part of the quadrant.

(b) We note that

$$\rho_n = \frac{\pi a}{2}\frac{\sin\theta_n}{\theta_n}$$

when n is finite, and that

$$\lim_{n\to\infty}\rho_n = \frac{\pi a}{2}.$$

Hence the function

$$\rho = \frac{\pi a}{2}\frac{\sin\theta}{\theta} \qquad (\theta \neq 0)$$

$$\rho = \frac{\pi a}{2} \qquad (\theta = 0)$$

is completely defined and continuous at every point, and is a function of the kind required.

(c) In the function derived above let $\theta = p\pi$, p being any number, integral or fractional, such that $p\pi$ can be constructed by elementary geometry. Then $\rho = (a\sin p\pi)/2p$. Since $\sin p\pi$ can be constructed by elementary geometry, we may find by elementary geometry the value of ρ for any admissible value of p.

It is evident that as θ increases through positive values the point (ρ, θ) traces a spiral in the first and second quadrants, passing through the origin for every value $\theta = n\pi$, n being integral. Furthermore it is easily shown that the horizontal axis is tangent to the curve at the origin. The curve is symmetric with respect to horizontal axis; hence there are two sets of whorls tangent to each other at the origin.

Some interesting properties of this spiral may be mentioned.

(1) the chord joining (ρ_1, α_1) and (ρ_2, α_2), where $\alpha_2 = 2\alpha_1$, is perpendicular to the line $\theta = \alpha_2$. Hence if we locate any point (ρ, α) on the curve, we may establish a sequence of points on the curve in a manner exactly like that by which $[P_n]$ was established. Every such sequence has $(\pi a/2, 0)$ as the limit point.

(2) The tangent to the curve at any point (ρ, α) intersects the line $\theta = 2\alpha$ on the circle $\rho = \pi a/2$. Hence if any line $\theta = \alpha$ be drawn through the origin, all the tangents to the curve at the points of intersection of this line and the curve pass through the same point, $(\pi a/2, 2\alpha)$.

(3) Let any line $\theta = \alpha$ intersect the successive whorls of the spiral, reading inward from the outermost intersection, at $R_0, R_1, R_2, \cdots$. Then $OR_0, OR_1, OR_2, \cdots$, form a harmonic sequence. This property is, of course, the same as that of the reciprocal spiral.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Introductory Mathematics by J. W. Lasley and E. T. Browne. New York, McGraw-Hill Book Co., 1933. xv+439 pages. \$2.75.

This book possesses many features that should commend it to those who believe in courses in generalized mathematics. It is, as it purports to be, a thorough introduction to algebra, trigonometry, and the calculus, and not merely a "survey" of these fields.

Perhaps the most striking feature is the admirably clear exposition of the theory. The treatment of Chapter IV on the "rate of change of functions"—leading up to the notion of the derivative—is especially commendable. Students will like the book because of the completeness of the discussion where theoretical difficulties are involved.

Part I—five chapters, 190 pages—deals with simple algebraic functions from the viewpoint of algebra, analytic geometry and differential and integral calculus. The algebra covers rather thoroughly much of the field usually included in College Algebra, including determinants of the second and third orders and the solution of numerical equations. In analytics only single-valued integral functions are considered so that no mention is made, for example, of the ellipse and the hyperbola. The calculus and its applications are thoroughly covered,

with all discussion based on algebraic functions solely, an excellent feature in a brief first course.

Part II—three chapters, 150 pages— is concerned with exponential, logarithmic and trigonometric functions, their derivatives and their applications. The solution of triangles is covered with sufficient detail.

The book is well printed and bound. There are plenty of exercises and such notions as momentum, work, etc. are carefully explained. The historical notes should also prove interesting.

EDWARD FLEISHER

The Theory of Spherical and Ellipsoidal Harmonics. By E. W. Hobson. Cambridge, University Press, 1931. xi+500 pages.

Partial Differential Equations of Mathematical Physics. By H. Bateman. Cambridge, University Press; New York, The Macmillan Co., 1932. xxii+522 pp.

These are both works of the "encyclopedic" type, falling in the general class of which we have had previously outstanding examples in Hobson's *Theory of Functions of a Real Variable* and Watson's *Bessel Functions*. Books of this type are extremely useful as reference works, both to those who are pursuing research in the given field or related fields, and to those who already have some acquaintance with the theory and wish to widen their knowledge in that direction. Neither is to be recommended primarily as an introduction to the field with which it deals.

It is inevitable that in a book of this encyclopedic type the exposition should suffer somewhat. In the first place, it is not humanly possible to cover so much ground, including researches that are still in process of development and thus lacking in completeness, and at the same time achieve that essential unity of treatment which stamps a mathematical treatise as a genuine work of art. Furthermore, it is rather natural that the tremendous labor of compiling a work dealing with such a mass of literature should leave the author in a less critical frame of mind with regard to details of exposition. Both works exhibit some of the defects here indicated, but these detract only in a minor way from their primary utility as specified above.

Both books deal with mathematical theories whose origin is to be found in problems of mathematical physics, and particularly in boundary value problems in connection with partial differential equations of physical significance. The work of the late Professor Hobson, as its title indicates, restricts itself to a comprehensive discussion of spherical and ellipsoidal harmonics, and the prevailing point of view in the exposition is that of the pure mathematician. Professor Bateman's book deals in part with the same material, necessarily more briefly, but treats also many other mathematical theories arising in connection with boundary value problems. Moreover, the form of exposition is oriented more toward the point of view of the mathematical physicist. For these reasons Hobson's book exhibits greater unity and would present less difficulty to a reader studying the field for the first time.

Among the important topics dealt with in Bateman's book, beyond those also treated by Hobson, may be mentioned the following: Fourier series, Green's theorem, Stokes's theorem, Green's function for various domains, Poisson's integral, conformal mapping, and Bessel functions. These topics, and others, are treated both in such of their theoretical aspects as lie closest to the physical applications, and also in connection with numerous particular applications. For a complete list of topics the reader is referred to the table of contents, which covers six pages in small type.

In conclusion let us reemphasize the point that the two authors have produced books which will be of great service to a large group of mathematicians and mathematical physicists, and that they have necessarily done this at the cost of much time and labor to themselves. They have earned the gratitude of the group in question and they deserve much credit for their achievement.

C. N. MOORE

Plane Trigonometry. By J. A. Northcott. New York, Ray Long and Richard R. Smith, Inc., 1933. x+152 pages. $1.60. Bound with Five-place Tables of E. S. Crawley, $2.40. Tables alone (F. S. Crofts Company), $1.00.

Plane Trigonometry. By W. L. Hart. New York, D. C. Heath and Company, 1933. vi+186+16 pages. $1.68. Bound with Three-, Four-, and Five-place Tables, $2.00. Tables alone, $1.32.

Plane Trigonometry. By C. A. Ewing. New York, McGraw-Hill Book Company, Inc., 1933. xii+165 pages. $1.20. Bound with Five-place Tables of R. D. Beetle, $1.60. Tables alone, $0.80.

These three texts lend themselves to joint review, for they present sufficient variety in attitude to afford interesting comparisons. Northcott "believes that analysis is the central theme" and that "computation should play only a subsidiary rôle." He even goes so far as to suggest that the computational theme might find ample room for development in a single concluding chapter. Hart "gives full recognition to both the numerical and the analytical sides of the subject," but is so bent on providing a flexible text that the two main themes are all but obscured. Ewing, anxious to meet the requirements of the College Entrance Examination Board, is well aware of the importance of both computation and analysis; but, for him, this means division of the composition into two distinct sections: Part One on the Solution of Triangles and Part Two on Trigonometric Analysis. Since Northcott and Ewing put computation and analysis in separate compartments, it is Hart who offers perhaps the best basis for an elastic presentation which aims at effective interplay of themes.

In Northcott, "a conscious effort has been made to keep the treatment brief." We find spots, however, where the discussion seems concise to the point of inadequacy. We mention the directions for computing with approximate data (page 17), and the explanation of limiting values of the functions (page 39). And it is difficult to understand the absence of the $\pm$ sign in the half-angle for-

mulas (page 64). We gain the impression that brevity has been secured through omission of explanatory material rather than through judicious compression. Yet we recognize the author's feeling for exposition and realize the unity he obtains by stressing the analytic aspect.

Hart starts with the acute angle but assures the reader that it is equally feasible to begin with the general angle. We are presented, throughout, with four- or five-place tables, and with choice of odd- or even-numbered problems. And the instructor, in graphing the functions, may measure angles in radians and take the unit the same on both axes (as does Northcott), or he may use degree measure and (with Ewing) make arbitrary selection of scale on the x-axis. There is never obtruded the question of right or wrong; since the intention is to please the teacher, a preference is no sooner implied than it is followed by an alternative procedure. In short, flexibility is carried to extremes; and the result is a heterogeneous presentation which of necessity lacks backbone. But a host of teachers will welcome the wealth of detailed material and be won by the freedom it affords.

Ewing's scheme—of opening with the acute angle and sticking to computation until it is disposed of—is carefully worked out and will appeal to those who favor this approach. Others, who like to plunge at once into the general angle, will turn to Hart, or to Northcott, according as they wish, or do not wish, to alternate theory with computation. Ewing deals satisfactorily with identities. But Hart is misleading in his rating of methods of proof, and Northcott considers solely the problem of transforming one expression into another, as if this were the only useful notion to be retained in identities. Also, Ewing differs from the others in that he concludes, not with a list of answers, but with a collection of recent examinations set by the College Board.

On certain points, it is especially interesting to make comparison of the three texts. Take, for example, the question of check formulas to be used in the logarithmic solution of oblique triangles. Ewing and Northcott advisedly employ the Mollweide equations, but without mention of other possible check formulas. Hart, after sounding appropriate warnings against the use of the Law of Sines as a check in Case I, makes no comment when he uses this law to check Case III. We should like to see the available check formulas more fully discussed both as to efficacy and inefficacy.

Another topic where comparisons are illuminating is that of computation. We have already expressed regret at the little Northcott gives us on computation with approximate values. Hart has more to say in this connection, but he does not prepare the ground for even a formally adequate treatment of the subject. It is little comfort to the student to be told (page 47) that "the degree of accuracy of the data and the nature of a problem must be carefully considered in deciding what places are *significant* in the results." And only a dearth of precise methods can explain the author's counsel (page 48) to round off "to a number of significant places which appears reasonably justified by the data."

Ewing contributes the maximum of detail on the subject of computaton. We

find explanations of abbreviated multiplication and division, of logarithms, of the slide rule, and some mention of computing machines. But, again, the rules are insufficient for the needs of practice. There is evidence that the vital point has been missed. Witness the attitude (page 4) toward absolute error and relative error. Moreover, as the treatment proceeds, we note less and less emphasis on rigor in computation. On page 98, conversion factors are given to the same number of decimal places, as if they were to enter additively. And we sense the same difficulty that we met in Hart when, at the very close of Part I in Ewing, we read: "the student should use his judgment as to the degree of accuracy of his results."

With such meager explanations as are found in the three texts under review, it seems clear that in questions of accuracy of computed results the student is left to his own feeble resources. So far as analysis is concerned, our authors meet the immediate requirements of the student; but, in the matter of computation, these texts resemble the majority of their predecessors in that they virtually ask the student to mark time. Yet, if training in computation is to serve any end, practical or theoretical, there can be no room for conjecture; instruction in this topic must provide intelligible and workable rules for obtaining the desired results from the given data.

As trigonometries go, these are all good texts. But it would hardly be honest appraisal to dismiss them with the observation that each has well-graded problems, neat figures—and an index. It should be pointed out that these are texts of the conventional type, and, therefore, teachable only in the sense that the teacher must supply the spark. And the matter becomes one of grave concern in that the formal instruction we deprecate continues often through Analytics and the Calculus. Thus is defeated the first object, which, as the reviewer sees it, is to develop in the student power: power to discern, to discriminate, and to dig deeply. We move toward this goal, not by furnishing the student a collection of rules and formulas, but rather by encouraging him to ask why and by helping him acquire the ability to answer his queries for himself.

C. A. GARABEDIAN

Mathematical Excursions. By Helen A. Merrill. Norwood, Mass., The Norwood Press, 1933. xi+143 pages.

This well named book is a very diverting volume, and most successful in carrying out its plan to be easily understood by those who are not learned. The variety of topics gives scope for manifold interests, and the very evident joy with which the author collected her illustrative material from near and far, and wrote the book, is conveyed to the reader throughout. To know how to do "multiplying without multiplication tables," to understand "how circulating decimals circulate," and to be able to write magic squares, for instance, will fascinate the young reader, as well as give pleasure to those already initiated.

A large part of the book is developed from number theory, and follows the lines of analysis. About a quarter of it, however, has material using algebra and

geometry in combination. It is especially interesting on "geometrical arithmetic" and on linkages. Under the chapter on different ways of writing numbers, many operations apparently difficult are divested of their camouflage and shown as simple applications of operations with dyadic numbers. The same direct simplicity characterizes the chapter on multiplying, one of whose attractions is its exhibition of the Russian Peasants' Method of multiplication. The chapter on incommensurable numbers, and that on π offer aid to the beginner who is often puzzled by those members of the number family. In the final chapter the statements of some of the famous unsolvable problems are given. Throughout the book many problems are solved, and one finds ample explanatory material. At the same time the author states, for the reader, a large collection of problems to be solved. For some of these the answers are listed at the end of the book.

The material, treatment, and the arrangement of this book make it a very suitable one for undergraduate mathematical clubs, as well as for pleasurable individual "excursions."

M. E. WELLS

Die Dreiteilung des Winkels. By Walter Breidenbach. Mathematisch-Physikalische Bibliothek, Reihe I, Band 78. Leipzig and Berlin, Teubner, 1933. iv+38 pages. RM 1.20.

The quadrature of the circle and the duplication of the cube have been considered by Beutel and by Herrmann in numbers 12 and 68 of this series. In this brochure the author brings out very clearly the degree of difficulty in the trisection of the angle. The proof of the impossibility of a construction by ruler and compasses is intentionally incomplete, the theorem that an algebraic expression is constructible if and only if it contains at most rational numbers and square roots being assumed.

The chief interest in the book lies in the large variety of solutions by higher plane curves, and especially in the reduction of these solutions to a unifying principle and to a single figure. The linkage trisectors of Kempe and of Sylvester are not included in the section on trisection instruments.

B. H. BROWN

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1932–1933

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim

is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Kentucky

The officers for the year 1932–1933 were: Dr L. W. Cohen, Director; Dr. H. H. Downing, Vice Director; Professor M. C. Brown, Secretary; Dr. C. G. Latimer, Librarian; Miss Buena Mathis, Treasurer for the first semester and Professor D. E. South, Treasurer for the second semester. These officers were elected May 19, 1932.

The meetings and programs were as follows:

October 13, 1932: "Topology" by Dr. L. W. Cohen.

November 10, 1932: "Correlation surfaces" by Professor D. E. South.

December 8, 1932: "Mathematical logic" by Dr. John Kuiper, Department of Philosophy.

December 9, 1932: Initiation banquet. Five new members were initiated.

January 12, 1933: "Equation balances" and "Mathematical epitaphs" by Professor E. L. Rees.

February 9, 1933: "Algebraic numbers and ideals" by Dr. C. G. Latimer.

March 9, 1933: "Newton's problem in the calculus of variations" by Dr. H. H. Downing.

April 5, 1933: "Mathematical models" by Dr. W. L. Moore of the University of Louisville.

April 15, 1933: Initiation banquet. Five new members were initiated.

May 5, 1933: "Genera of binary quadratic forms" by D. B. Palmeter; "Methods of calculation of tables of binary quadratic forms" by J. H. D. Teller.

The chapter donated $100 in books to the library and voted to donate a second hundred dollars worth of books in the near future.

M. C. BROWN, *Secretary*

Pi Mu Epsilon of Iowa State College

The officers for 1932–1933 were: Marianne Pruess, Director; Guy Strong, Vice Director; Glenn Walrath, Treasurer; Ruth Frizzell, Secretary; George Fink, Librarian; Professor D. L. Holl, Faculty Advisor; Dr. E. R. Smith, Head of the Department of Mathematics, Permanent Secretary.

We now have seventy-five active members. We had one initiation the past year on February 1, 1933, at which time we initiated eight new members.

At the annual college Honor's Day in the fall, 1932, we awarded a check of $15 and a certificate to Harold F. Graves, a junior electrical engineering student. We give this prize each year to the student having the highest scholastic average who has completed the course in calculus. This student is also taken into Pi Mu Epsilon.

The meetings and programs were as follows:

October 19, 1932: "German schools" by Dr. E. S. Allen.

December 7, 1932: Review of a paper by B. F. Kimball on "Three theorems applicable to vibration theorems" by R. H. Cook.

January 10, 1933: "Showing that lead will float and cork will sink" by Walter Fraser; Review of a book entitled "Mathematical Wrinkles" by Charles Armstrong.

February 1, 1933: Initiation and banquet.

February 15, 1933: Illustrated lecture on "What the Egyptians knew about mathematics 3500 years ago" by Marian E. Daniells.

March 8, 1933: "Solution of a diffraction equation" by Guy Strong; "Early history of mathematics in America" by Marianne Pruess.

April 19, 1933: "The use of the Wronskian in solving differential equations for linear independence" by C. C. Hurd; "Origin of the calendar" by Ruth Frizzell.

May 16, 1933: "An Iowa journal of mathematics" by Gertrude Herr; Review of the book "Numerology" by E. T. Bell given by Dr. C. Gouwens.

RUTH FRIZZELL, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the Cooper Union Institute of Technology

The officers for 1932–1933 were: W. W. Rigrod, '33, President; L. Green, '34, Vice President; J. F. Skelly, '34, Secretary; C. A. Wamser, '34, Assistant Secretary; J. B. Watkins, '33, Treasurer; H. J. Queen, '33, Librarian; M. Greenspan, '33, Publication Representative.

The Club has a membership of two hundred fifty.

The meetings and programs were as follows:

November 9, 1932: "The nature of complex numbers" by Professor W. B. Fite of Columbia University.

December 7, 1932: "Theory and use of the slide rule" by H. J. DiGiovanni, '33.

December 14, 1932: "Practical applications of trigonometry to surveying" by J. B. Lyttle, '34.

January 11, 1933: "Theory of least squares" by F. Shapiro, '35.

February 1, 1933: "Mathematical fallacies" by T. W. Reynolds, '35.

February 8, 1933: "Calculating machines" by A. F. Barry, '35.

March 1, 1933: "Transcendental numbers" by H. Turkin, '35.

March 15, 1933: "Determination of position in navigation" by B. K. Hartman, '35.

April 12, 1933: "Geometry of the complex domain" by M. Greenbaum, '34. Election of officers for 1933–1934.

C. H. LEHMANN, *Faculty Advisor*

The Junior Mathematics Club of the University of Chicago

The officers for 1932–1933 were: Miss Julia W. Bauer, President; Mr. Clyde H. Graves, Secretary-Treasurer; Mr. Joseph D. Novak, Program Chairman; Miss Frances E. Baker, Social Chairman.

The purpose of the Junior Mathematics Club is to give the undergraduates as well as the graduates the opportunity of hearing papers in different branches of mathematics which members of the group have prepared and wish to present to the club. Consequently, not only do Fellows in mathematics, National Research Council Fellows, and graduate students working on special problems present papers to the club, but also undergraduates who are specializing in mathematics and who have been encouraged to address the club. Occasionally a professor from a related field has been asked to talk upon a subject of interest which will show the application of mathematics. Therefore, we may say our aim is to broaden the student's knowledge of mathematics and to supplement class room instruction by giving students an opportunity to present and to listen to papers on various phases of mathematics.

The club sponsors practically all of the social activities of the department of mathematics. Its membership for the past year has been about 52. Any graduate student or undergraduate majoring in mathematics or mathematical astronomy is eligible to membership. The meetings which are preceded by teas are held on every other Wednesday in Eckhart Hall Common Room.

The meetings and programs were as follows:

October 12, 1932: Address of welcome by Professor G. A. Bliss; Short talk on the library facilities by Professor L. M. Graves; Report of the Association meetings held in California in August by Professor M. I. Logsdon.

October 25, 1932: "The historical background of certain mathematical periods" by Dr. R. G. Sanger.

November 9, 1932: "Some necessary conditions underlying the early development of mathematics" by Professor H. E. Slaught.

November 23, 1932: "The knot problem" by Miss Anna A. Stafford.

December 7, 1932: "A problem closely related to the shortest distance problem in the calculus of variations" by Dr. M. R. Hestenes.

January 4, 1933: "Transformations of ruled surfaces" by Mr. Joseph D. Novak.

January 18, 1933: "Evolution of the concept 'Integer'" by Mr. H. S. Clair.

February 1, 1933: "Symmetry and design from the group standpoint" by Miss Frances Baker.

February 15, 1933: "Determination of sets of integral elements for certain rational division algebras" by Mr. G. C. Webber.

March 1, 1933: "The Egyptian's use of mathematics" by Dr. J. A. Wilson of the Oriental Institute of the University of Chicago.

March 15, 1933: "Correlations and null-systems" by Mr. Clyde H. Graves.

April 12, 1933: "The isoperimetric properties of the sphere" by Dr. Max Coral.

April 26, 1933: "Some applications of mathematics to the study of personality" by Professor L. L. Thurstone of the Psychology Department.

May 10, 1933: "A brief history of the infinitesimal calculus" by Mr. H. H. Harmon.

May 24, 1933: "Nomograms" by Dr. William C. Krumbein of the Geology Department.

June 7, 1933: "Women in mathematics" by Dr. Ruth G. Mason.

During the year a tea dance and five bridge parties were sponsored by the club.

CLYDE H. GRAVES, *Secretary-Treasurer*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 65. *Proposed by J. M. West, Pennsylvania State College.*

If the vertices of a triangle taken counterclockwise have the abscissas x_1, x_2 and x_3, and if the slopes of the opposite sides are m_1, m_2 and m_3, then prove that the area equals $\frac{1}{2}(x_1-x_2)(x_1-x_3)(m_2-m_3)$, as well as either of the two similar expressions obtainable from this by cyclic permutation of the subscripts.

E 66. *Proposed by J. Rosenbaum, The Milford School, Milford, Connecticut.*

Prove that the sum of the squares of the medians of a tetrahedron equals four-ninths of the sum of the squares of the edges. (A median of a tetrahedron is a line joining a vertex with the centroid of the opposite face.)

E 67. *Proposed by E. C. Kennedy, University of Texas.*

Give a scheme for writing down mechanically the sides of an unlimited number of dissimilar right triangles whose sides are integers. After the first set, the values are to be written down, not merely indicated, without any calculations whatever. No addition, subtraction, multiplication, division, involution or evolution, mental or otherwise, is allowed.

One way is to start with the triangle whose sides are 21, 220 and 221.

E 68. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In the triangle ABC, D is the midpoint of BC. The equilateral triangles,

ABP, ACQ and ADR are drawn in the plane of triangle ABC, the vertices of each being listed counterclockwise. Prove that R is the midpoint of PQ.

E 69. *Proposed by Raphael Robinson, University of California at Berkeley.*

Instead of a product of powers, $a^b c^d$, a printer accidentally prints the four digit number, $abcd$. The value is however the same. Find the number and show that it is unique.

Editor's Note. This problem may be found in Dudeney's *Amusements in Mathematics* (Thomas Nelson and Sons), page 20.

SOLUTIONS

E 36. [1933, 295] *Proposed by B. H. Brown, Dartmouth College.*

Show that the thirteenth of the month is more likely to be Friday than any one of the other days of the week.

Solution by Raphael Robinson, University of California at Berkeley.

Leap years are determined in a way which depends on the number of the year modulo 400. But in 400 years, including 97 leap years, there are exactly 20,871 weeks. Hence the calender repeats every 400 years. (If there were not an exact integral number of weeks in 400 years, there would be an equal chance for the thirteenth of the month to fall on each day of the week.) In 400 years the thirteenth of the month occurs just 4800 times, distributed among the different days of the week as follows:

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
687	685	685	687	684	688	684

Hence, in the long run, the thirteenth of the month is most likely to fall on Friday.

Solved also by Churchill Eisenhart and the proposer.

E 37. [1933, 295] *Proposed by Arthur Haas, Thomas Jefferson High School, N.Y.*

In the following multiplication of a three-place number by a two-place number, all but three of the sixteen digits have been replaced by the letter x. Reconstruct the problem and show that the solution is unique.

$$\begin{array}{r} x\;x\;x \\ x\;x \\ \hline x\;x\;x \\ x\;x\;4 \\ \hline x\;x\;x\;0\;1 \end{array}$$

Solution by Grace Shover, Columbus, Ohio.

The values of some of the x's are apparent by inspection, so we insert their values and reletter the others thus:

$$\begin{array}{rrrrr} & & a & b & c \\ & & & d & e \\ \hline & & f & 6 & 1 \\ & 9 & g & 4 & \\ \hline 1 & 0 & h & 0 & 1 \end{array}$$

Since the product of c and e ends in 1, they can only be 1 and 1, or 3 and 7, or 7 and 3, or 9 and 9, which gives us the following four cases for consideration.

Case 1. $c=e=1$. $(abc)(e)=f61$. $b=6$. $a=f$. $(f61)(d)=9g4$. $d=4$. $g=4$. And no value of f will fit.

Case 2. $c=3$. $e=7$. $7(abc)=f61$. $b=2$. $a=1$. $abc=123$. $d(123)=9g4$. $d=8$. $g=8$. $h=7$. No contradictions arise and $123\times 87=10701$ is a solution.

Case 3. $c=7$. $e=3$. $3(ab7)=f61$. $b=8$. $d(a87)=9g4$. $d=2$. $a=4$. But then $3\times 487=f61$ and no one-digit value of f will fit.

Case 4. $c=9$. $e=9$. $9(ab9)=f61$. $b=2$. $9(a29)=f61$. $f=9a+2<10$, which is impossible, since a is not zero.

Since only one solution appeared in trying all possible cases, it is unique.

Solved also by Mrs. Annabel S. Boyce, W. E. Buker, M. L. Constable, S. A. Corey, A. P. Free, F. C. Gentry, E. C. Kennedy, H. R. Leifer, Theodore Lindquist, Lee Lorch, Roy MacKay, Janet McDonald, C. T. Oergel, Raphael Robinson, R. M. Sutton, Simon Vatriquant and B. C. Zimmerman.

E 38. [1933, 295] *Proposed by J. R. Musselman, Western Reserve University.*

It is well known that the midpoints of the sides of any plane quadrilateral constitute the vertices of a parallelogram. Determine the most general conditions under which the parallelogram becomes (a) a rhombus, (b) a rectangle, and (c) a square.

Solution by Lazarus Medveson, Jr., Albuquerque, New Mexico.

From the theorem which states that the line joining the midpoints of two sides of a triangle is parallel to and equal to half of the third side, there follow these general conditions:

(a) If the diagonals of the quadrilateral are equal, the parallelogram is a rhombus.

(b) If the diagonals of the quadrilateral are perpendicular, the parallelogram is a rectangle.

(c) If the diagonals of the quadrilateral are both equal and perpendicular, the parallelogram is a square.

Solved also by F. C. Gentry, Theodore Lindquist, Lee Lorch, Roy MacKay, O. J. Ramler and Simon Vatriquant.

E 39. [1933, 295] *Proposed by W. R. Ransom, Tufts College.*

Obtain both roots of the equation, $x^2-365.04x+45.04=0$, correct to four significant digits, by means of four-place logarithm tables.

Solution by Maud Willey, Long Beach, Mississippi.

The customary solution by the quadratic formula gives $x=182.52\pm\sqrt{33269}$, but four-place logarithms give $\sqrt{33269}=182.4$, and hence give but one significant digit of the smaller root. However, the larger root is 364.9 and the product of the roots is the given number, 45.04. A simple logarithmic division then gives the smaller root as .1234.

R. N. Walter arrives at the same results by the method due to C. H. Gräffe, which is explained on page 121 of L. E. Dickson's *Elementary Theory of Equations*.

Solved also by the proposer.

E 41. [1933, 296] *Proposed by V. F. Ivanoff, San Francisco.*

A variable circular arc of constant length, l, has one end fixed in position and direction. Find the locus of its other end.

Solution by O. J. Ramler, Catholic University, Washington, D. C.

Employing a system of polar coordinates, we take the center of the variable arc on a line perpendicular to the polar axis at the origin. Let the fixed end of the variable arc be at the origin, and let the polar axis be tangent to the arc at this point. Let the free end of the arc have the polar coordinates, (r, θ).

Then if R is the variable radius of the variable arc of constant length l, the central angle in radians is l/R, and $\theta=l/2R$. Also, $r=2R\sin\theta$. When R is eliminated between these two equations, there results the equation of the locus,[1]

$$r\theta = l\sin\theta.$$

The curve is a double spiral, symmetric about the polar axis, with an unlimited number of loops all tangent to the polar axis at the origin.

From the nature of the problem it is quite obvious that when $\theta=0$, $r=l$, thus illustrating nicely an important limit of the calculus, namely:

$$\operatorname*{Lim}_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Solved also by Lazarus Medveson, Jr., C. C. Richtmeyer; Simon Vatriquant, and R. N. Walter.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

[1] See *An Unusual Spiral* by L. S. Johnston, this issue of the Monthly, p. 596.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3646. *Proposed by N. A. Court, University of Oklahoma.*

Determine the surface generated by the common perpendicular of two skew lines a and b, when a describes a flat pencil while b remains fixed.

3647. *Proposed by H. Grossman, New York.*

Prove that

$$\sum_{n=0}^{\infty} \frac{(-4\pi^2 r^2)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-4\pi^2 r^2)^n}{(2n+2)!} = 0,$$

where r is any integer.

3648. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Derive by the method of instantaneous centers a construction for the tangent at any point of the Archimedean spiral.

3649. *Proposed by F. T. H'Doubler, Springfield, Mo.*

Solve the functional equation

$$f(xy) = [f(x)]^{y^\beta}[f(y)]^{x^\beta},$$

where β is a real constant and $f(x)$ is a real continuous and single valued function of the real variable x.

3650. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given in a plane any five points which are known to lie upon an ellipse; construct by ruler and compasses the principal axes of the ellipse. Show how the construction may fail if the points are chosen arbitrarily.

3651. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

Two persons play a game using 24 cards numbered 1 to 6, placed face up, (there being four cards of each number), by alternately turning over any card not previously turned. The first player to make the sum of the numbers on the cards turned over exceed 30, loses. With best play, should the player who plays first always win, or always lose?

3652. *Proposed by A. F. Stevenson, University of Toronto.*

The practice of certain cigarette manufacturers of supplying playing cards, poker hands, etc. with their cigarette packets, and of offering various articles in exchange for a complete set of these cards, suggests the following problem:

Assuming that each packet of cigarettes contains one of a set of 52 cards, and that these cards are distributed among the packets at random, (the num-

ber of packets available being infinite) what is the average minimum number of packets that must be purchased in order to obtain a complete set of cards?

3653. *Proposed by L. S. Johnston, University of Detroit.*

Given the triangle ABC and the points A', B', and C' on BC, CA, and AB respectively such that $AC'/C'B=BA'/A'C=CB'/B'A=m/n$ (m less than n for convenience). Let AA' and BB' intersect at C'', BB' and CC' intersect at A'', and CC' and AA' intersect at B''. Prove that

$$AB'':B''C'':C''A'=BC'':C''A'':A''B'=CA'':A''B'':B''C',$$

and calculate the division ratios in terms of m and n.

SOLUTIONS

3585. [1932, 608]. *Proposed by R. E. Gaines, University of Richmond.*

If a tangent at P_1 to the cardioid $\rho=a(1+\cos\theta)$ cuts the curve again at P_2 and P_3, and if the normal at P_1 cuts the curve again at P_4, P_5, P_6, then (1) $\rho_2\rho_3/\rho_1$ is constant. Also (2) $\rho_1+\rho_2+\rho_3=\rho_4+\rho_5+\rho_6$; (3) if the chord $P_2P_3=2^{1/2}a$, it subtends a right angle at the origin.

Solution by F. Underwood, University College, Nottingham, England.

Let the point P_1 be $\{a(1+\cos\alpha),\alpha\}$. Then the tangent and normal at P_1 are respectively,

$$\frac{2a}{\rho}\cos^3\frac{\alpha}{2}=\cos\left(\theta-\frac{3\alpha}{2}\right)$$

and

$$\frac{2a}{\rho}\cos^2\frac{\alpha}{2}\sin\frac{\alpha}{2}=\sin\left(\frac{3\alpha}{2}-\theta\right).$$

The values of ρ for the points where these meet the cardioid again are given by putting

$$\cos\theta=\rho/a-1\ ;\ \sin\theta=\{\rho(2a-\rho)\}^{1/2}/a.$$

Hence ρ_1, ρ_1, ρ_2, ρ_3 are the roots of the equation

$$\rho^4-2a\rho^3-3a^2\rho^2\cos\frac{3\alpha}{2}\cos\frac{\alpha}{2}+4a^3\rho\cos\frac{3\alpha}{2}\cos^3\frac{\alpha}{2}+4a^4\cos^6\frac{\alpha}{2}=0,$$

and ρ_1, ρ_4, ρ_5, ρ_6 are the roots of

$$\rho^4-2a\rho^3-a^2\rho^2\sin\frac{3\alpha}{2}\sin\frac{\alpha}{2}$$
$$+4a^3\rho\sin\frac{3\alpha}{2}\cos^2\frac{\alpha}{2}\sin\frac{\alpha}{2}+4a^4\cos^4\frac{\alpha}{2}\sin^2\frac{\alpha}{2}=0.$$

Hence

$$(1) \qquad \rho_2\rho_3/\rho_1 = \frac{\rho_1^2\rho_2\rho_3}{\rho_1^3} = \frac{4a^4\cos^6\dfrac{\alpha}{2}}{a^3(1+\cos\alpha)^3} = \frac{a}{2}\,.$$

Also $2\rho_1+\rho_2+\rho_3=2a=\rho_1+\rho_4+\rho_5+\rho_6$; and therefore

$$(2) \qquad \rho_1+\rho_2+\rho_3=\rho_4+\rho_5+\rho_6.$$

Since

$$(3) \qquad \rho_2+\rho_3=2(a-\rho_1), \qquad 2\rho_2\rho_3=a\rho_1,$$

we must have $0\leqq\rho_1\leqq\frac{1}{2}a$ in order for ρ_2 and ρ_3 to be real. If $\rho_1=\frac{1}{2}a$, $\alpha=2\pi/3$; and $\rho_2=\rho_3=\frac{1}{2}a$, $\theta_2=\theta_3=4\pi/3$. We then compute

$$\cos\theta_3\cos\theta_2=\frac{5\rho_1}{2a}-1, \qquad \sin\theta_3\sin\theta_2=+\frac{3\rho_1}{2a},$$

where in the second equation above the minus sign has been rejected in taking a square root; for we see from this equation that neither factor on the left can be zero except for $\rho_1=0$. When $\rho_1=\frac{1}{2}a$ the sign is seen to be plus and it will remain plus for ρ_1 in the interval $0\leqq\rho_1\leqq\frac{1}{2}a$.

Set $\phi=\theta_3-\theta_2\geqq 0$, then

$$(4) \qquad 4\rho_1=a(1+\cos\phi), \qquad 0\leqq\rho_1\leqq\tfrac{1}{2}a,\ \pi\geqq\phi\geqq 0.$$

From the law of cosines and (3) and (4) we find

$$(5) \qquad P_2P_3=2a\sin\tfrac{1}{2}\phi.$$

Hence if $P_2P_3=2^{1/2}a$, $\phi=\frac{1}{2}\pi$, $\rho_1=\frac{1}{4}a$, and $\cos\alpha=-\frac{3}{4}$. The proof shows also that the converse is true.

3587. [1933, 52] *Proposed by P. R. Rider, Washington University.*

The axes of two right circular cylinders of indefinite length intersect perpendicularly. Find the volume common to the two cylinders. (This problem actually arose in designing a tank car.)

Solution by N. B. Moore, California Institute of Technology.

If the two cylinders are of equal radius R, then every section of the common volume parallel to, and at a distance x from the plane of the axes, is a square of area $4(R^2-x^2)$, and the common volume is

$$V_{R,R}=8\int_0^R(R^2-x^2)dx=(16/3)R^3.$$

If the cylinders have radii r, $R(0\leqq r\leqq R)$, then every such section is a rectangle of area $4(R^2-x^2)^{1/2}(r^2-x^2)^{1/2}$, and

$$V_{r,R} = 8\int_0^r (R^2 - x^2)^{1/2}(r^2 - x^2)^{1/2}dx.$$

By making the transformation $x = r \sin \theta$, and denoting by k the ratio r/R, it is easily shown that

$$V_{r,R} = 8k^2R^3I(k),$$

where

$$I(k) \equiv \int_0^{\pi/2} \frac{d\theta}{\Delta\theta} - (1 + k^2)\int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{\Delta\theta} + k^2\int_0^{\pi/2} \frac{\sin^4\theta\, d\theta}{\Delta\theta},$$

$$\Delta\theta = (1 - k^2 \sin^2\theta)^{1/2}.$$

$I(k)$ can be evaluated by using the following well-known relations, where the notation is that commonly used for complete elliptic integrals of first and second kinds:

$$\int_0^{\pi/2} \frac{\sin^4\theta\, d\theta}{\Delta\theta} = \frac{2(1 + k^2)}{3k^2}\int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{\Delta\theta} - \frac{1}{3k^2}\int_0^{\pi/2} \frac{d\theta}{\Delta\theta},$$

$$\int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{\Delta\theta} = (1/k^2)\int_0^{\pi/2} \frac{d\theta}{\Delta\theta} - (1/k^2)\int_0^{\pi/2} \Delta\theta\, d\theta,$$

$$\int_0^{\pi/2} \frac{d\theta}{\Delta\theta} = K, \text{ and } \int_0^{\pi/2} \Delta\theta\, d\theta = E.$$

The common volume is thus readily obtained as

$$V_{r,R} = (8/3)R^3[(1 + k^2)E - (1 - k^2)K],$$

or,

$$V_{r,R} = \tfrac{1}{2}V_{R,R}[(1 + k^2)E - (1 - k^2)K].$$

Solved also by C. A. Hutchinson, W. W. Johnson, J. B. Meyer, V. F. Murray, C. E. Rhodes, C. Grace Shover, and the proposer. J. E. Thompson referred to a solution in Hancock's *Elliptic Integrals*, p. 89, ex. 6. A similar problem is given in Byerly's *Integral Calculus*, sec. ed., p. 281.

3588. [1933, 52] *Proposed by N. H. McCoy, Smith College.*

Find an explicit expression for f_n if $f_0 = 1$ and

$$f_{n+1} = (x^2 + y^2)f_n + 2ty\frac{\partial f_n}{\partial x} + t^2\frac{\partial^2 f_n}{\partial x^2},$$

where t is a parameter. Show that

$$(f_n)_{x=y=0} = K_n t^n, \quad K_n = \left.\frac{d^n(\sec 2t)^{1/2}}{dt^n}\right|_{t=0}.$$

Solution by J. V. Uspensky, Stanford University.

The power series

$$\omega = \sum_0^\infty \frac{\tau^n}{n!} f_n,$$

provided it has a non-vanishing convergence radius, by virtue of the conditions

$$f_{n+1} = (x^2 + y^2)f_n + 2ty\frac{df_n}{dx} + t^2\frac{d^2f_n}{dx^2}, \quad f_0 = 1,$$

represents that solution of the equation

$$\frac{\partial\omega}{\partial\tau} = (x^2 + y^2)\omega + 2ty\frac{\partial\omega}{\partial x} + t^2\frac{\partial^2\omega}{\partial x^2},$$

which for $\tau = 0$ reduces to 1. One easily verifies that such a solution is given by

$$\omega = \{\sec(2t\tau)\}^{1/2}\exp\left[\frac{x^2 + y^2}{2t}\tan(2t\tau) + \frac{xy}{t}\{\sec(2t\tau) - 1\}\right].$$

Expanding it into a power series in τ we can find the explicit expression of f_n as a homogeneous function of $p = x^2 + y^2$, $q = xy$ and t of the dimension n. The fact that

$$(f_n)_{x=y=0} = K_n t^n$$

where

$$K_n = \left.\frac{d^n(\sec 2\tau)^{1/2}}{d\tau^n}\right|_{\tau=0}$$

follows immediately from the expression for ω.

Solved also by L. Zander.

3591. [1933, 52] *Proposed by B. F. Kimball, Schenectady, N. Y.*

Let the nth difference of $\log x$ with difference interval one, be denoted by $\Delta^n \log x$. Show that

$$\lim_{n\to\infty} n^x \log n \Delta^n \log x = \Gamma(x).$$

Solution by Morgan Ward, California Institute of Technology.

Denote by (R) that portion of the plane of the complex variable z bounded by a cut along the negative axis of reals and a fixed circle of arbitrarily large radius, and let the argument of z be greater than $-\pi$ and less than π. Then $\log z$ and $\Gamma(z)$ are one-valued analytic functions of z throughout (R). I shall show that if x is any fixed point within (R),

$$(1) \qquad (-1)^{n+1} n^x \log n \Delta^n \log x = \Gamma(x) + O\left(\frac{1}{\log n}\right),$$

the constant implied in the O-symbol necessarily depending on x. The sign $(-1)^{n+1}$ was omitted by the proposer.

First of all, if $n \geqq 1$,

$$(2) \qquad (-1)^{n+1}\Delta^n \log x = \int_0^1 \frac{(n-1)!dt}{(x+t)(x+t+1)\cdots(x+t+n-1)}.$$

For

$$\Delta^n \log x = \Delta^{n-1}[\log(x+1) - \log x] = \Delta^{n-1}\int_0^1 \frac{dt}{x+t} = \int_0^1 \Delta^{n-1}(x+t)^{-1}dt,$$

and by the familiar formula of the calculus of finite differences for the difference of a factorial,

$$\Delta^{n-1}(x+t)^{-1} = \frac{(-1)^{n+1}(n-1)!}{(x+t)(x+t+1)\cdots(x+t+n-1)}.$$

Now it is well known that if z is fixed,

$$\lim_{n\to\infty} \frac{n^z 1\cdot 2\cdots(n-1)}{z(z+1)\cdots(z+n-1)} = \Gamma(z).$$

If we rewrite the expression on the left as $n^z\Gamma(n)\Gamma(z)/\Gamma(z+n)$ and use the asymptotic formulas for $\Gamma(n)$, $\Gamma(z+n)$ which are valid if z is in (R), we obtain the more precise result

$$\frac{(n-1)!}{z(z+1)\cdots(z+n-1)} = \frac{\Gamma(z)}{n^z} + O\left(\frac{1}{n^{z+1}}\right), \quad |\arg z| < \pi.$$

Now assume that x is in (R). Then we may write $x+t$ for z in this expression and substitute in the integral in (2). There results

$$\begin{aligned}(-1)^{n+1}\Delta^n \log x &= \int_0^1 \left(\frac{\Gamma(t+x)}{n^{t+x}} + O\left(\frac{1}{n^{t+x+1}}\right)\right)dt \\ &= \frac{1}{n^x}\int_0^1 \frac{\Gamma(t+x)}{n^t}dt + O\left(\frac{1}{n^{x+1}\log n}\right).\end{aligned}$$

On integrating by parts, we find that

$$(3) \qquad \begin{aligned}(-1)^{n+1}\Delta^n \log x &= \frac{\Gamma(x)}{n^x \log n} - \frac{\Gamma(x+1)}{n^{x+1}\log n} \\ &\quad + \frac{1}{n^x \log n}\int_0^1 \frac{\Gamma'(t+x)}{n^t}dt + O\left(\frac{1}{n^{x+1}\log n}\right).\end{aligned}$$

It is clear that

$$\int_0^1 \frac{\Gamma'(t+x)}{n^t}dt = O\left(\frac{1}{\log n}\right).$$

Therefore, if we multiply (3) by $n^x \log n$, we obtain finally

$$(-1)^{n+1} n^x \log n \Delta^n \log x = \Gamma(x) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{\log n}\right) + O\left(\frac{1}{n}\right)$$
$$= \Gamma(x) + O\left(\frac{1}{\log n}\right).$$

I should like to express my thanks to Professor Otto Dunkel for suggesting a simplification of my initial solution.

Solved also by the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Fourth International Conference of Applied Mechanics will be held at Cambridge, England, July 3 to 9, 1934. All persons interested in the subject are invited to attend and submit papers. Information relative to types of papers that may be presented can be obtained from the Secretary, Fourth International Conference of Applied Mechanics, Engineering Laboratory, Cambridge.

Professor V. Karapetoff of the Department of Electrical Engineering of Cornell University has been appointed Lieutenant Commander in the Naval Reserve and has been assigned to the Volunteer Naval Reserve for engineering duties.

The Rumford Medal of the American Academy of Arts and Sciences has been awarded to Professor Harlow Shapley of Harvard University for researches in the luminosity of stars and galaxies.

Dr. Charles F. Roos, formerly permanent secretary of the American Association for the Advancement of Science, has been granted indefinite furlough from the fellowship of the Guggenheim foundation, to enable him to accept an appointment as specialist in economic balance for the National Recovery Administration. He assumed office July 27.

The University of Wisconsin has conferred an honorary doctorate on Professor Arnold Sommerfeld of the University of Munich.

An account of the life and works of the late professor J. C. Fields of the University of Toronto has been prepared by Professor J. L. Synge of the same university, and published in the April, 1933, issue of the Journal of the London Mathematical Society.

Assistant Professor H. W. Brinkmann of Harvard University has been appointed to an associate professorship at Swarthmore College.

Dr. Dirk Brouwer has been promoted to an associate professorship of astronomy at Yale University.

Mrs. Grace Murray Hopper has been appointed instructor at Vassar College.

NECROLOGY

CORRIGENDA

Volume XL, 1933.

P. 120, line 12, for "Bulter," read "Butler."
P. 124, line 9, for "A. L. Meder," read "A. E. Meder."
P. 174, line 3, for "W. M. Aylor," read "M. W. Aylor."
P. 178, problem E-10, the skeleton division should be as in vol. 39 (1932), p. 548.
P. 188, A correction.
P. 309, 13th line from bottom, for "American" read "America."
P. 340, A correction to vol. 40 (1933), p. 226.
P. 406, in equations (1), for "$\frac{\partial U}{\partial \xi}$, $\frac{\partial U}{\partial \eta}$, $\frac{\partial U}{\partial \zeta}$" read respectively "$\frac{\partial U}{\partial \xi_i}$, $\frac{\partial U}{\partial \eta_i}$, $\frac{\partial U}{\partial \zeta_i}$."
P. 407, line 5, in the numerator of the first fraction read "e_3" for "e_2."
P. 412, 18th line from bottom, for "4 − 224a" read "4 − 224a′."
P. 420, 5th line from bottom, for "Whalen," read "Whelan."
P. 440, A Note. An addition to a reading list in elementary theory of equations, vol. 40 (1933), pp. 77–84.
P. 445, line 15, for "Medgycy" read "Medgyesy."
P. 504, 10th line from bottom, "Columbia College, South Carolina" and "Lander College" should be interchanged.
P. 605, line 18, for "Bauer, Julia W.", read "Bower, Julia W."

CONTENTS

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BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Annual Meeting of the Association, Cambridge, Mass., Dec. 27-29, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merged with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 9.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

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FOR THE ACADEMIC YEAR

1933–1934

For a list of all officers and trustees for preceding years, see page 67.

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Closed for printing November 24, 1933

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ATTLEBORO. Holt.
BELMONT. Douglass, Rutledge.
BOSTON. Andrew, Barber, Bruce, Garabedian, Gould, Laurentine, Leavens, Mode, Spear, Weaver, Wilson.
BROOKLINE. Miller.
CAMBRIDGE. Bailey, Beatley, Birkhoff, Bradley, Brown, Coolidge, Cooper, Crum, Feenberg, Franklin, Gaylord, Graustein, Huntington, Kennelly, Morse, Passano, Rice, Robinson, Seidel, Stabler, Starrett, Street, Walsh, Widder, Wilson, Woods, Wright, Zeldin.
DANVERS. Majella.
DORCHESTER. Davis, Heins, Quigley.
GROTON. Nash.
LEXINGTON. Moyle.
LOWELL. Aurelius.
MEDFORD. Miller.
NATICK. Willis.

NORTHAMPTON. Benedict, McCoy, Munroe, Rambo, Wood.
PETERSHAM. Moriarty.
PITTSFIELD. Washburne.
READING. Skofield.
SOUTH HADLEY. Doak, Martin.
SWAMPSCOTT. Evans.
TUFTS COLLEGE. Mergendahl, Ransom.
WELLESLEY. Copeland, Merrill, Morton, C. E. Smith, Stark, Young.
WESTON. Burke.
WILLIAMSTOWN. Agard, Dorwart, Hardy, Wells.
WOLLASTON. Dennison.
WORCESTER. Brown, Gay, Melville, Morley, Rice, Wheeler.

MICHIGAN. (64)

ALBION. Ingalls, Sleight.
ALMA. Clack.
ANN ARBOR. Ayres, Baten, Bradshaw, Churchill, Coe, Corliss, Craig, Denton, Elder, Field, Ford, Glover, Hildebrandt, Hopkins, Kaltenborn, Karpinski, Love, Nyswander, Raiford, Rainich, Rood, Rouse, Running, Schorling, Swanson, Wilder.
BAY CITY. Shellenbarger.
DETROIT. Baldwin, Darnell, Fisk, Folley, Johnson, Johnston, McCarthy, Mary Paula, Muehlman, Mullen, Nelson, Pixley, Thome.
EAST LANSING. Crowe, Grove, Olson, Plant, Powell, Speeker.
FLINT. Stout.
HILLSDALE. Herron.
HOUGHTON. Roman.
IRONWOOD. Field.
KALAMAZOO. Ackley, Blair, Everett, Walton.
MARQUETTE. Spooner.
MOUNT PLEASANT. Richtmeyer.
YPSILANTI. Barnhill, Erikson, Lindquist, Lyman, Matteson.

MINNESOTA. (47)

COLERAINE. Fattu, Tangjerd.
COLLEGEVILLE. Danzl, Hansen, Winkelmann.
DULUTH. Strane.
GILBERT. Schey.
MANKATO. Robbins.
MINNEAPOLIS. Brink, Brooke, Bussey, Carlson, Dalaker, Fischer, Gibbens, Gunstad, Guttman, Hart, Hartig, Jackson, Jensen, Kirchner, Ness, Priester, Quaid, Scammon, Schnell, Shuman, Shumway, Thorp, Underhill, Walder, Wilder.
MOORHEAD. Leonard.
NORTHFIELD. Gingrich, Solum, White.
ROCHESTER. Cruise.
ST. JOSEPH. Claudette.
ST. PAUL. Alice Irene, Blackall, Bush, Chellevold, Kingery, Mary Aloysius, Taylor.
VIRGINIA. Hancock.

MISSISSIPPI. (16)

BLUE MOUNTAIN. Hutchins.
CLEVELAND. Hickey.
GRENADA. Harris.
HATTIESBURG. Parker.
JACKSON. Babbitt, McCoy, Mitchell.
LONG BEACH. Willey.
RAYMOND. McDonald.
STATE COLLEGE. Caruthers, C. D. Smith.
TOUGALOO. Howe.
UNIVERSITY. Bickerstaff, Hume, Scott.
WESSON. Felder.

MISSOURI. (42)

CANTON. Ingold.
CAPE GIRARDEAU. Knepper.
CARTHAGE. Murto.
CLAYTON. Haertter.
COLUMBIA. Callaway, Ingold, Robinson, Wahlin, Westfall.
FAYETTE. Fleet.
FULTON. Christian, Sweazey.
KANSAS CITY. Cutting, Pierson.
KIRKSVILLE. Cosby, Jamison.
KIRKWOOD. Harris.
MAYSVILLE. Saunders.
PARKVILLE. Wells.
ROLLA. Hinsch.
ST. CHARLES. Karr.
ST. LOUIS. Brick, Buell, Case, Dunford, Dunkel, Grummann, Huntington, King, Middlemiss, O'Donnell, Osborn, Rider, Roever, Siroky, E. Stephens, J. Y. Stephens.
SPRINGFIELD. Finkel, H'Doubler.
WARRENSBURG. Scarborough.
WEBSTER GROVES. Clarke, Pennell.

MONTANA. (5)

BOZEMAN. Hurst.
HELENA. Canning.
MISSOULA. Carey, Lennes, Merrill.

NEBRASKA. (20)

BEAVER CROSSING. Thompson.
HASTINGS. McDill.
LINCOLN. Basoco, Brenke, Camp, Candy, Collins, Gaba, Howie, Pierce, Runge.
OMAHA. Bettinger, Campbell, Earl, Fitzpatrick, Gunn.
PERU. Hill.
WAYNE. Boyce, Hove.
YORK. Feemster.

NEVADA. (2)

RENO. Searcy, Wood.

NEW HAMPSHIRE. (18)

CONCORD. Conwell.
DURHAM. Bauer, Slobin, Wilbur.
EXETER. Butterfield, Funkhouser, Sweet.
FRANCONIA. Nelson.
HANOVER. Beetle, Brown, Forsyth, Mathewson, Morgan, Perkins, Robinson, Silverman, Wilder.
PLYMOUTH. G. M. Smith.

NEW JERSEY. (50)

BELLEPLAIN. Durell.
BLOOMFIELD. Hussey.
EAST ORANGE. Nordgaard, Stanwick.
HIGHTSTOWN. Litterick.
HOBOKEN. Murray.
JERSEY CITY. J. P. Smith.
LAWRENCEVILLE. Kimball, Mikesh.
LEONIA. Gafafer.
MENDHAM. Stafford.
MONTCLAIR. Davis, Mallory.
MOORESTOWN. Wood.
NEWARK. Conkling.
NEW BRUNSWICK. Bunyan, Meder, Morris, Nelson, Walter, Wilson.
PRINCETON. Adams, Alexander, Ball, Blumenthal, Eisenhart, Flood, Gillespie, Hacker, Knebelman, Lefschetz, Lehmer, Levine, McShane, Peterson, Schmeiser, Schultz, Thomas, Torrance, Veblen, von Neumann, Walker, Wedderburn, Wilder, Willson.
SOUTH ORANGE. Loveridge.
TRENTON. Sanford, Shuster.
UNION. Brown.
WORTENDYKE. F. E. Smith.

NEW MEXICO. (11)

ALBUQUERQUE. Bauer, Graham, MacKay, Munn, Munro, Newsom.
ARTESIA. Harp.
LAS VEGAS. Roberts, Rodgers.
SILVER CITY. Mickelson.
SOCORRO. Reece.

NEW YORK. (241)

ALBANY. Alice Irene, Beaver, Birchenough, Do Bell, Lester, Lowenstein, Stokes.
ALFRED. Polan, Seidlin, Titsworth, Whitford.
ALLEGANY. McLaughlin.
ANNANDALE-ON-HUDSON. Garabedian, Phalen.
AURORA. Hollcroft, Rusk.
BALDWIN. Grove.
BELLPORT. Walsh.
BRONX. Tanzola.
BROOKLYN. Berry, Bowden, Charosh, Cowles, Deutsch, Fleisher, Francis, Griffin, Haas, Hertzler, Hildebrandt, R. A. Johnson, Karnow, Kennison, Koch, Langman, Lepowsky, Lieber, Locke, Mary Thecla, Moore, Ruderman, Schmellner, Schuyler, Thompson, Walter, Welkowitz, Whitford, Young-Woodbridge.
BUFFALO. Archer, Gehman, Harrington, Montague, Pound, Rice.
CLINTON. Brown, Carruth, Ferry, Fitch, Patterson.
CORONA. Hanson.
ELMIRA. Suffa, Wright.
FLUSHING. Lehmann.
FOREST HILLS. Walker.
GENEVA. W. H. Durfee, W. P. Durfee, Hubbs, Rood.
HAMILTON. Aude, A. W. Smith.
HOUGHTON. Davison.
ITHACA. Agnew, Barbour, Boothroyd, Carver, Dye, Gillespie, Hurwitz, B. W. Jones, Karapetoff, Paradiso, Randolph, Ranum, Snyder, Trevor.
JORDAN. Howe.
KENMORE. Brockett.
KINDERHOOK. Magee.
NEW YORK. Adams, Allen, Allison, Anderson, Archibald, Berger, Bergstresser, Berkeley, Berry, Bradley, Breckenridge, Brewster, A. B. Brown, Burgess, Bushey, G. A. Campbell, G. C. Campbell, Clark, Cooley, Doermann, Edmondson, Eisele, Farnum, Feld, Fiske, Fite, Flanders, Foster, Frank, Frankel, Fry, Fuller, Gentzler, Gill, Girard, Graham, Hall, Harper, Hawkes, Hayes, Henderson, Hill, Hirsch, Hopper, Hughes, Hurwitz, Jablonower, Joffe, P. C. Jones, Kasner, Koopman, Larkin, Linehan, MacGregor, Maiden, Miller, Mirick, Molina, Morehouse, Mullins, Paaswell, Packer, Payne, Pedersen, Penn, Peters, Pimpton, Pooler, Post, Pride, Putnam, Quilty, Raudenbush, Reddick, Rees, Reeve, Ritt, Rosinger, Saurel, Schelkunoff, W. S. Schlauch, Schub, Seely, Shaw, Shewhart, Siceloff, Simons, Skelding, D. E. Smith, R. F. Smith, R. R. Smith, Streator, Tilley, Turner, Upton, Velton, Wahlert, Walker, Walther, Webster, Wechsler, Weisner, Whelan, Whitford, Winters, Wood, Wright.
NISKAYUNA. Male.
ONEONTA. Sanford, Schoonmaker.
PARISH. Church.
POTSDAM. Waltz.
POUGHKEEPSIE. Cummings, Wells.
ROCHESTER. Betz, Byrnes, Eastham, Gale, Harding, Kimball, Long, Price, Smyth, Watkeys.
RYE. Vivian.
ST. BONAVENTURE. Nickol.
SCARSDALE. Lawton, Mac Neish, Weaver.
SCHENECTADY. Fox, Kimball, Libman, Morse, Oergel, Snyder, Ulrich, Vedder.
STATEN ISLAND. Andersen, Drago.
SYRACUSE. Campbell, Carroll, Decker, Harwood, Lindsey, Ryan, Taylor.
TROY. Crockett, McGiffert.
WEST POINT. Echols, Jones.
YONKERS. Hubert, John, Yanosik.

NORTH CAROLINA. (26)

ASHEVILLE. Peck.
CHAPEL HILL. Browne, Henderson, Lasley, Linker, Mackie.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. Mebane.
DURHAM. Dearborn, Dressel, Elliott, Hickson, Rankin, Roberts, Winton.
ELON COLLEGE. Amick.
GREENBORO. Barton, Pegram, Ragsdale, Strong.
GREENVILLE. Graham, ReBarker.
MARS HILL. Robinson.

WILMINGTON. Downing.
WINGATE. Hendricks.

NORTH DAKOTA. (6)

FARGO. Householder, I. W. Smith.
GRAND FORKS. Leith, Staley.
UNIVERSITY. Hitchcock.
VALLEY CITY. Meyer.

OHIO. (120)

ADA. Whitted.
AKRON. Bender.
ASHLAND. Black.
ATHENS. Reed.
BEREA. Baur, Dustheimer.
BLUFFTON. Hirschler.
BOWLING GREEN. Mathias, Overman.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Barnett, Brand, Hancock, Justice, Kennedy, Kersten, Kindle, Lubin, Merriman, Moore, Mullings, Rhodes, Salkover, E. S. Smith, Yowell.
CLEVELAND. Baker, Boyce, O. E. Brown, Burington, Burwell, Focke, Freas, Hadley, Johnson, Jonah, Justin, Morris, Musselman, Nassau, Oldenburger, Sauté, Simon, Thomas.
COLUMBUS. Amos, Bailey, Bamforth, Beatty, Blumberg, Crandell, Hildner, Horn, M. E. Jones, Kuhn, LaPaz, MacDuffee, Manson, Morris, Newlin, Radó, Rasor, Rickard, Shover, Singer, Toops, Walters, Wildermuth.
DEFIANCE. MacCullough.
DELAWARE. Crane, Jose, Rowland.
EAST CLEVELAND. Getchell.
FINDLAY. Roots.
GAMBIER. Allen, Bumer.
GRANVILLE. Ladner, Wiley.
HILLIARD. Weaver.
HIRAM. Clarke, Jerome.
KENT. Manchester, Rogers, Stelson.
LOUDONVILLE. Feinler.
MARIETTA. Cope, Rea.
MOUNT ST. JOSEPH. Corona.
NEW CONCORD. White.
NEW LEXINGTON. Hoops.
NORTH CANTON. Schug.
NORWOOD. Wishard.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Yeaton.
OXFORD. Anderson, Pollard, Spenceley, Tappan, Wolfe.
PAINESVILLE. Lewis.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Brandeberry, Dancer, Lemme, Mercedes, O'Toole, Winslow.
WESTERVILLE. Glover, Menke.
WILMINGTON. Spinks.
WOOSTER. Knight, Williamson, Yanney.
YELLOW SPRINGS. Dwyer.
YOUNGSTOWN. Foard.

OKLAHOMA. (27)

ADA. Knight, Wallace.
ALVA. Hall.
LANGSTON. Tinner.
NORMAN. Brixey, Court, Duval, Erwin, Hassler, G. La Fon, J. E. La Fon, McFarland, Meacham, Reaves.
SHAWNEE. Short.
STILLWATER. Allen, Barnett, Flanders, Garretson, Gundersen, H. W. Smith, Zant.
TULSA. Byrd, Veatch, West.
WEATHERFORD. McCormick.
WETUMKA. Gentry.

OREGON. (9)

ALBANY. Ramsey.
CORVALLIS. Beaty, Kirkham, McAlister, Milne, Williams.
EUGENE. De Cou.
PORTLAND. Griffin, Merriss.

PANAMA. (1)

PANAMA CITY. Linares.

PENNSYLVANIA. (130)

ALLENTOWN. Deck, Hallett.
ANNVILLE. Wagner.
BALA-CYNWYD. Sensenig.
BARNESBORO. Fisanick.
BEAVER FALLS. Cleland, McCormick.
BETHLEHEM. Ashbaugh, Cairns, Cutler, Ewing, Fort, Latshaw, Rau, Raynor, Reynolds, Shook, Smail, Van Arnam.
BRIDGEVILLE. Aberle.
BRYN MAWR. Lehr, Wheeler.
BUTLER. Robb.
CALIFORNIA. Foberg.
CARLISLE. Ayres, Landis.
COLLEGEVILLE. Clawson, Manning.
EASTON. Hall, Hatch, W. M. Smith.
ERIE. Benedicta, Wells.
GROVE CITY. Grimes.
HARRISBURG. Whited.
HAVERFORD. Gummere, Reid, Wilson.
HUNTINGDON. Hess, Shively, Stayer.
KUTZTOWN. Knedler, Kunkel.
LANCASTER. Charles, Long, Worthington.
LATROBE. Seubert.
LEETSDALE. Buker.
LEWISBURG. Gold, Lindemann, MacCreadie, Richardson.
LINCOLN UNIVERSITY. Wright.
LOCKHAVEN. S. J. Smith.
MEADVILLE. Akers, Beisel, R. E. Smith.
PHILADELPHIA. Arnold, Bond, Caris, Chambers, Constable, Davis, Evans, Kline, Latshaw, Linton, Mitchell, Partridge, Rosengarten, Rothermel, Roulton, Safford, Shohat, Spencer, Tartler, Winston.
PITTSBURGH. Baird, Calkins, Cowley, Dines, Hicks, Hoover, Johnson, Karpov, Moskowitz, Neelley, Olds, Riggs, Rosenbach, Saibel, Swartzel, Taber, Taylor, Whitman.
SCRANTON. Bertrand.
SELINSGROVE. Boeder.

SEWICKLEY. Miller.
SHIPPENSBURG. Kieffer.
SLIPPERY ROCK. Lady.
STATE COLLEGE. Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Hagen, Hamilton, Moody, F. W. Owens, H. B. Owens, Rupp, Sheffer, Shibli, Wagner, West.
SWARTHMORE. Brinkmann, Dresden, Kovalenko, Marriott, Williams.
SWISSVALE. Foraker.
UPPER DARBY. McDonough.
WASHINGTON. Atchison, Bert, Moore, Rasel, Shaub, Thomas.
YORK. Baker.

PHILIPPINE ISLANDS. (3)

LAGUNA. Salvosa.
LEYTE. Icamen.
MANILA. Jimenez.

PORTO RICO. (2)

MAYAGUEZ. Sanchez-Diaz.
SAN GERMAN. Ramos.

RHODE ISLAND. (21)

EAST GREENWICH. Sperry.
EAST PROVIDENCE. Tyler.
NEWPORT. Chase.
PROVIDENCE. Adams, Adkins, Archibald, Astrachan, Bennett, Cameron, Carlen, Currier, Gilman, Hill, Manning, Oakley, Quade, Richardson, Rosskopf, Smiley, Tamarkin, Watt.

SOUTH CAROLINA. (13)

CHARLESTON. Bond, Coleman.
COLLEGE PLACE. Weber.
COLUMBIA. Coleman, Jackson, Williams.
GAFFNEY. Thompson.
GREENVILLE. Earle.
HARTSVILLE. Reaves.
ROCK HILL. Grant, Pugh.
SALUDA. Ramage.
SPARTANBURG. Peck.

SOUTH DAKOTA. (11)

BROOKINGS, MacDougal, Miller, Wente.
FAIRVIEW. Olson.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Bowles, Cook.
SIOUX FALLS. Fuller.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.

TENNESSEE. (14)

CHATTANOOGA. Perry.
CLEVELAND. Hutto.
JACKSON. Carr.
KNOXVILLE. Bond, Ghormley.
MARYVILLE. Knapp.
NASHVILLE. Blair, S. I. Jones, N. P. Miser, W. L. Miser, Peterson, Wren.
PULASKI. Meade.
TOWNSEND. Keller.

TEXAS. (65)

ABILENE. Burnam, Tate.
ALPINE. Gilley.
AUSTIN. Banks, Batchelder, Benedict, Cooper, Craig, Decherd, Dodd, Ettlinger, Haskell, Lubben, Moore, Muller, Norbert, Robinson, Vandiver.
BORGER. May.
BROWNSVILLE. De la Garza.
CANYON. Murray.
CENTER POINT. Rees.
COLLEGE STATION. Blumberg, Edmonson, Halperin.
DALLAS. Cell, Dice, E. H. Jones.
DENTON. M. C. Brown, Hughes, Oldham.
EDINBURG. Searcy.
EL PASO. Kennedy.
FORT WORTH. Howard, Sherer.
GALVESTON. Underwood.
GEORGETOWN. Wapple.
HASKELL. Nelson.
HOUSTON. Blau, Bray, Dean, Evans, Ford, Lovett, W. A. Rees, Streetman.
LUBBOCK. Sparks, Thompson, Underwood.
NACOGDOCHES. Cross, Ferguson, Oxsheer.
PAMPA. Rice.
PRAIRIE VIEW. Randall, Turner, Wilson.
SAN ANTONIO. Hurry, McNelly, Schnepp.
STEPHENVILLE. Cromwell, McSweeny, Redden.
TYLER. Holmes.
WAXAHACHIE. Newton.
WICHITA FALLS. Adams.

UTAH. (4)

EPHRAIM. Horsfall.
SALT LAKE CITY. Gibson, Hayes, Pehrson.

VERMONT. (10)

BURLINGTON. Bullard, Butterfield, Millington, Swift, Thomas.
MIDDLEBURY. Hazeltine, Wiley.
NORTHFIELD. Dix, Holmes.
WINOOSKI. Alliot.

VIRGINIA. (42)

ASHLAND. Simpson.
BLACKSBURG. Hatcher, O'Shaughnessy, Rasche, Williams.
BLUEFIELD. Wright.
CHARLOTTESVILLE. Wade.
EMORY. Miller.
FARMVILLE. Taliaferro.
GREENWAY. Hollis.
HAMPTON. Perkins.
LANGLEY FIELD. Pinkerton.
LEXINGTON. Byrne, Paxton, Purdie, L. W. Smith.
LYNCHBURG. Berry, Larew, Pattillo, Wiggin.
MONTEREY. Colaw.
RICHMOND. Gaines, Harris, M. L. Smith, Wheeler.
SALEM. Carpenter.
SOUTH BOSTON. Patten.
STAUNTON. Taylor.
SWEET BRIAR. Morenus.

UNIVERSITY. Aylor, Blincoe, Buchanan, Echols, Linfield, Luck, Oglesby, Puckett, Sparrow.
WILLIAMSBURG. Calkins, Russell, Stetson.
WOODSTOCK. Wunder.

WASHINGTON (17).

CLARKSTON. Keeler.
OVINGTON. Benander.
PULLMAN. Butler, Isaacs.
SEATTLE. Ballantine, Carlson, Cramlet, McFarlan, Moritz, Mullemeister, Neikirk, Winger.
SPOKANE. Amos.
TACOMA. Hanawalt, Martin.
WALLA WALLA. Bratton.
YAKIMA. Whitney.

WEST VIRGINIA. (18)

ESKDALE. Todd.
HARPERS FERRY. Drew.
HUNTINGTON. Hackney.
INSTITUTE. Lacy.
INWOOD. Mish.
KEYSER. Wilson.
MONTGOMERY. W. F. Smith.
MORGANTOWN. Colwell, Davis, Eiesland, Reynolds, Stewart, Turner, Vehse.
SMITHERS. Bell.
VAN. Morris.
WHEELING. Bagby, Weimer.

WISCONSIN. (45)

BELOIT. Conwell, Huffer.
GREEN BAY. Kalcik.
MADISON. Allen, Bennett, Evans, Hart, Hartung, Ingraham, Langer, Lowney, March, Rose, Skinner, I. S. Sokolnikoff, Van Vleck, Wilson.
MILWAUKEE. Battig, Bear, Beckwith, Ericson, Evans, Knight, Ledesma, Lewandowski, Marden, Mary Felice, Norris, Parkinson, Pettit, Quarles, Rasor, Roth, Vass.
OSHKOSH. Beenken, Price.
PLATTEVILLE. Warner.
RIPON. Woodmansee.
RIVER FALLS. Eide.
SUPERIOR. C. W. Smith.
WAUKESHA. Hopkins.
WEST ALLIS. Wolf.
WEST DE PERE. De Cleene, Sromovsky.
WISCONSIN RAPIDS. McMillan.

WYOMING. (4)

LARAMIE. Barr, Bellamy, Neubauer, Rechard.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINA. (1)

BUENOS AIRES, Baidaff.

AUSTRIA. (1)

VIENNA. Putnam.

BELGIUM. (1)

LOUVAIN. Vanhee.

BRITISH HONDURAS. (1)

ORANGE WALK. Zimmerman.

BURMA. (1)

RANGOON. Campbell.

CHILE. (1)

SANTIAGO. Salas-Edwards.

CHINA. (4)

CANTON. MacDonald, Woo.
NANKING. Chang.
PEKING. Konantz.

FRANCE. (3)

LE MANS. Thébault.
PARIS. Fréchet, Hadamard.

GERMANY. (1)

GÖTTINGEN. Bond.

GREAT BRITAIN. (6)

CAMBRIDGE. Hardy, James.
DUBLIN. Rowe.
EDINBURGH. Horsburgh.
NOTTINGHAM. Piaggio.
OXFORD. Frecheville.

INDIA. (3)

ALLAHABAD CITY. Mitra.
BOMBAY. Dalal.
MADURA. Lockwood.

ITALY. (5)

BOLOGNA. Bortolotti, Pincherle.
CAGLIARI. Crudeli.
ROME. Enriques, Laboccetta.

JAPAN. (1)

SENDAI. Hayashi.

KOREA. (1)

PYENGYANG. Parker.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

PALESTINE. (1)

RAM ALLAH. Tarazi.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SIAM. (1)

BANGKOK. Hadlock.

SOUTH AFRICA. (3)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.
RONDEBOSCH. Muir.

SOUTH AUSTRALIA. (1)

ADELAIDE. Wilton.

SWITZERLAND. (1)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHATEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (2)

CONSTANTINOPLE. Mourad.
ISTANBUL. Harshbarger.

UKRAINE. (1)

KIEFF. Kryloff.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF TRUSTEES AND OFFICERS

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement of such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-President such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings, provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

Presidents

E. R. Hedrick ... 1916
Florian Cajori ... 1917
E. V. Huntington ... 1918
H. E. Slaught ... 1919
D. E. Smith ... 1920
G. A. Miller ... 1921
R. C. Archibald ... 1922
R. D. Carmichael ... 1923
H. L. Rietz ... 1924
J. L. Coolidge ... 1925
Dunham Jackson ... 1926
W. B. Ford ... 1927–1928
J. W. Young ... 1929–1930
E. T. Bell ... 1931–1932
Arnold Dresden ... 1933–

Vice-Presidents

E. V. Huntington ... 1916
G. A. Miller ... 1916
D. N. Lehmer ... 1917, 1918
Oswald Veblen ... 1917
J. W. Young ... 1918, 1926
R. G. D. Richardson ... 1919
H. L. Rietz ... 1919
Helen A. Merrill ... 1920
E. J. Wilczynski ... 1920
R. C. Archibald ... 1921
R. D. Carmichael ... 1921, 1922
B. F. Finkel ... 1922
A. B. Chace ... 1923
L. P. Eisenhart ... 1923
J. L. Coolidge ... 1924
Dunham Jackson ... 1924, 1925
A. A. Bennett ... 1925, 1933
W. B. Ford ... 1926
A. J. Kempner ... 1927, 1928
Clara E. Smith ... 1927
F. D. Murnaghan ... 1928
E. T. Bell ... 1929, 1930
W. C. Graustein ... 1929, 1930
Arnold, Dresden ... 1931
C. N. Moore ... 1931
W. H. Bussey ... 1932
G. C. Evans ... 1932
E. B. Stouffer ... 1933

Secretary-Treasurer

(Appointed by the Trustees after 1918)

(W. D. Cairns ... 1916–

Committee on Official Journal

(Appointed by the Trustees.)

H. E. Slaught ... 1916–
R. D. Carmichael ... 1916–1918
W. H. Bussey ... 1916–1918
R. C. Archibald ... 1919–1921
W. A. Hurwitz ... 1919–1921
A. A. Bennett ... 1922
H. P. Manning ... 1921–1922
W. B. Ford ... 1923–1925
J. L. Coolidge ... 1923
A. J. Kempner ... 1924–
W. H. Bussey ... 1926–1931
W. B. Carver ... 1932–

Elected Members of the Board

D. N. Lehmer ... 1916–1918, 1922–1924, 1930–1932
R. E. Moritz ... 1916–1918
K. D. Swartzel ... 1916
Oswald Veblen ... 1916, 1920–1922, 1926–1931
R. C. Archibald ... 1916–1917, 1923–1930
Florian Cajori ... 1916, 1918–1923, 1926–1930
M. B. Porter ... 1916–1917
J. W. Young ... 1916–1917, 1920–1922
B. F. Finkel ... 1916–1921, 1930–
E. H. Moore ... 1916–1921, 1923–1928
Alexander Ziwet ... 1916–1918
E. R. Hedrick ... 1917–1922, 1924–1929, 1932–
J. N. Van der Vries ... 1916–1918
Helen A. Merrill ... 1917–1919
D. E. Smith ... 1917–1919, 1921–1926
Elizabeth B. Cowley ... 1918–1920
G. A. Miller ... 1918–1920, 1922–1924
E. J. Wilczynski ... 1918–1919, 1922–1926
L. P. Eisenhart ... 1919–1922, 1925–1930
E V. Huntington ... 1917, 1919–1927, 1933–
E. L. Dodd ... 1920
R. D. Carmichael ... 1920, 1924–1929
A. A. Bennett ... 1921, 1930–1932
H. L. Rietz ... 1921–1923, 1925–1930
C. F. Gummer ... 1921–1925
Dunham Jackson ... 1923–1929
Clara E. Smith ... 1923–1925
A. B. Chace ... 1924–1925
J. L. Coolidge ... 1926–1931
E. T. Bell ... 1927–1928
E. P. Lane ... 1928–1933
W. B. Ford ... 1929–
E. R. Smith ... 1929
W. L. Hart ... 1930–
Lao G. Simons ... 1930–1931
L. L. Dines ... 1931–
T. C. Fry ... 1931–
J. W. Glover ... 1931–
H. E. Buchanan ... 1932–
W. R. Longley ... 1932–
E. J. Moulton ... 1933–

The Carus Mathematical Monographs

The CARUS MONOGRAPHS are already fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS, and her son, DR. EDWARD H. CARUS.

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No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSOR DAVID EUGENE SMITH and DOCTOR JEKUTHIEL GINSBURG. (To be published in January 1934.)

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CHANCELLOR ARNOLD BUFFUM CHACE, of Brown University, rendered signal honor to the ASSOCIATION by publishing under its auspices his RHIND MATHEMATICAL PAPYRUS. The entire receipts from the sale of this great work will be used to start an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHACE FUND.

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Volume II, 140 plates (11¼″ x 14″) in two colors with Text and Introductions, contains the Photographic Facsimile, Hieroglyphic Transcription, Transliteration, and Literal Translation.

This exposition of the oldest mathematical document in the world will be of great value, not only to students of mathematics but to any one interested in the work of a civilization existing nearly 4,000 years ago.

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